Smectics, Symmetry Breaking and Surfaces

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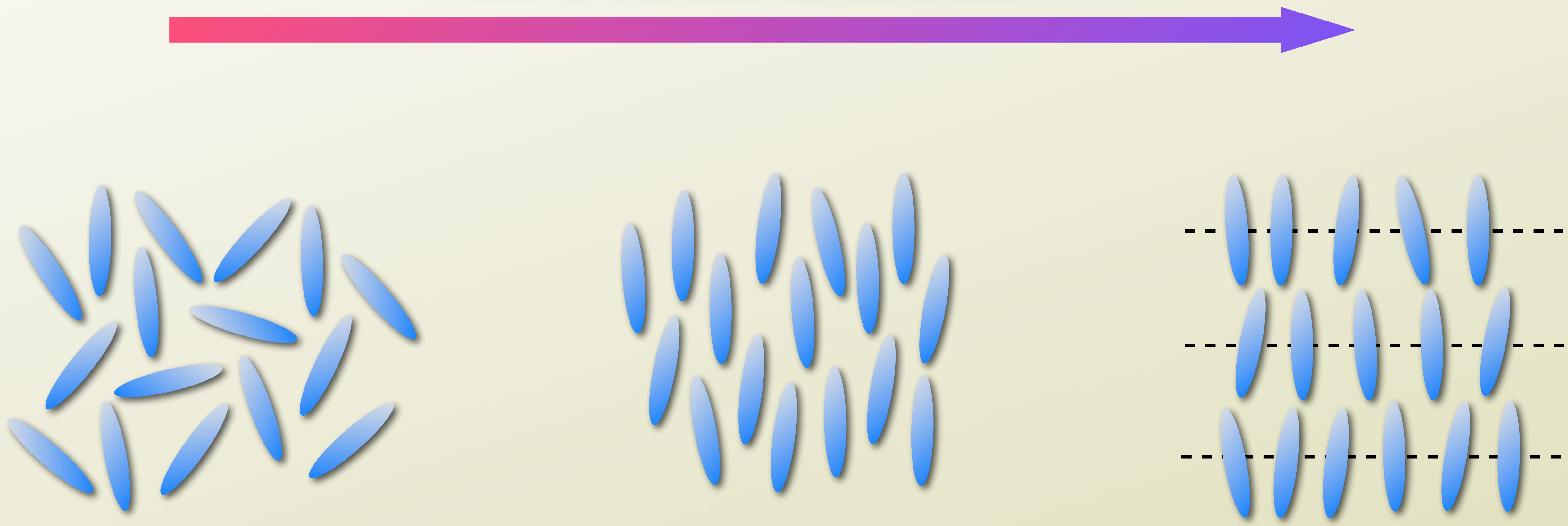
Department of Physics & Astronomy
University of Pennsylvania

Photo by Michi Nakata

University of Virginia, March 25th 2010

Liquid Crystal Mesophases

cool or increase concentration



Isotropic | Nematic | Smectic-A

uniaxial directional order

one-dimensional positional order

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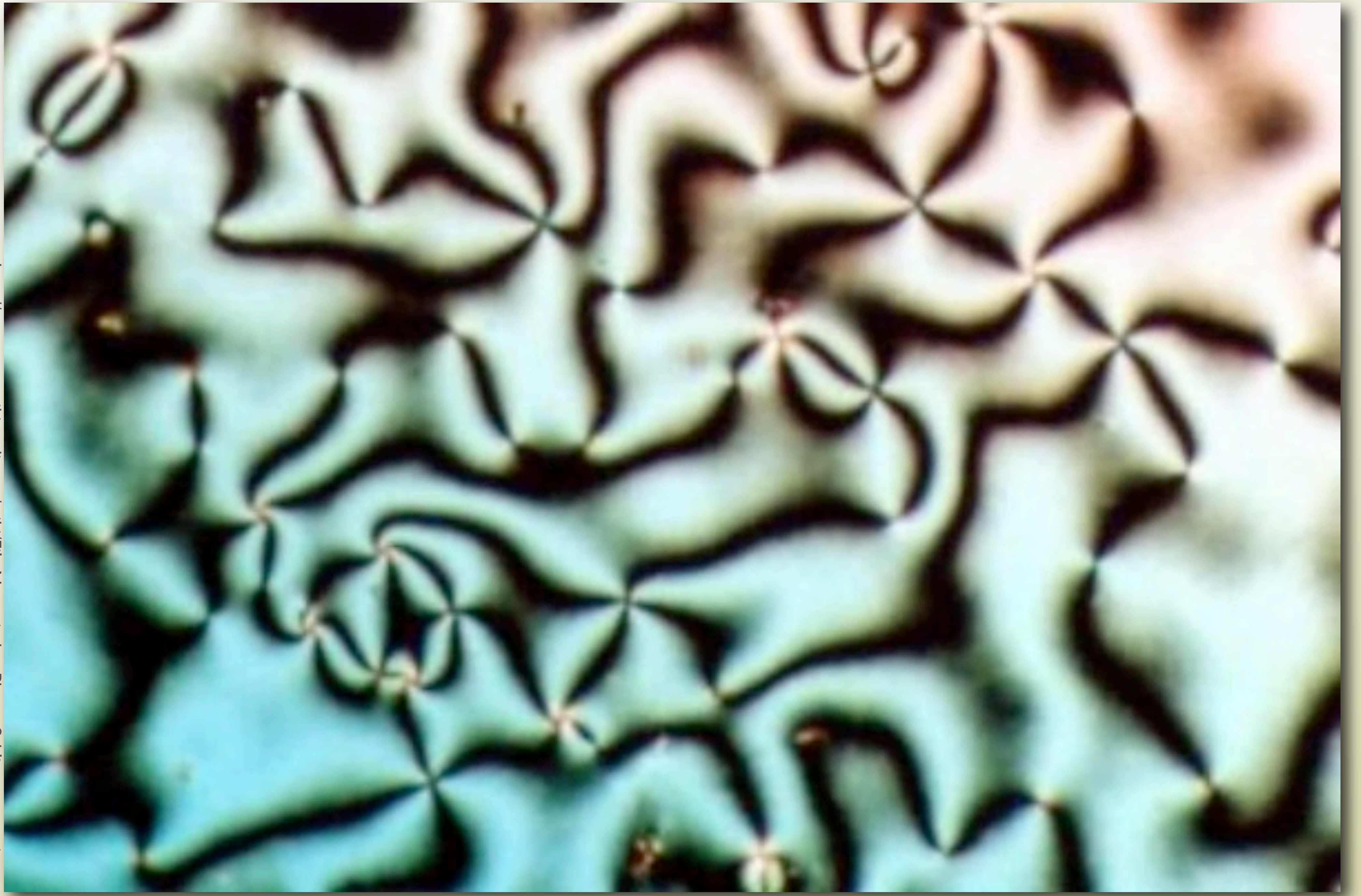
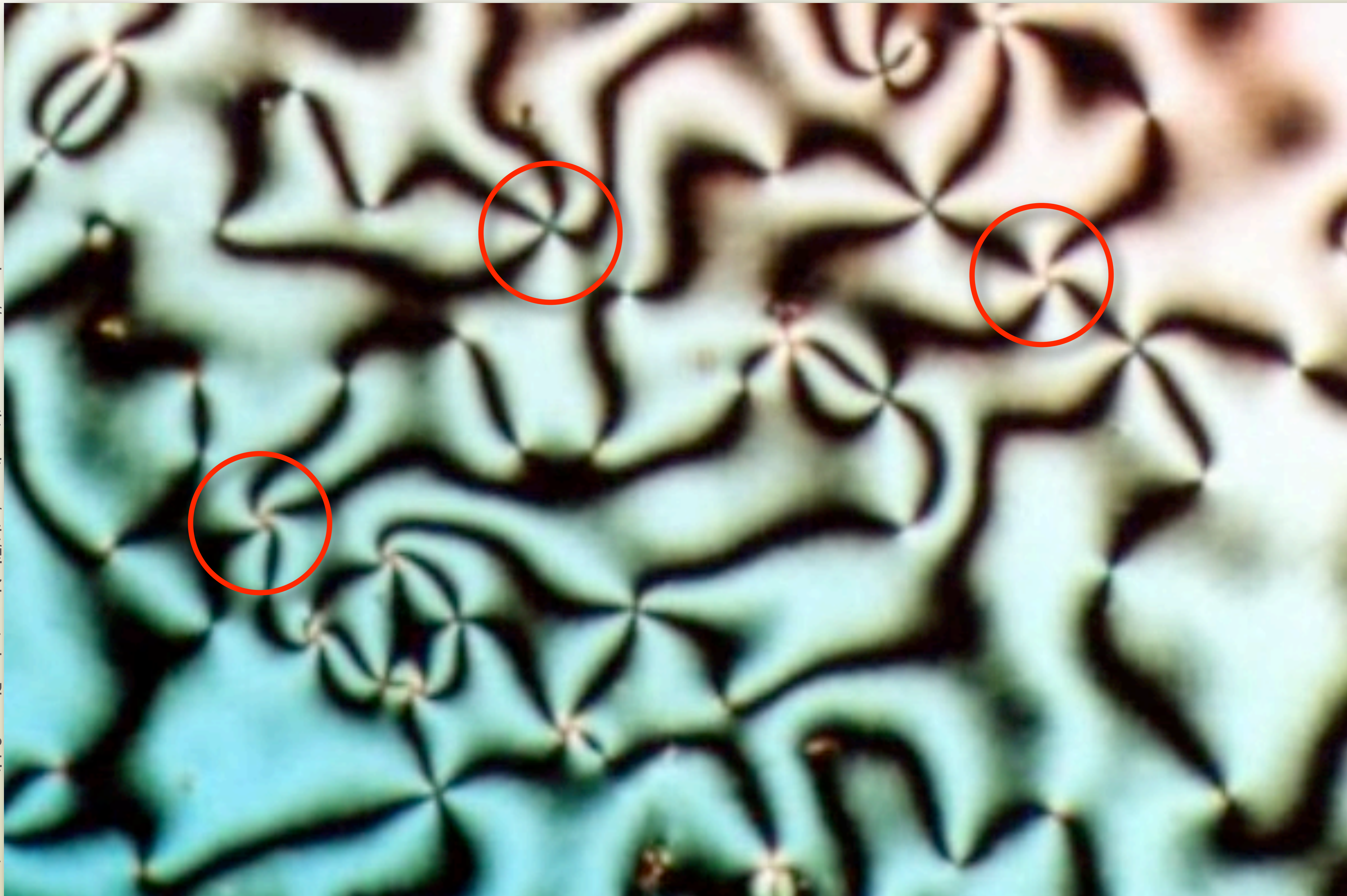
NEMATICS IN TWO DIMENSIONS



http://commons.wikimedia.org/wiki/File:Nematische_Phase_Schlierentextur.jpg

Nematics in Two Dimensions



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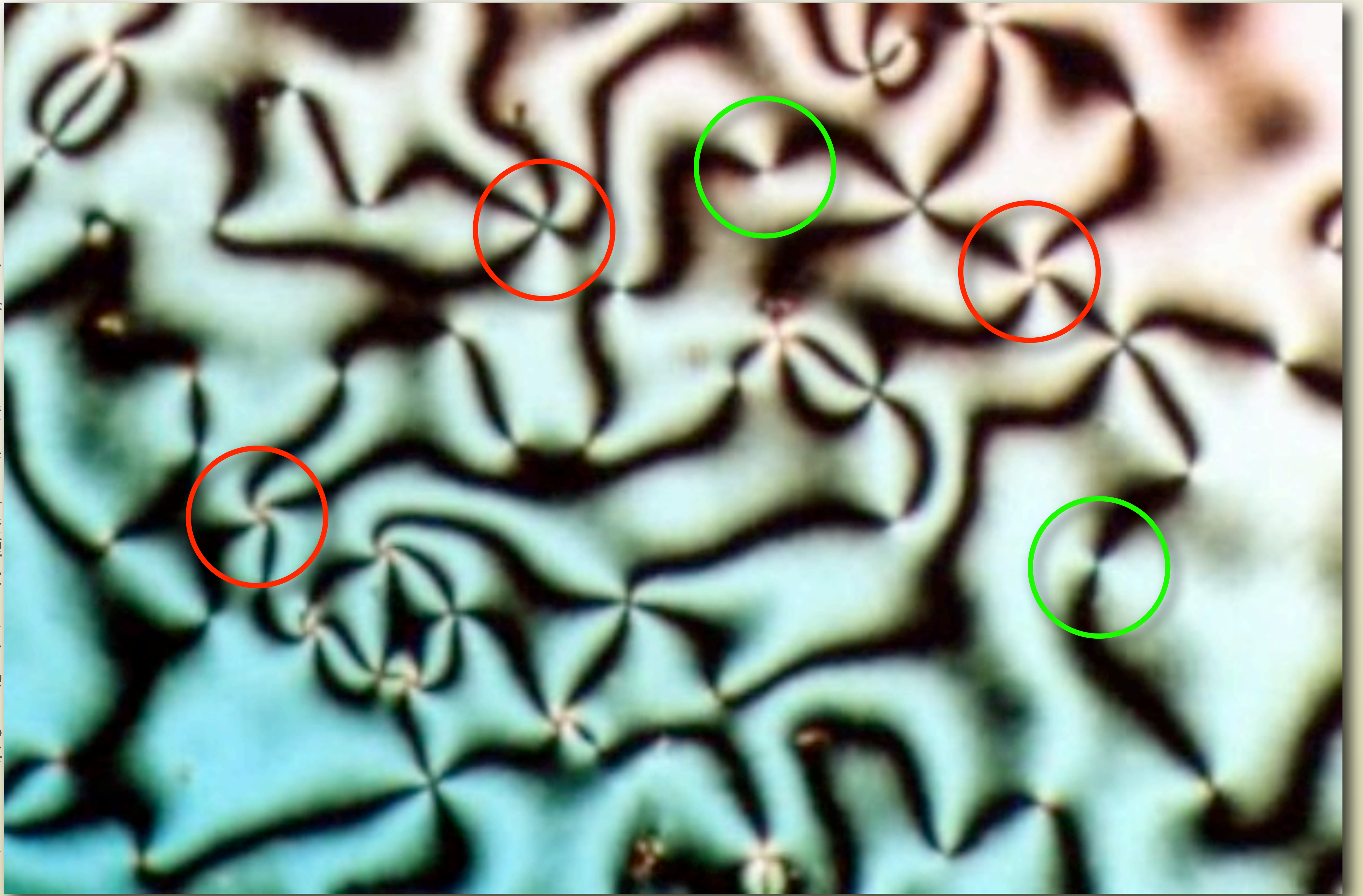

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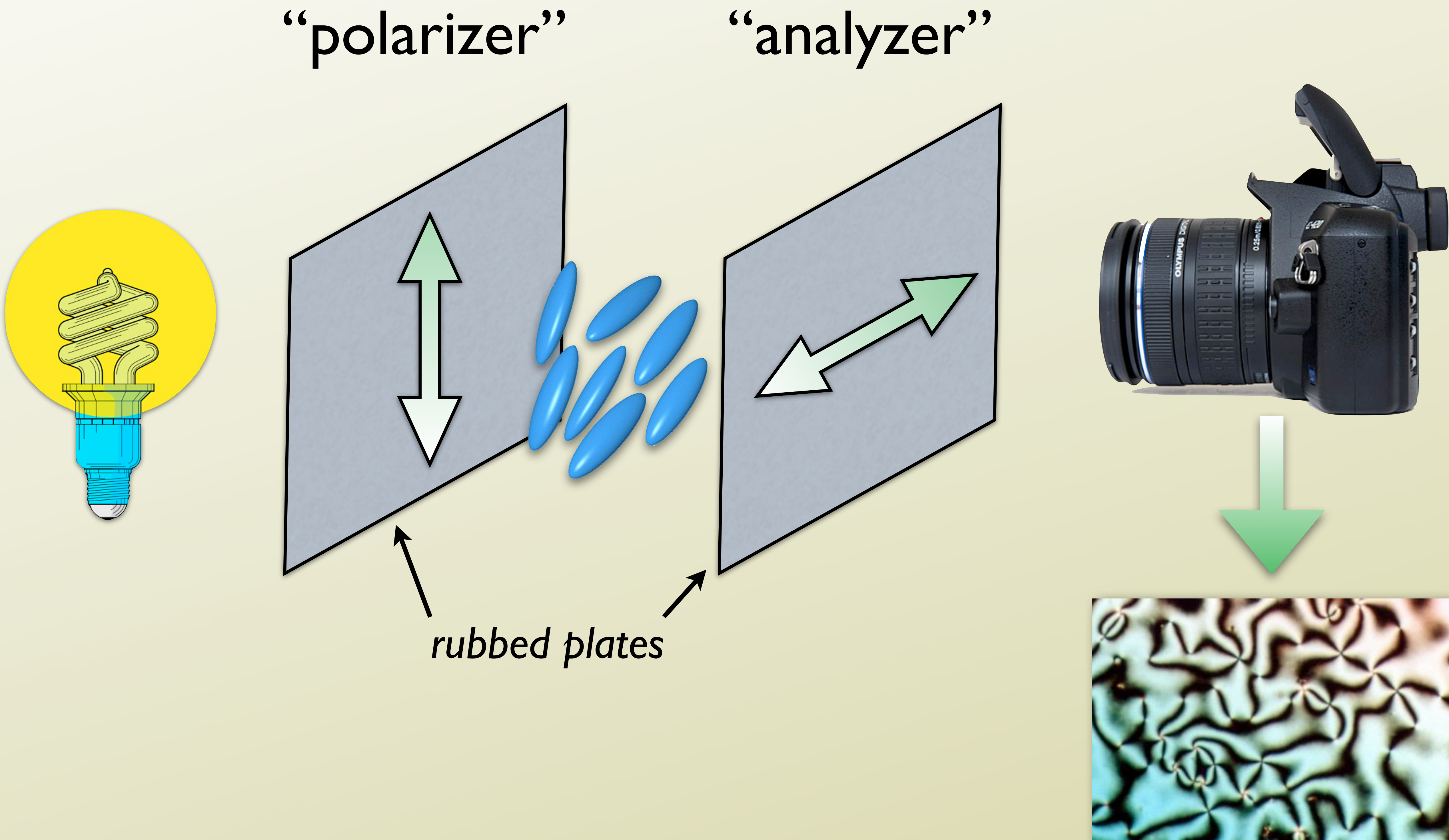
NEMATICS IN TWO DIMENSIONS: *WHAT ARE WE SEEING?*



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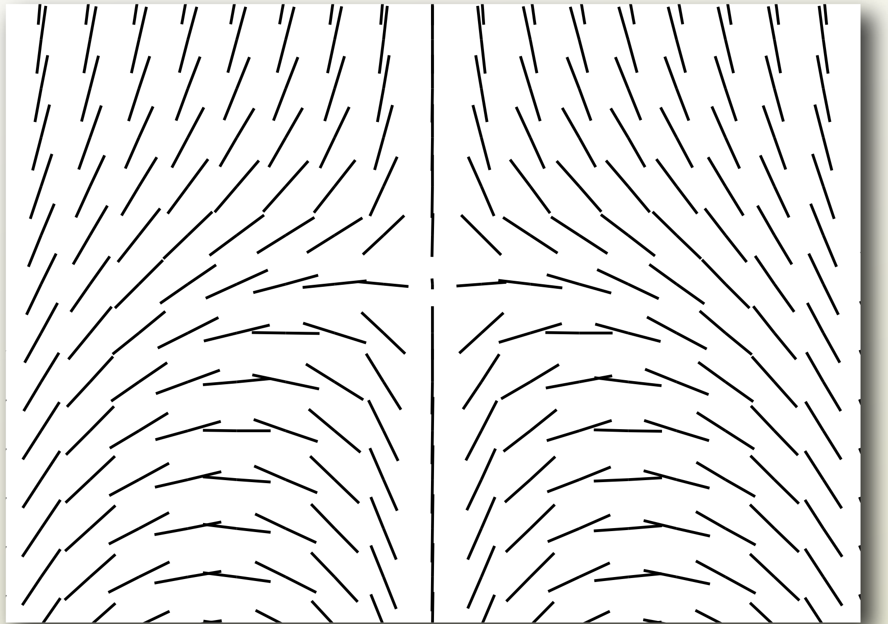

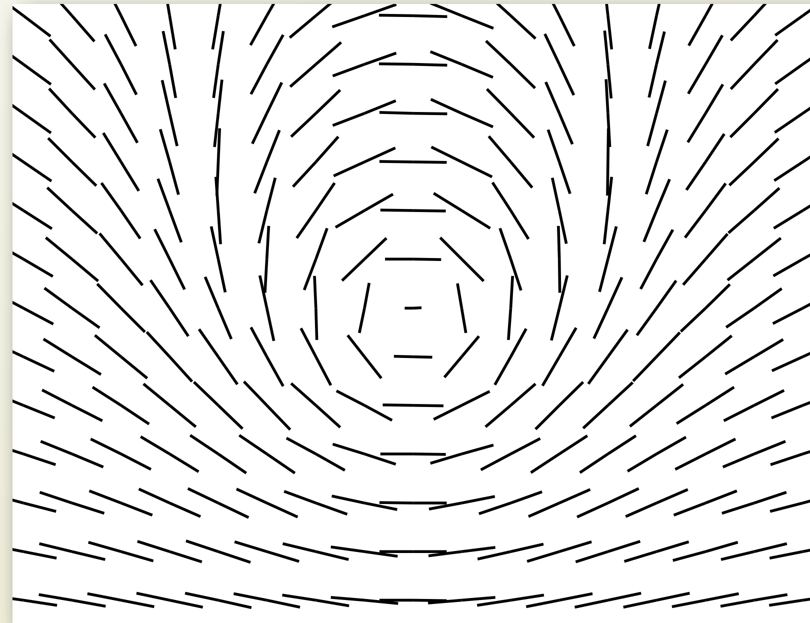

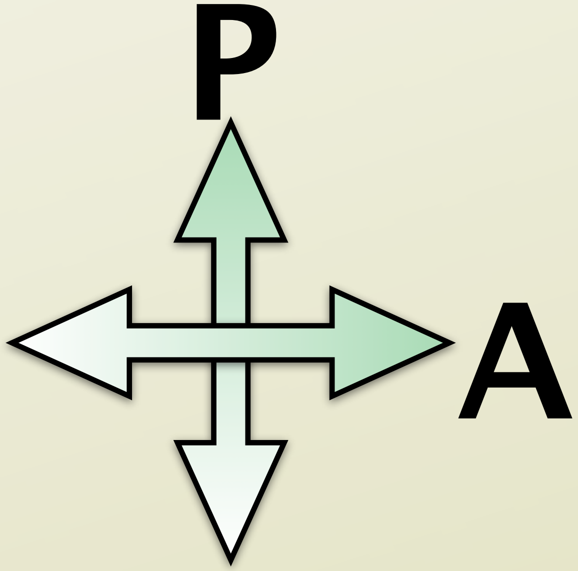

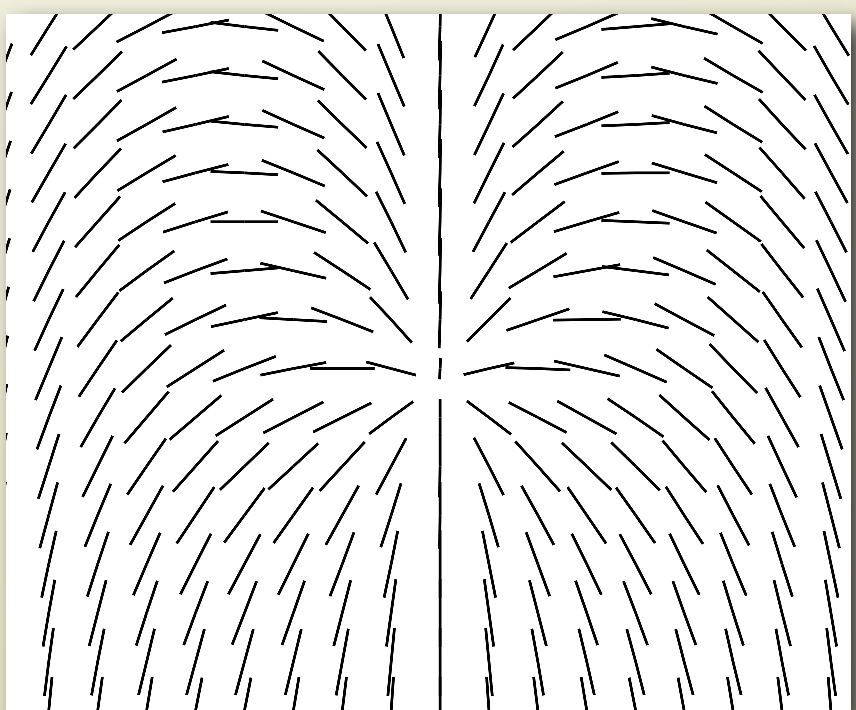

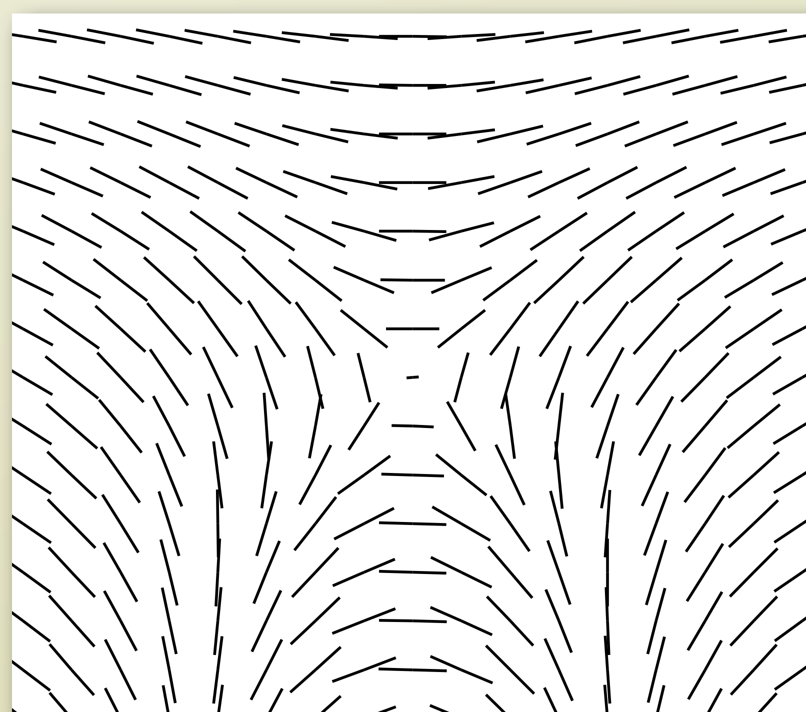

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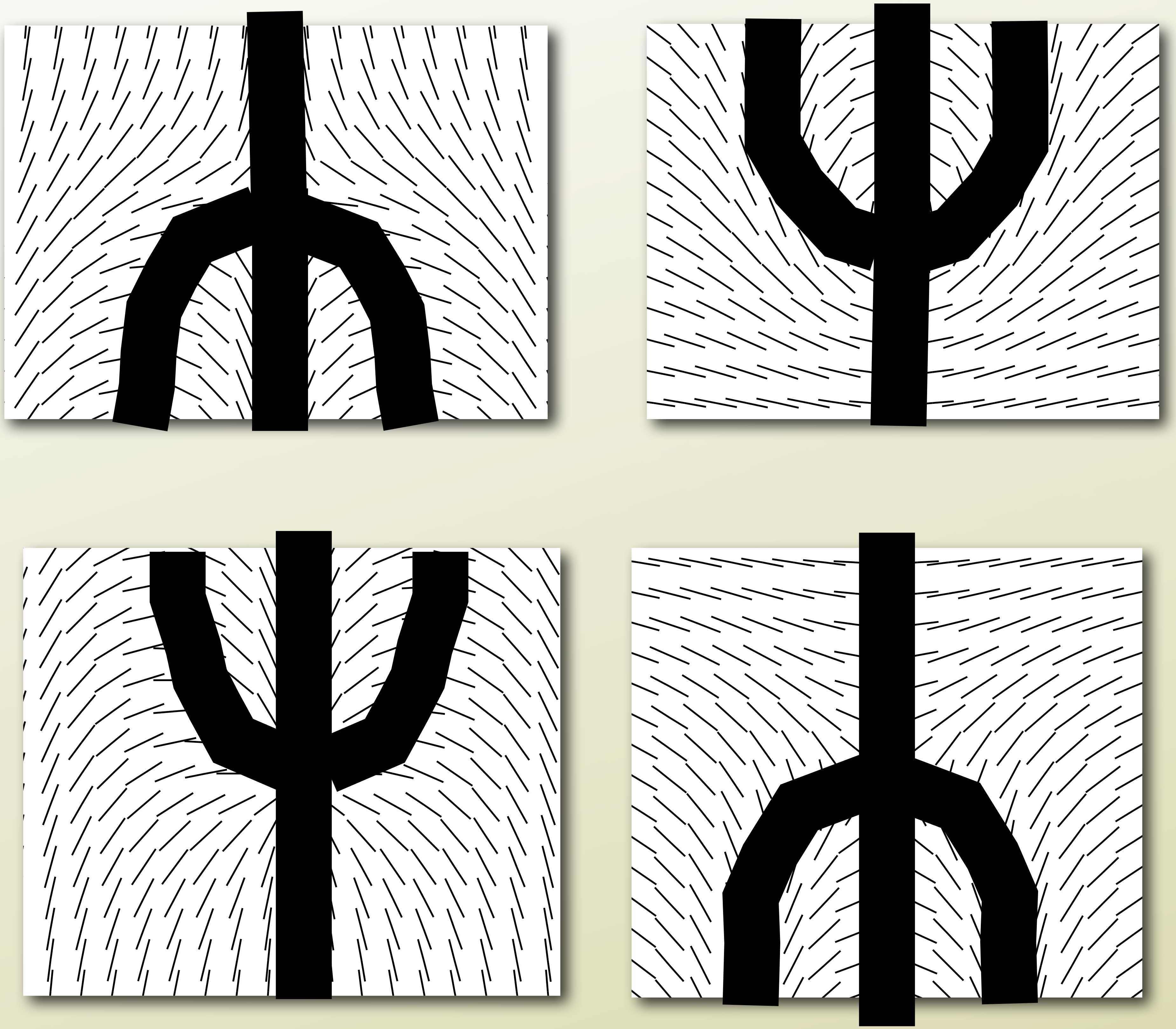

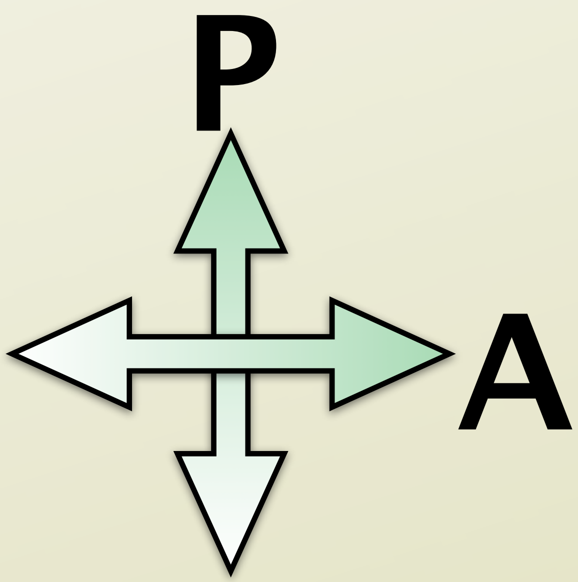

the brushes are the preimages of the polarizer and analyzer direction

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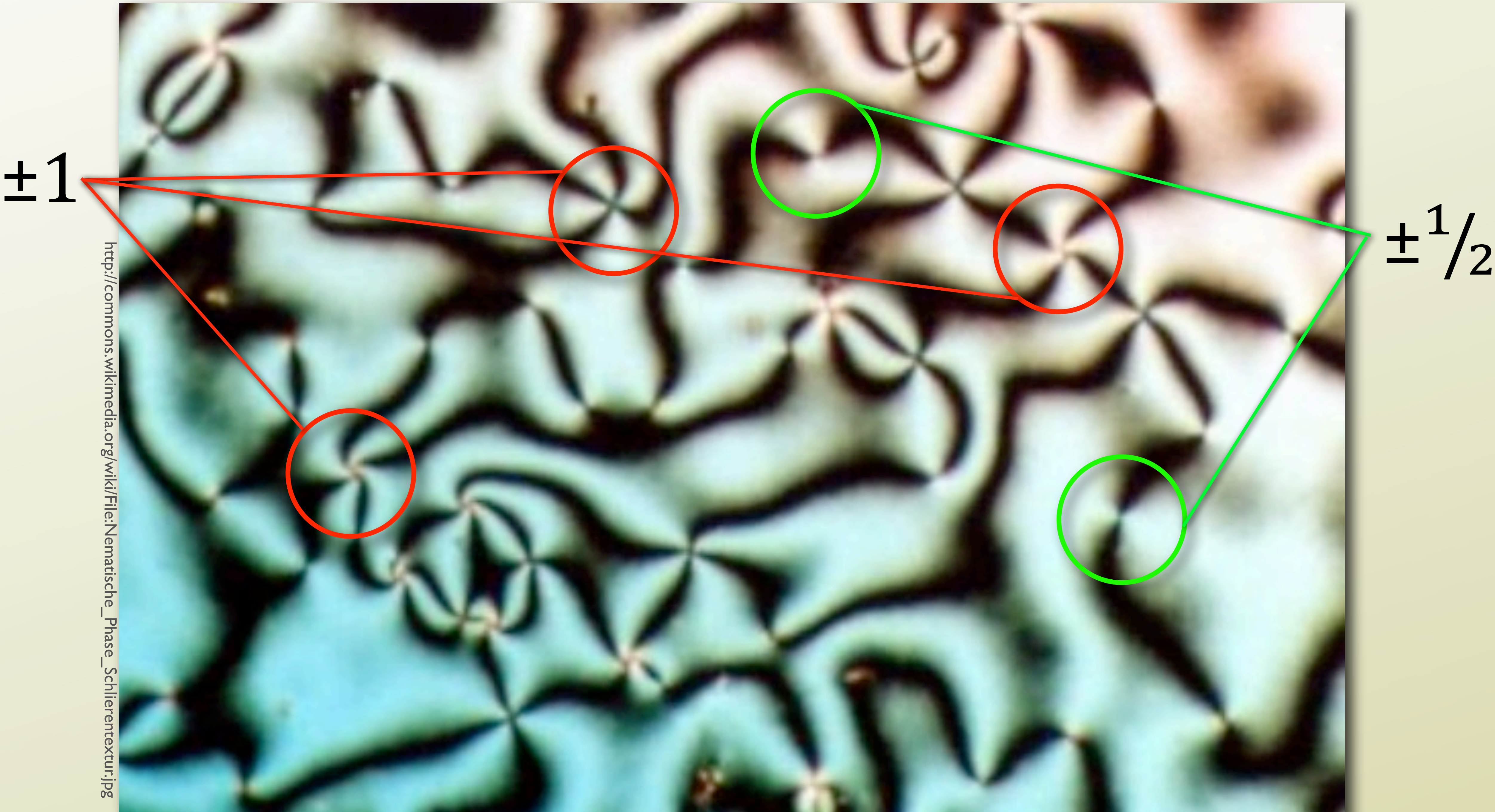
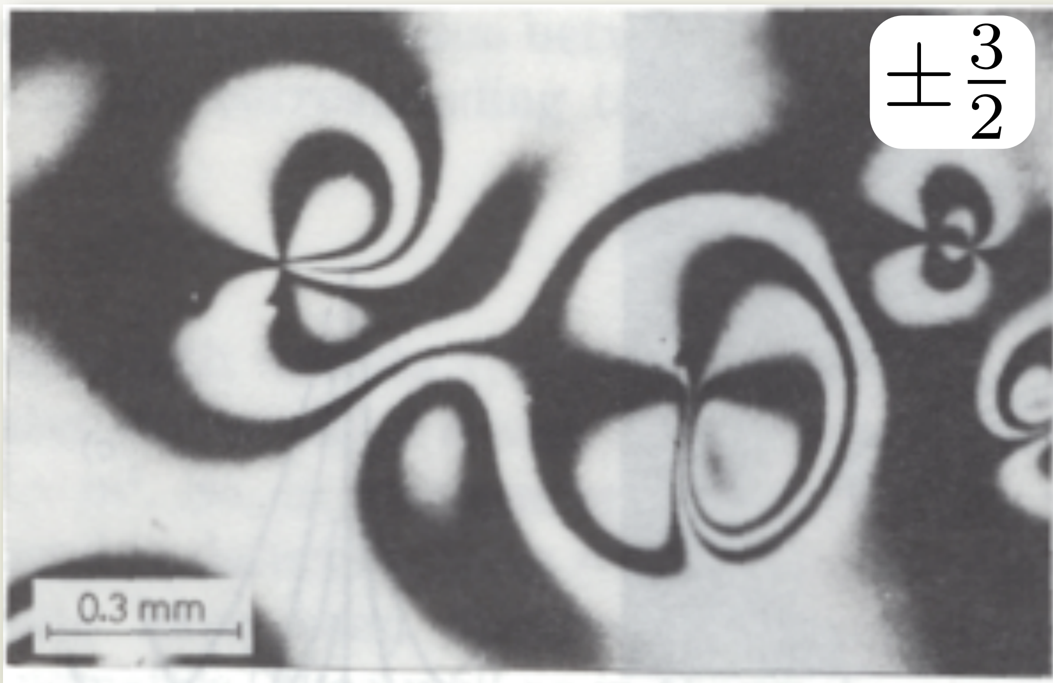
NEMATICS IN TWO DIMENSIONS

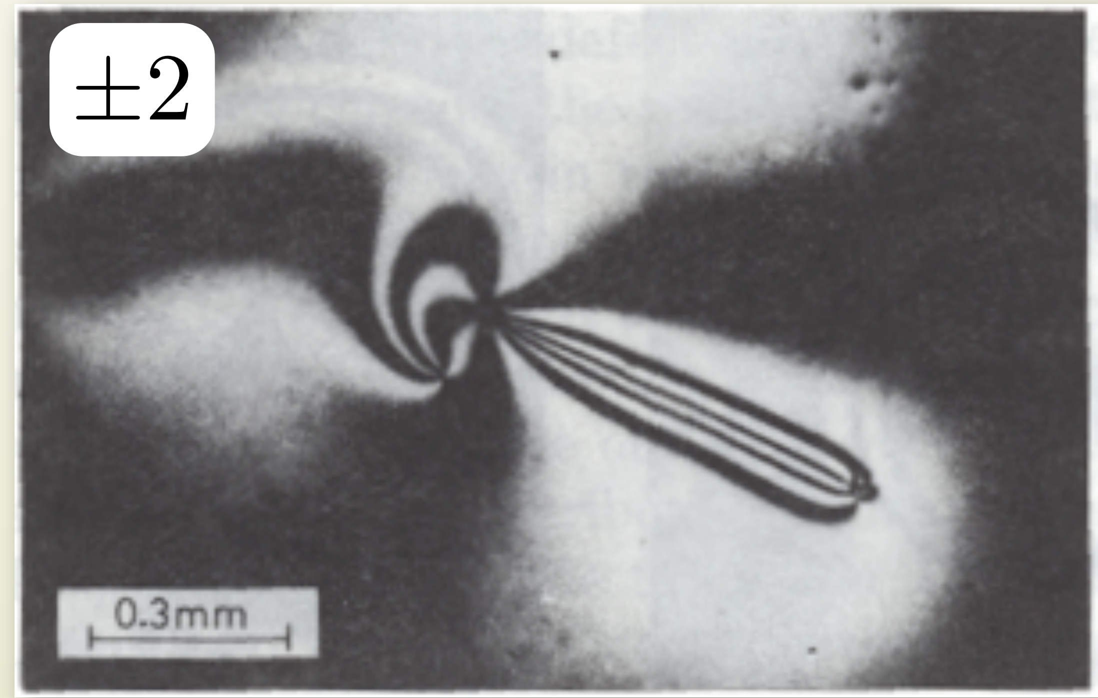


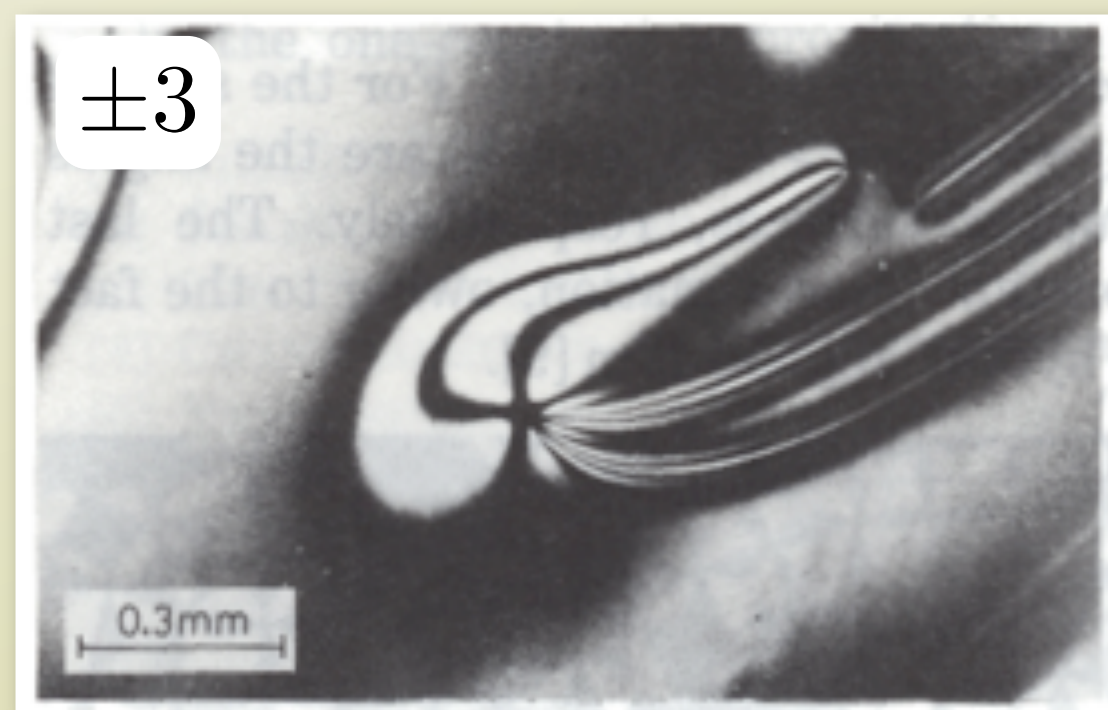
http://commons.wikimedia.org/wiki/File:Nematische_Phase_Schlierentextur.jpg

Maps from $\mathbb{R}^2\backslash\{0\} \rightarrow \mathbb{R}P^1$

HIGHER CHARGES?

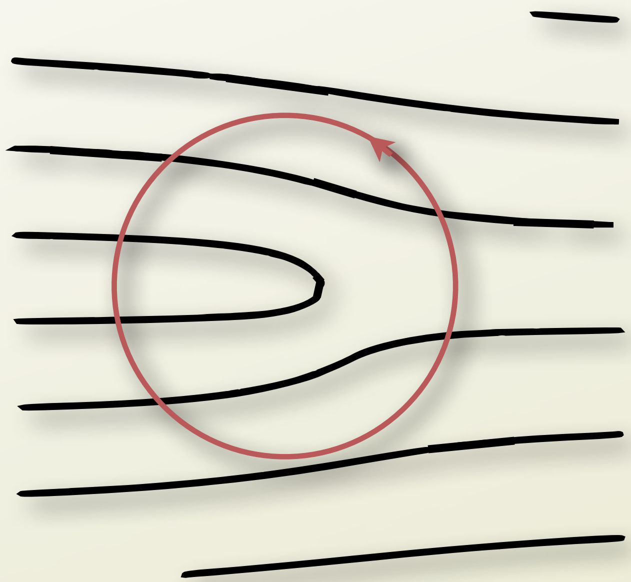


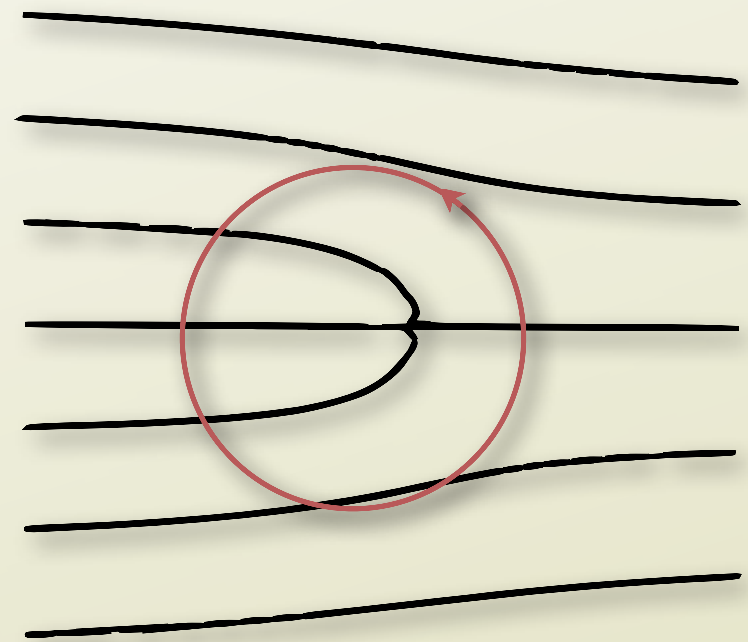


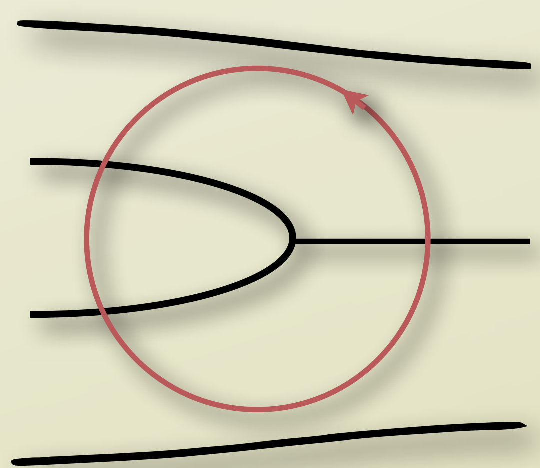


Lavrentovich & Natishin, *EPL* **12**, 135 (1990)

Dislocations: defects in the translational order

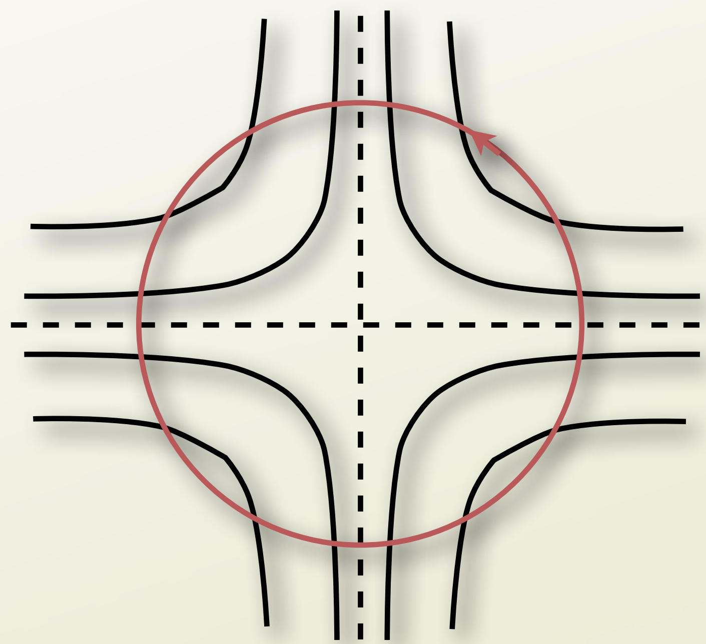


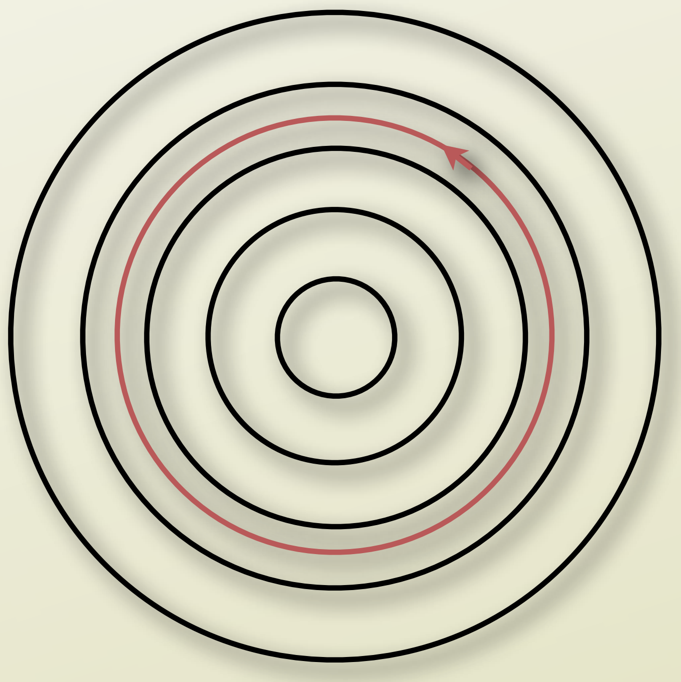


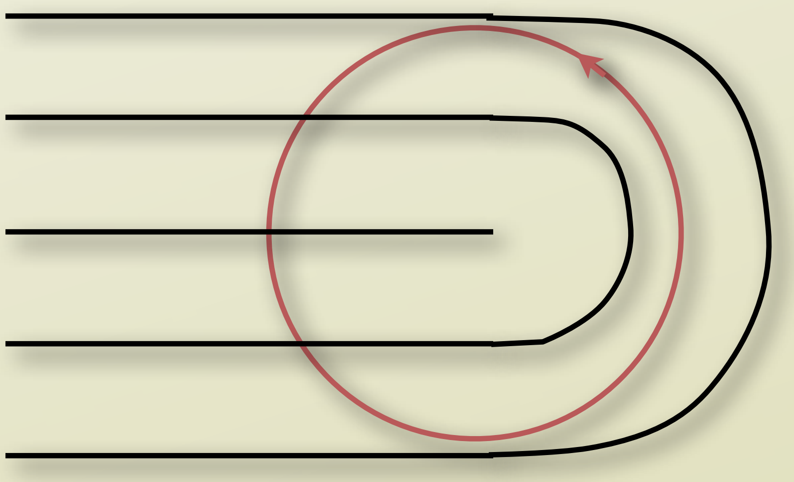


Maps from $\mathbb{R}^2 \backslash \{0\} \to S^1$

DISCLINATIONS: DEFECTS IN THE ORIENTATIONAL ORDER







Maps from $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}P^1$

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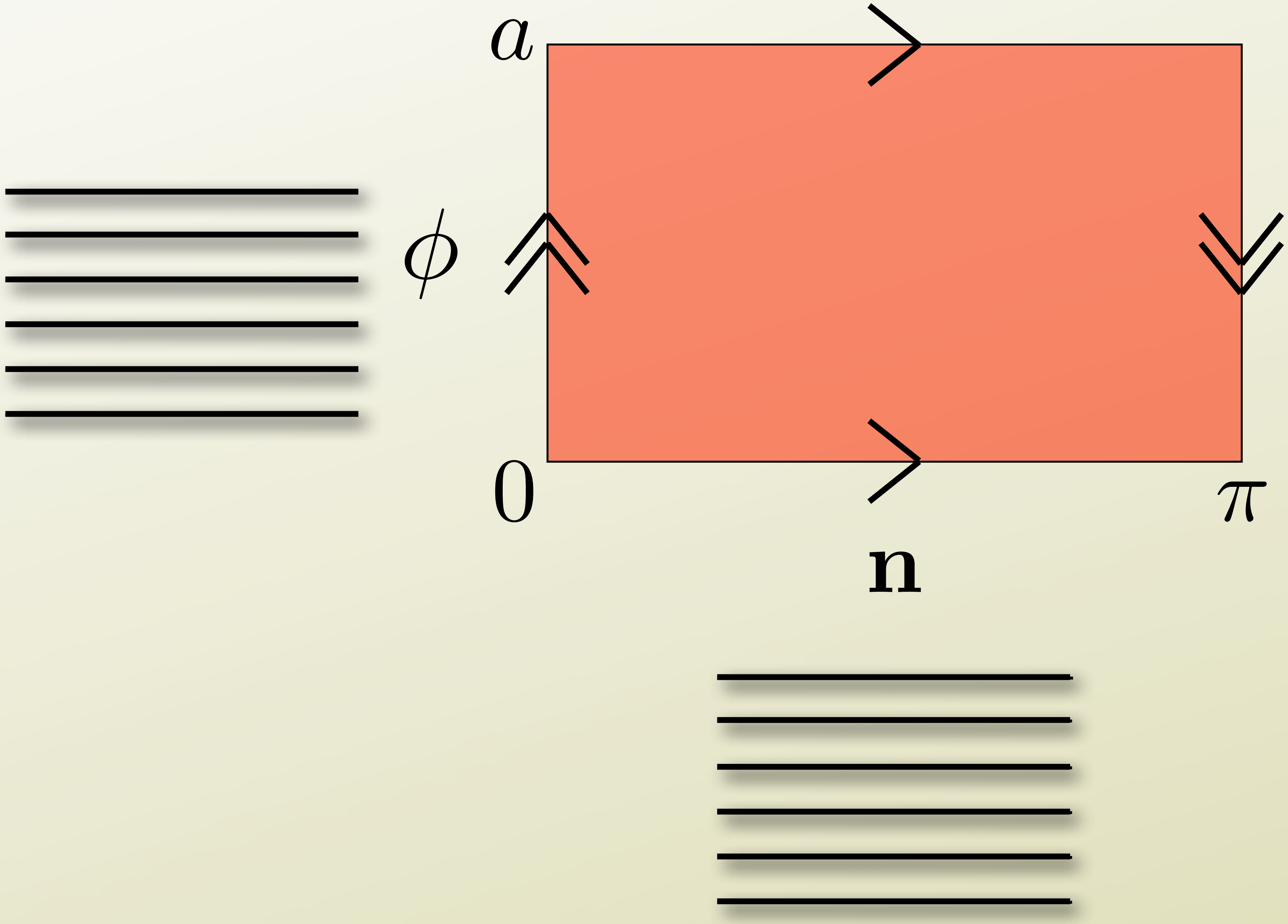
Ground State Manifold: Fundamental Group



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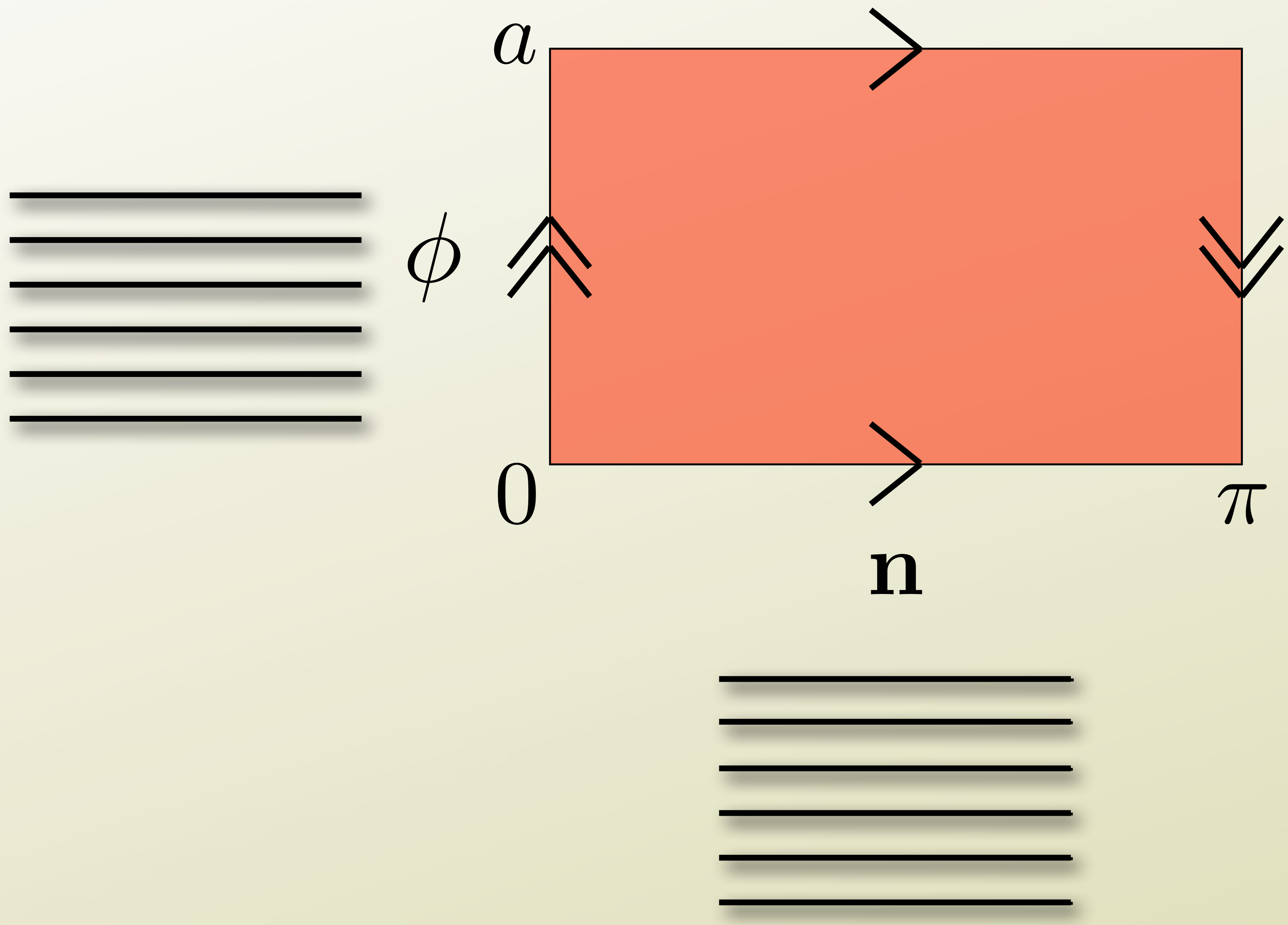
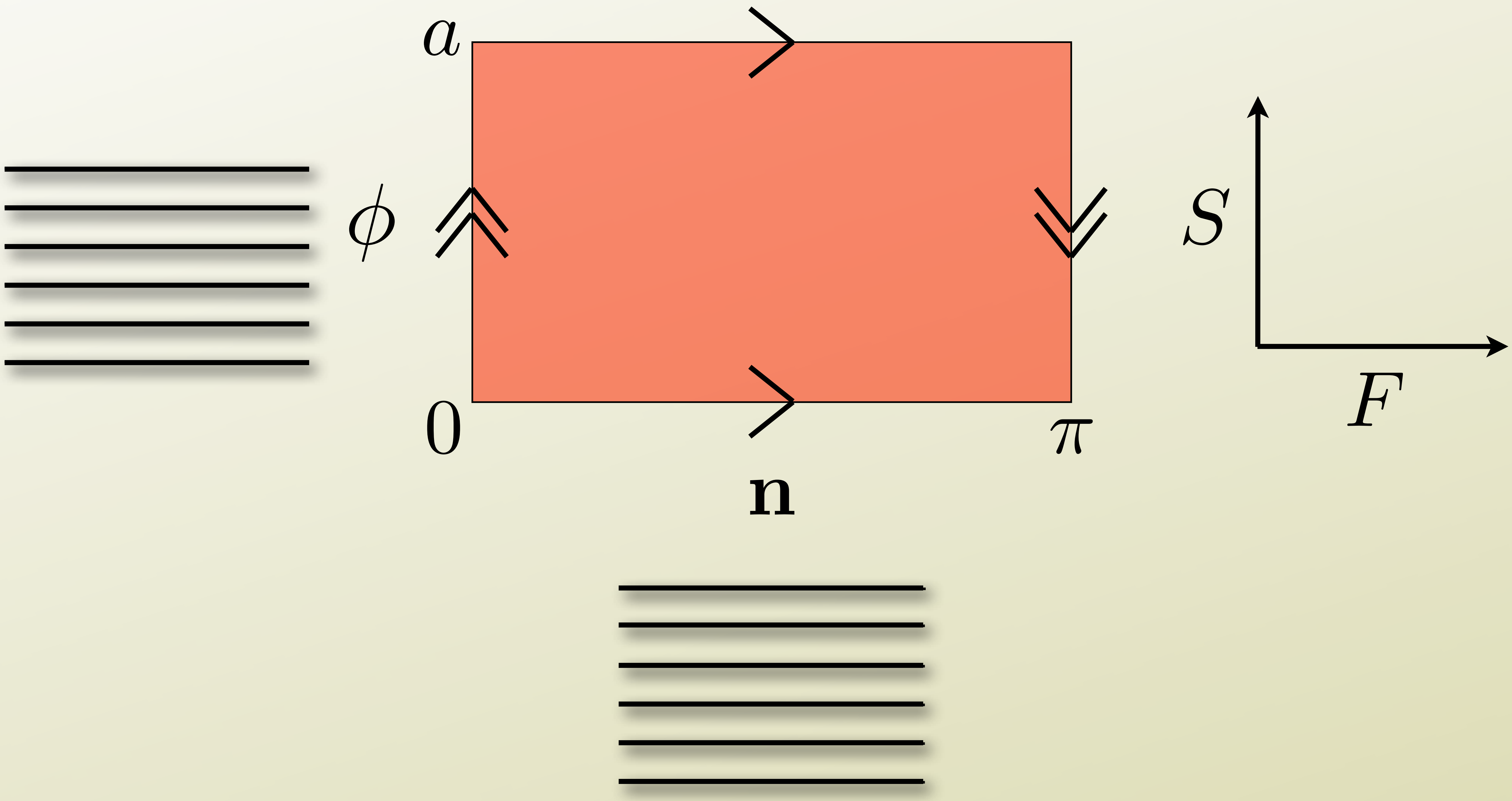


Ground State Manifold: Fundamental Group



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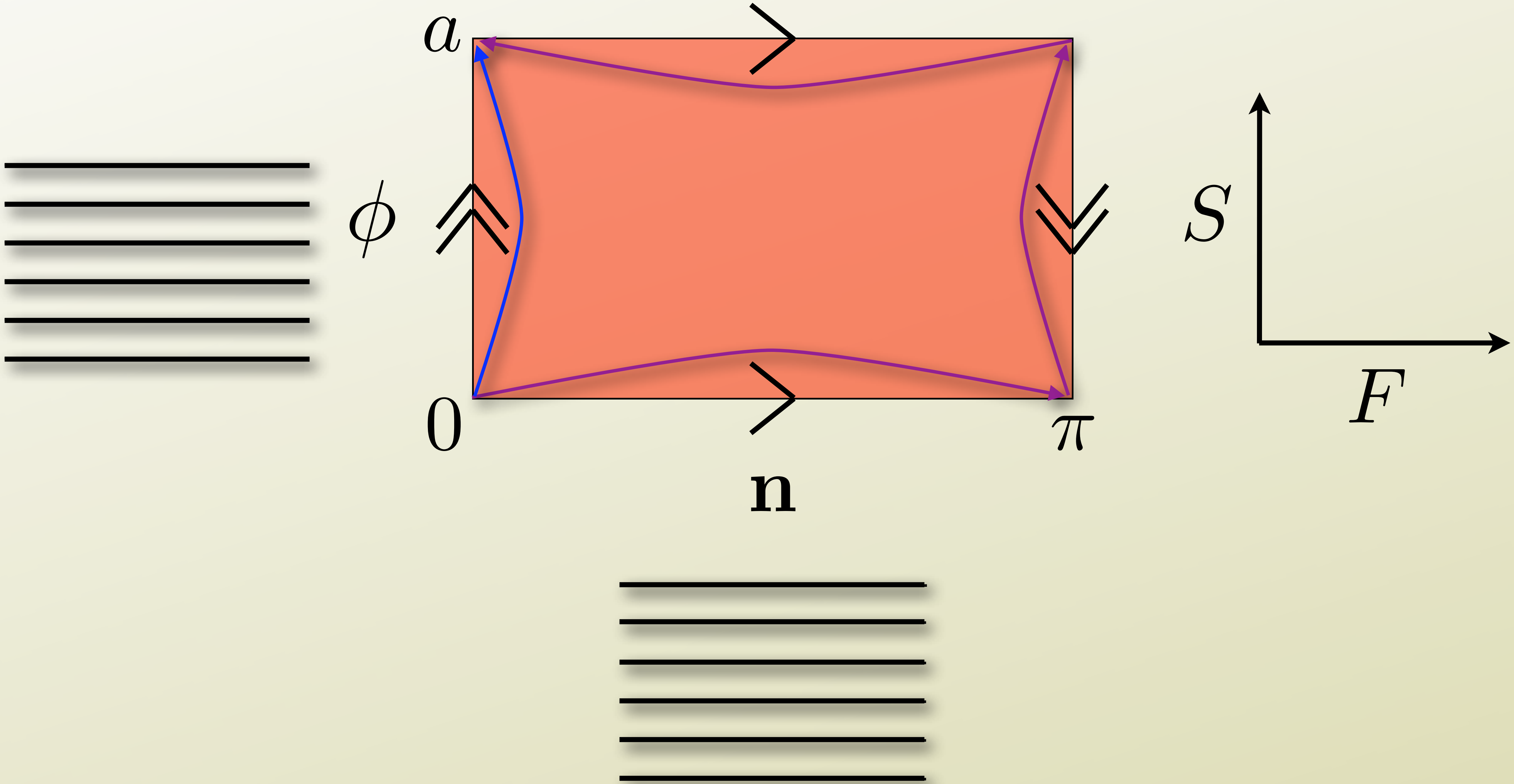

Maps from $\mathbb{R}^2 \backslash \{0\} \to \langle S, F | F S^{-1} F^{-1} = S \rangle$

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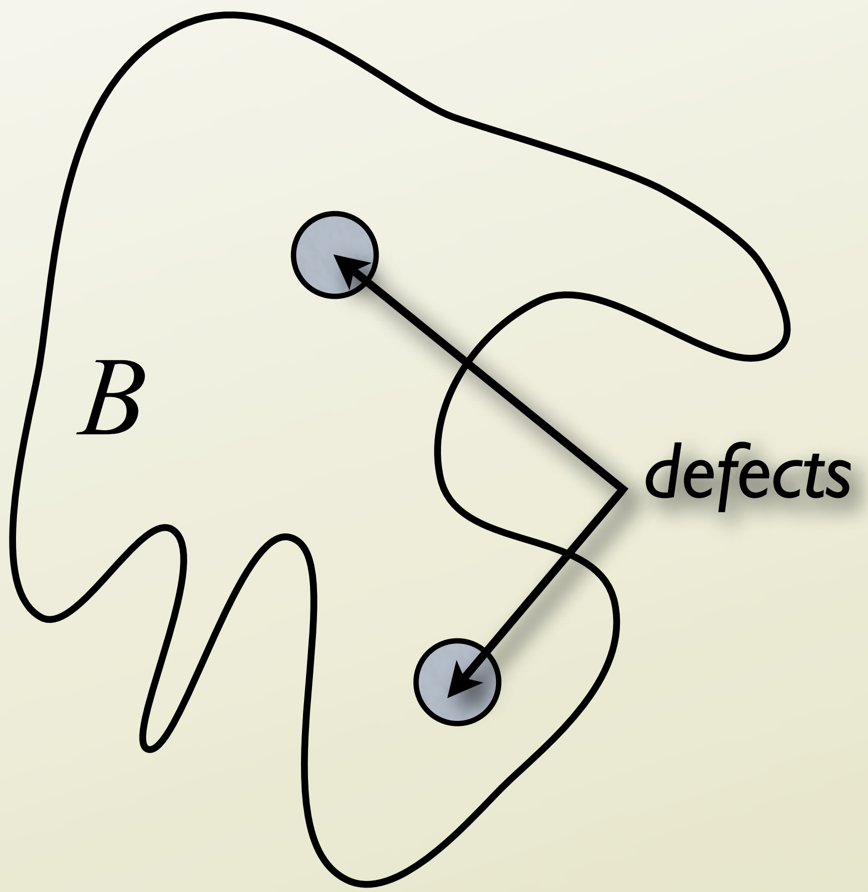
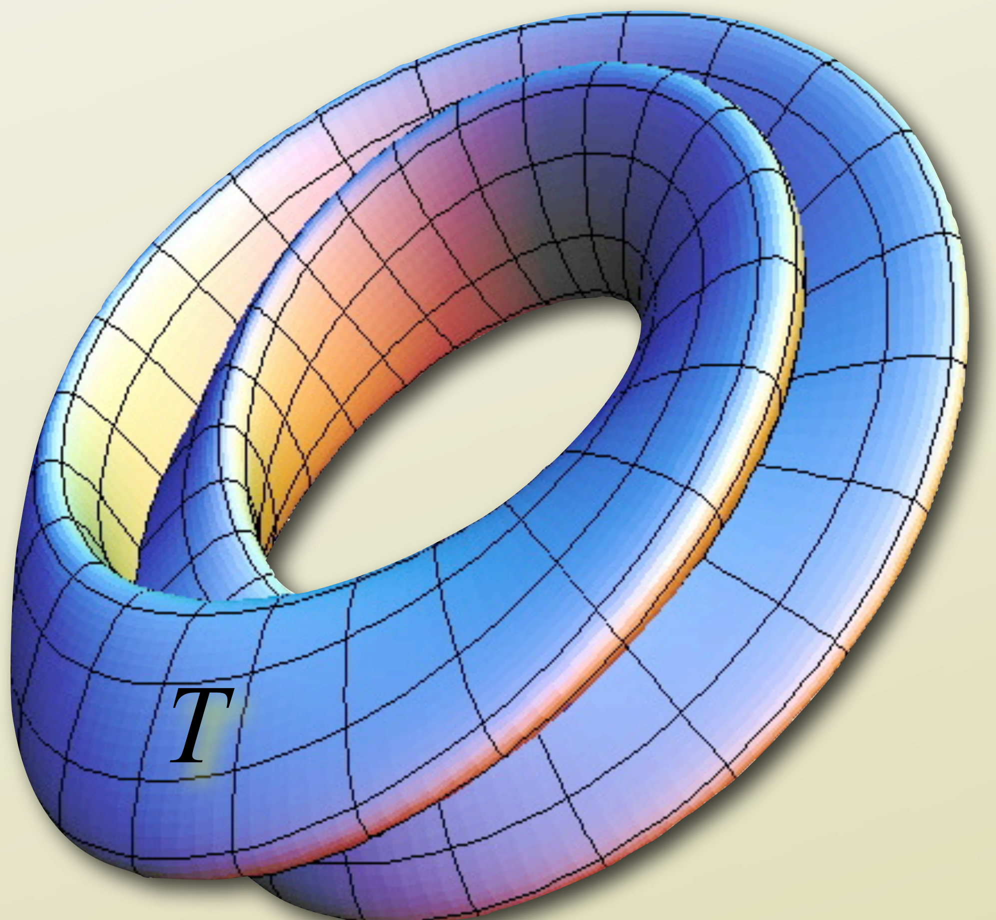
DEFECTS AND HOMOTOPY: QUICK REVIEW

Sample



Ground State Manifold



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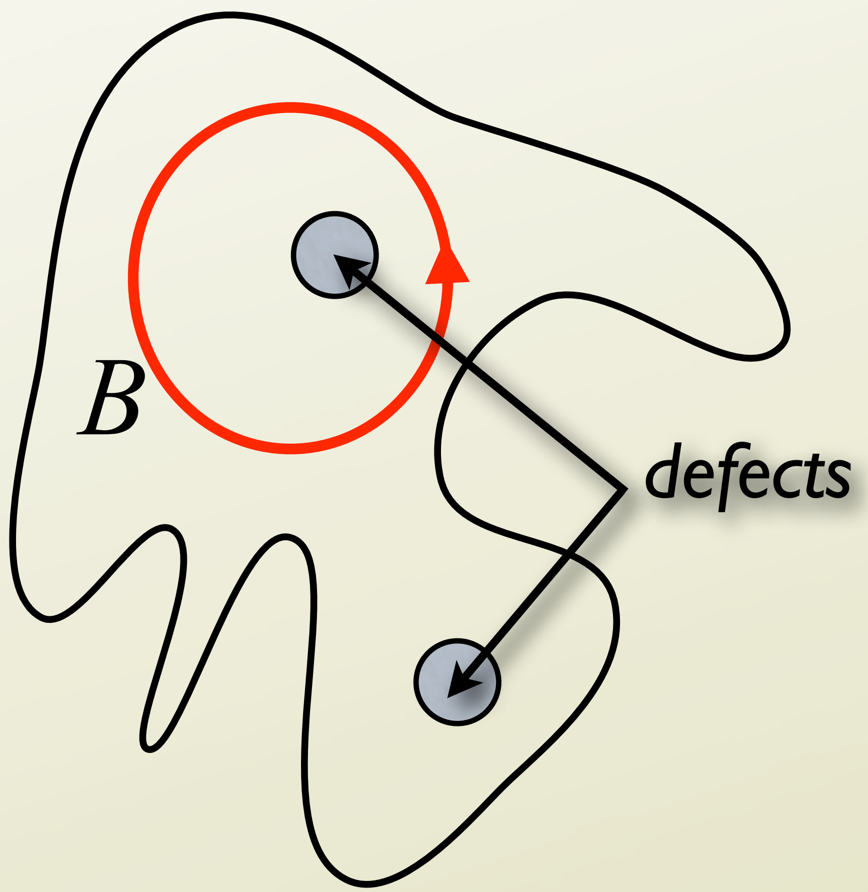
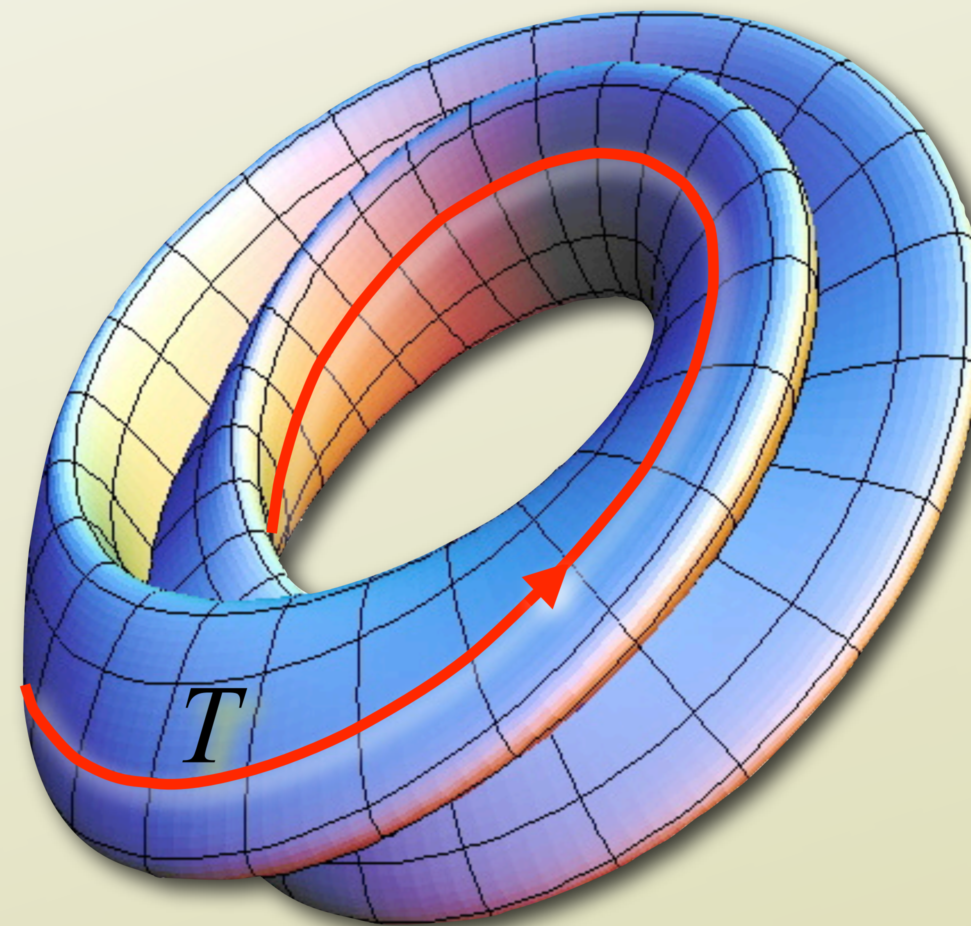
Defects and Homotopy: Quick Review

Sample



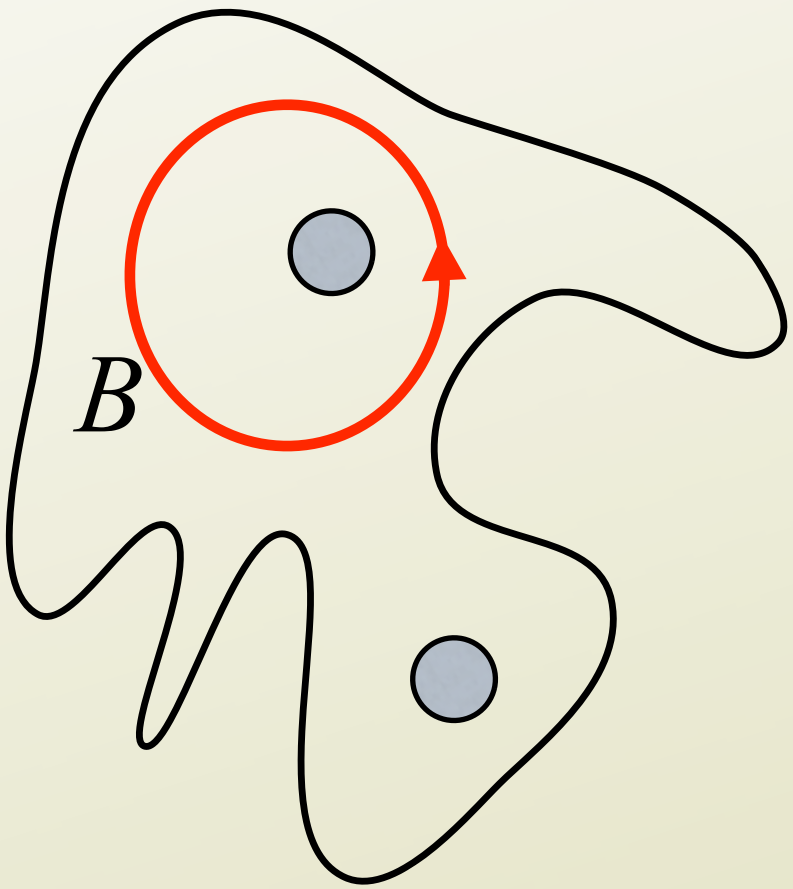
Ground State Manifold



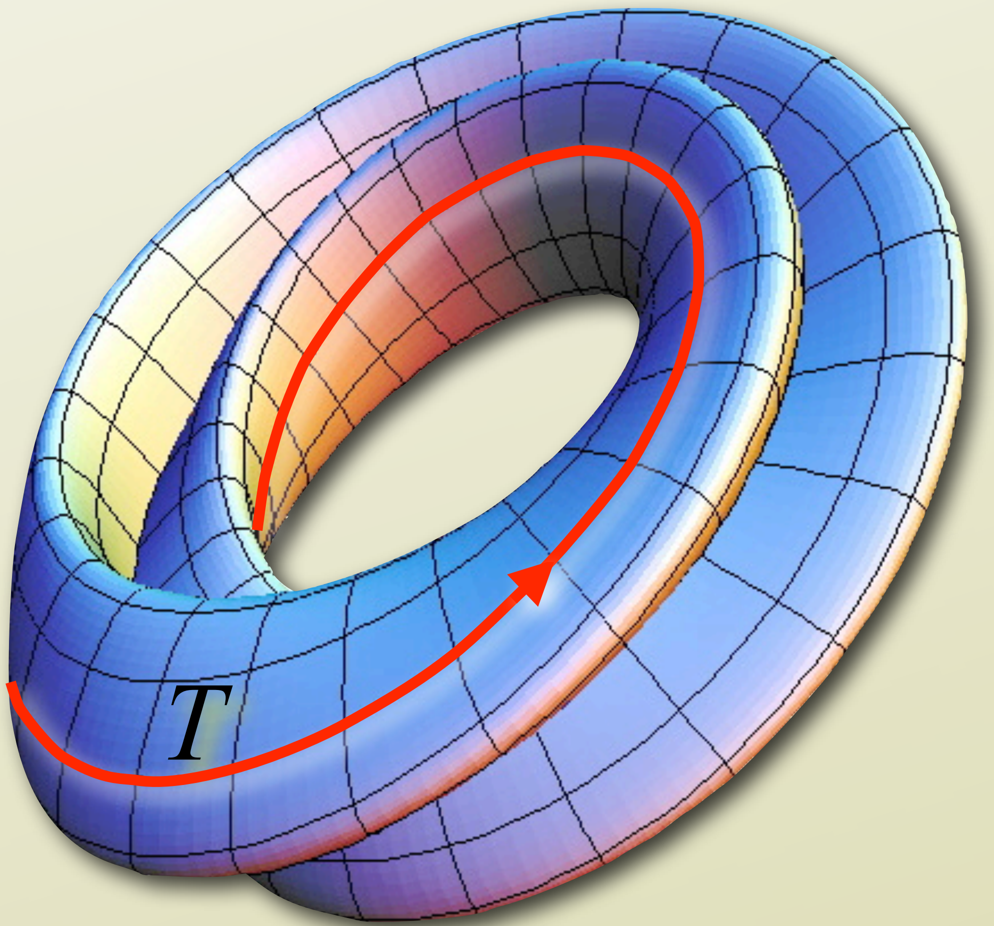
Maps from $\pi_1(B) \to \pi_1(T)$

Defects and Homotopy: Quick Review

fix conjugacy class in B



free homotopy on T



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DEFECTS AND HOMOTOPY: QUICK REVIEW

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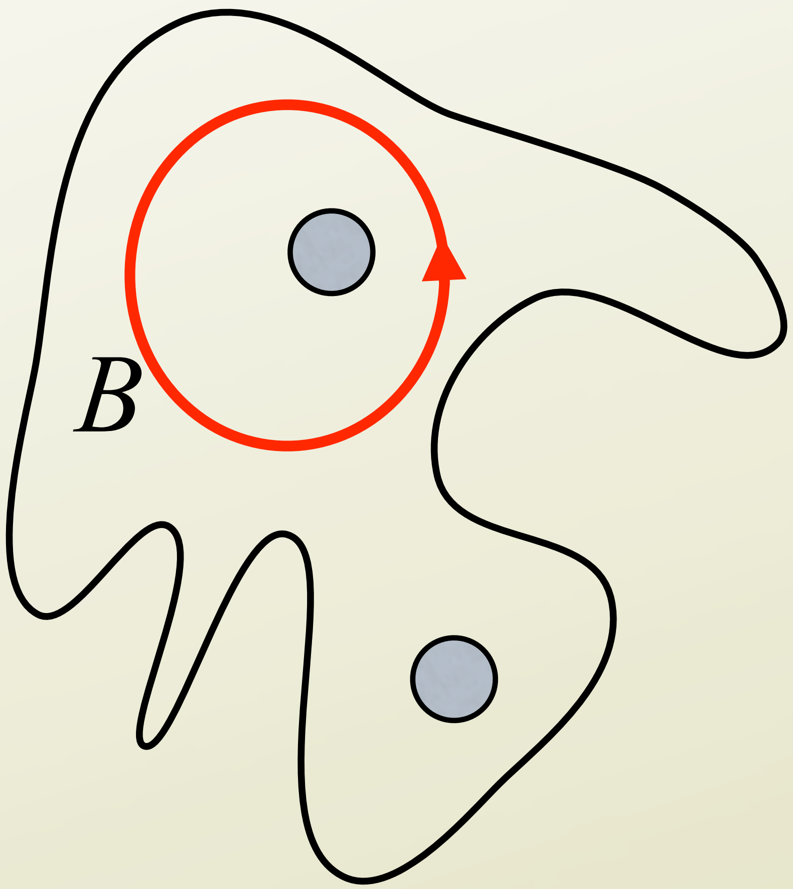

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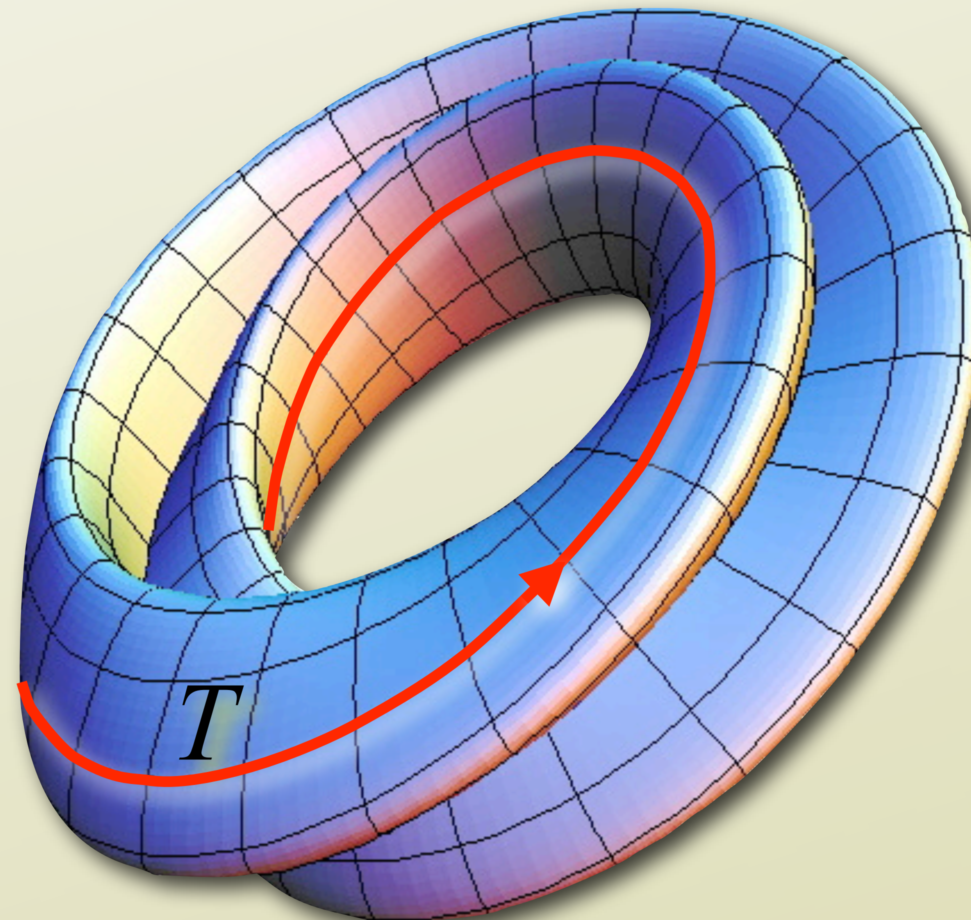

Maps from $B \to \mathrm{Cl}(\alpha), \alpha \in \pi_1(T)$

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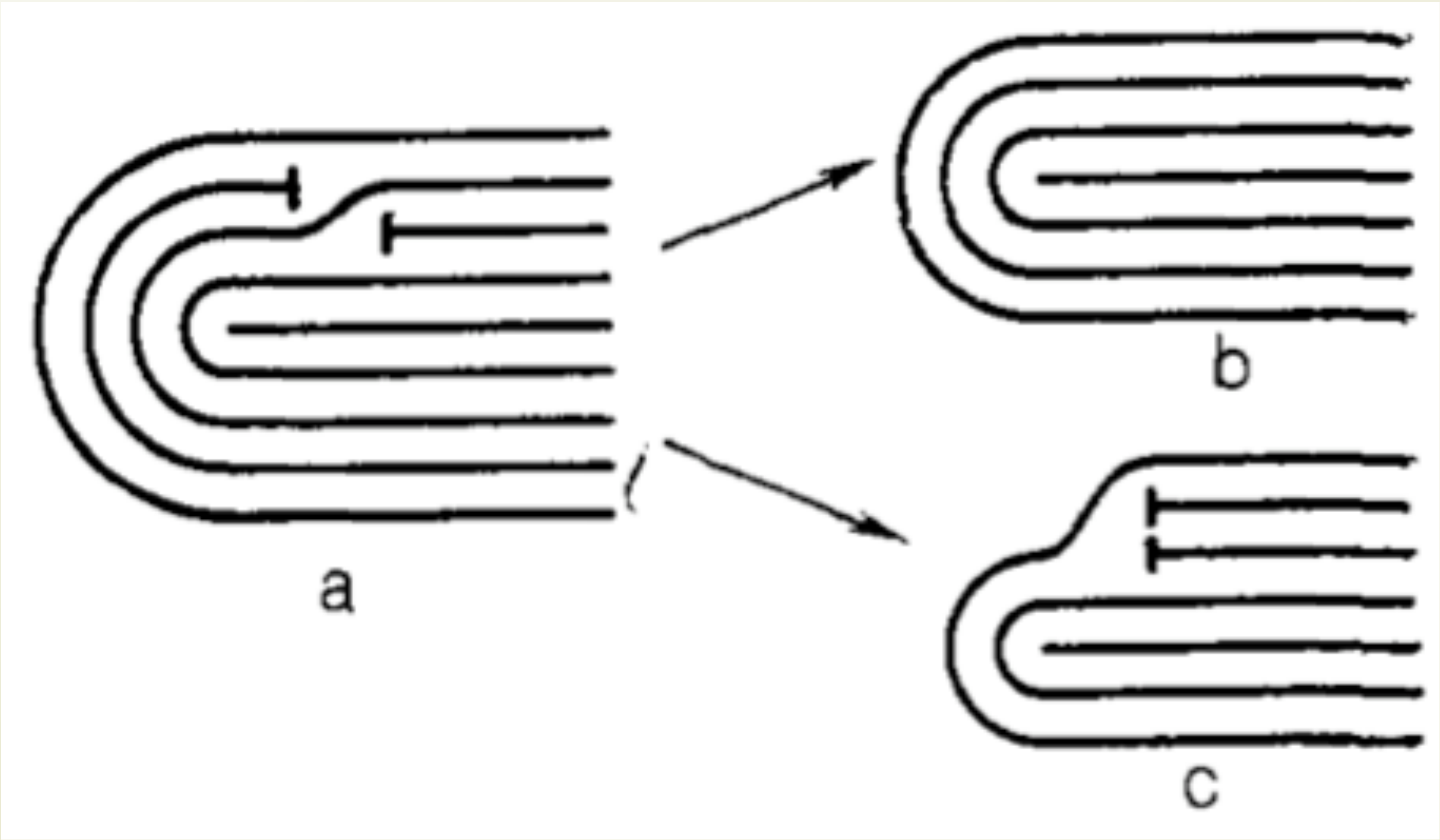
Defects and Homotopy: Quick Review



F

FS^2

Maps from $B \to \mathrm{Cl}(\alpha), \alpha \in \pi_1(T)$

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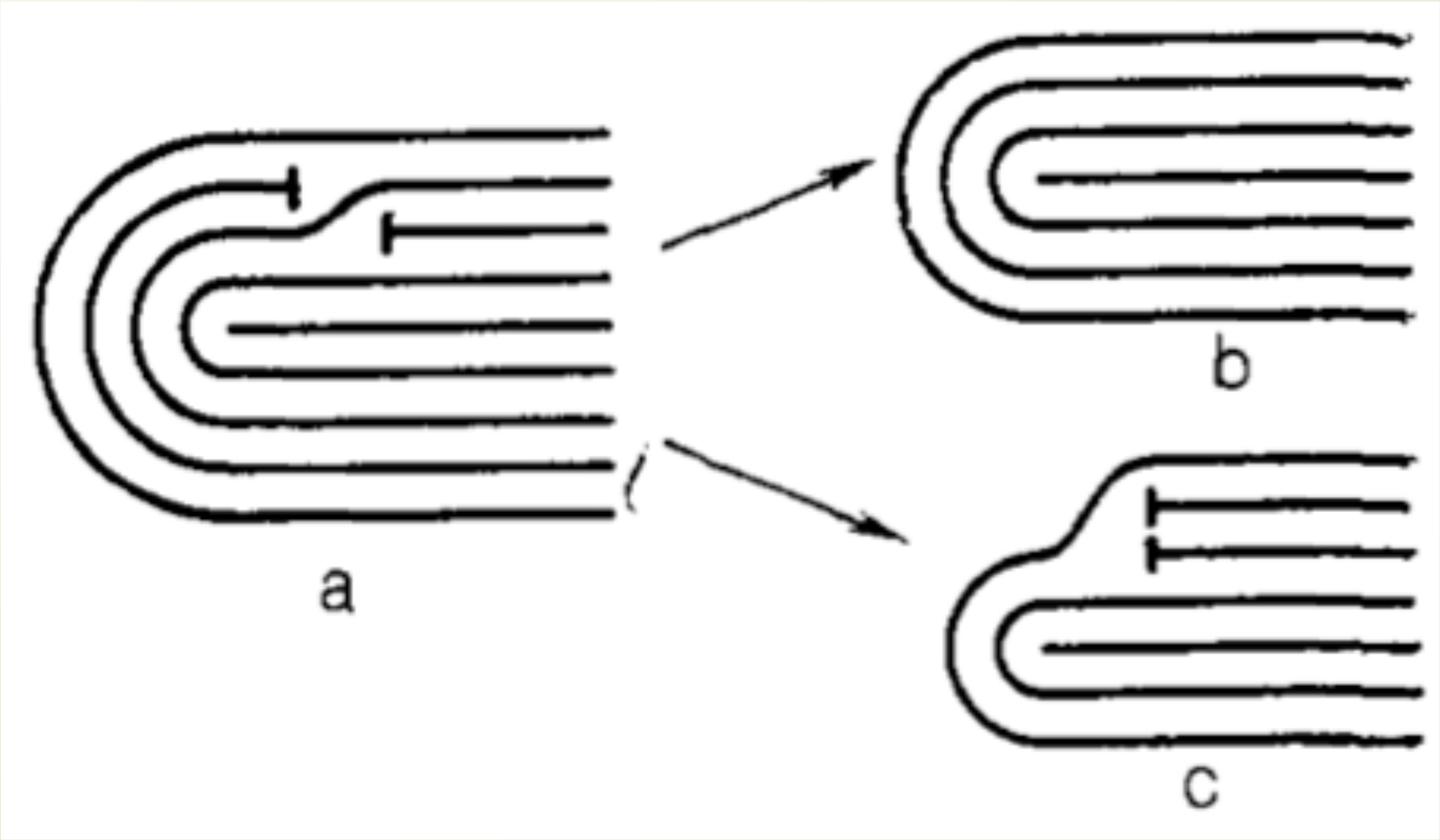
DEFECTS AND HOMOTOPY: QUICK REVIEW



F

FS^2

$$S(FS^2)S^{-1} = SFS = F$$

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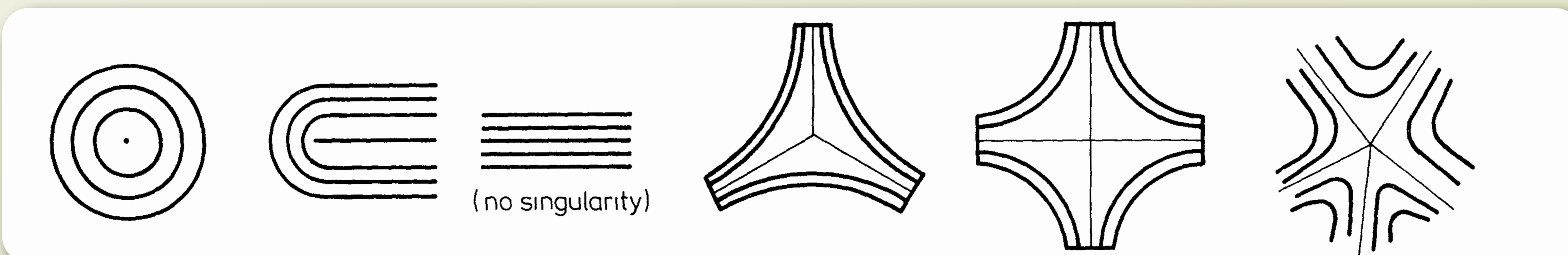
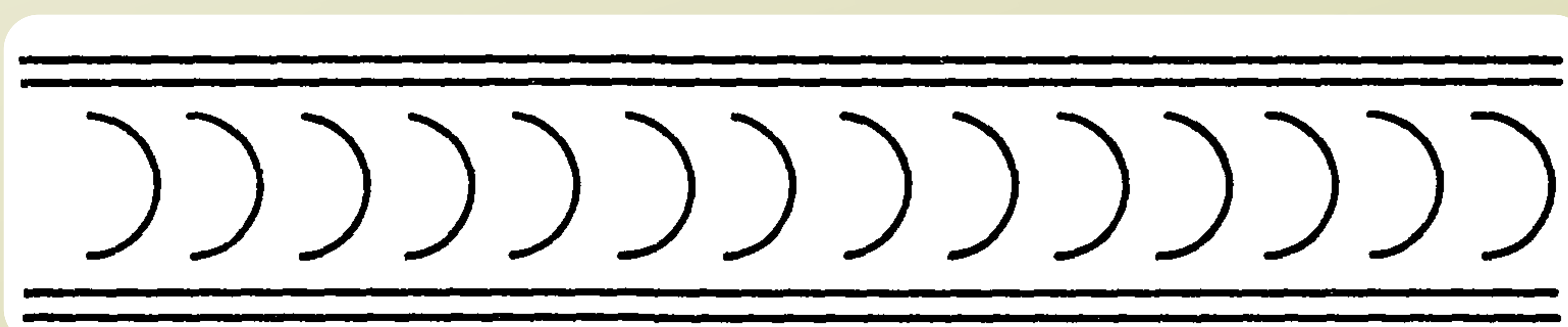
FUNDAMENTAL GROUP: NOT THE WHOLE STORY

Theorem (Poénaru)
Let $\mathbf{n}$ be a field of directors [a line field] in $\mathbb{R}^2$ with an isolated singularity at 0, defining a measured foliation. Then $I(\mathbf{n}) \leq 1$. In particular, a vector field ξ on $\mathbb{R}^2$, with an isolated singularity at 0, such that $\nabla \times \xi = 0$, has the property that $I(\xi) \leq 1$.





N. D. Mermin, *Rev. Mod. Phys.* **51**, 591-648 (1979); V. Poénaru, *Commun. Math. Phys.* **80**, 127-136 (1981)

Fundamental Group: Not the whole story

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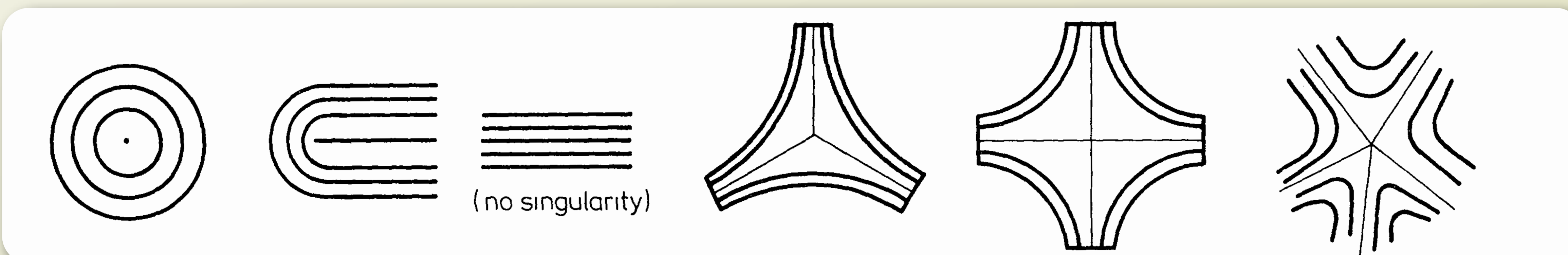
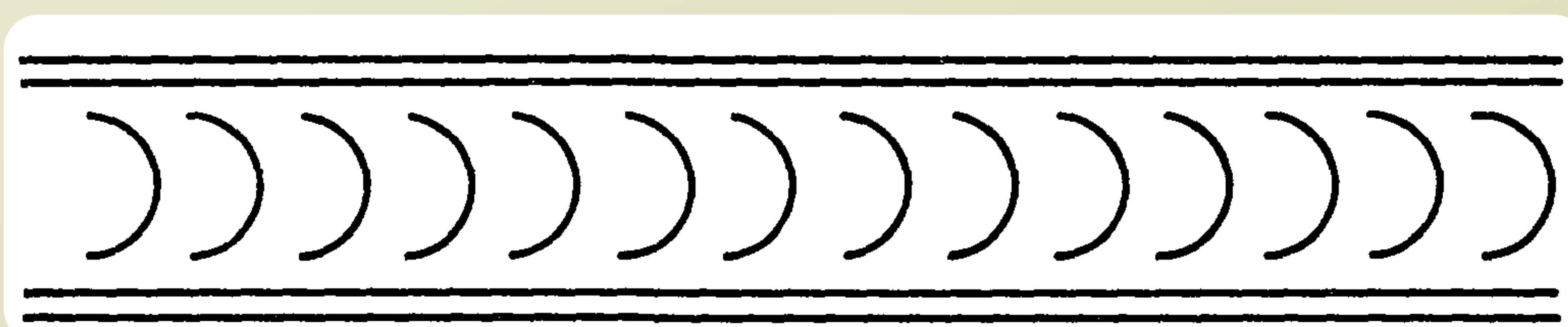
Measured:



Not:



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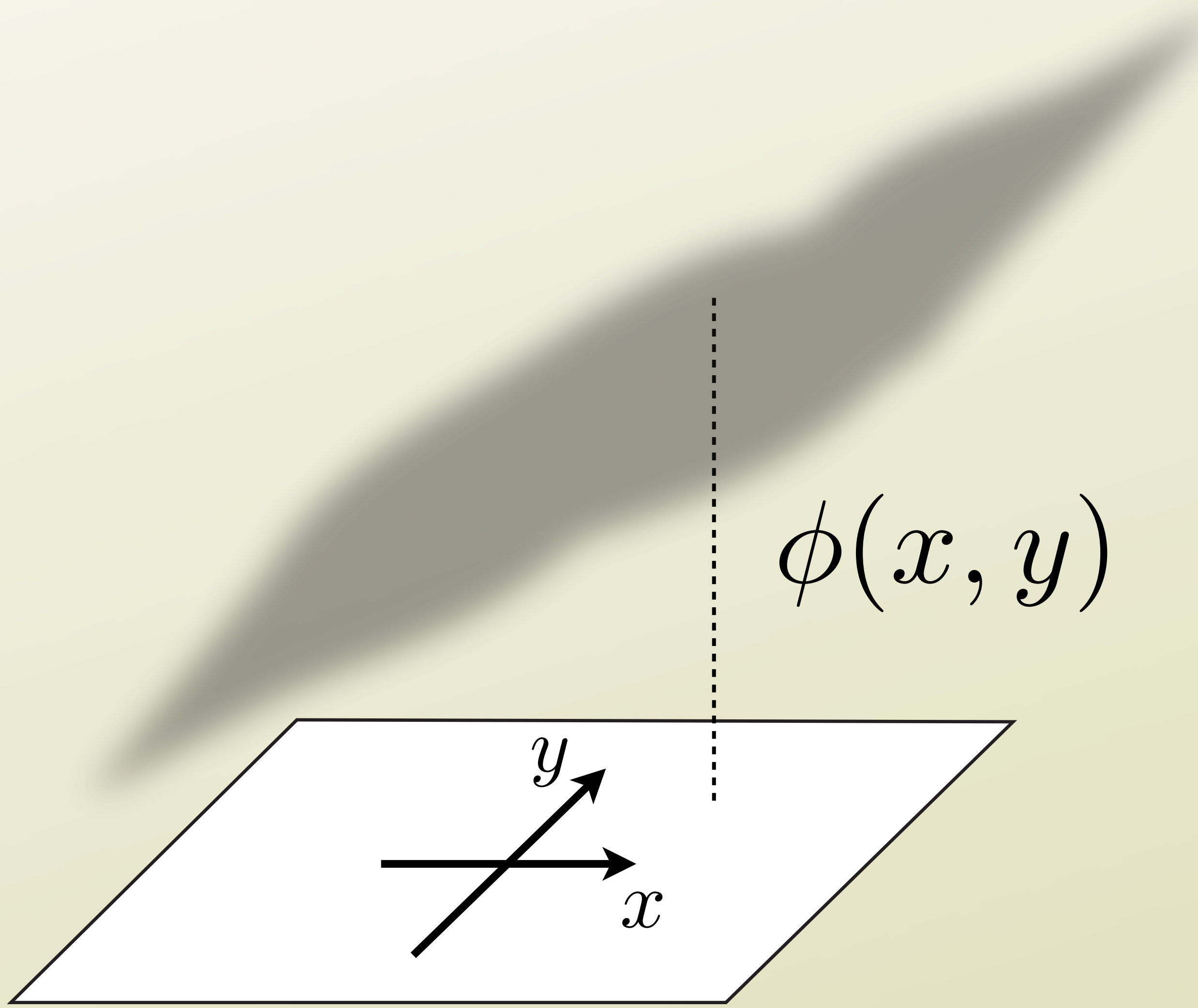
Smectic Phase Field as a Height Function

Chen, Alexander, Kamien, *PNAS* **106**, 15577-15582 (2009)

SMECTIC PHASE FIELD AS A HEIGHT FUNCTION



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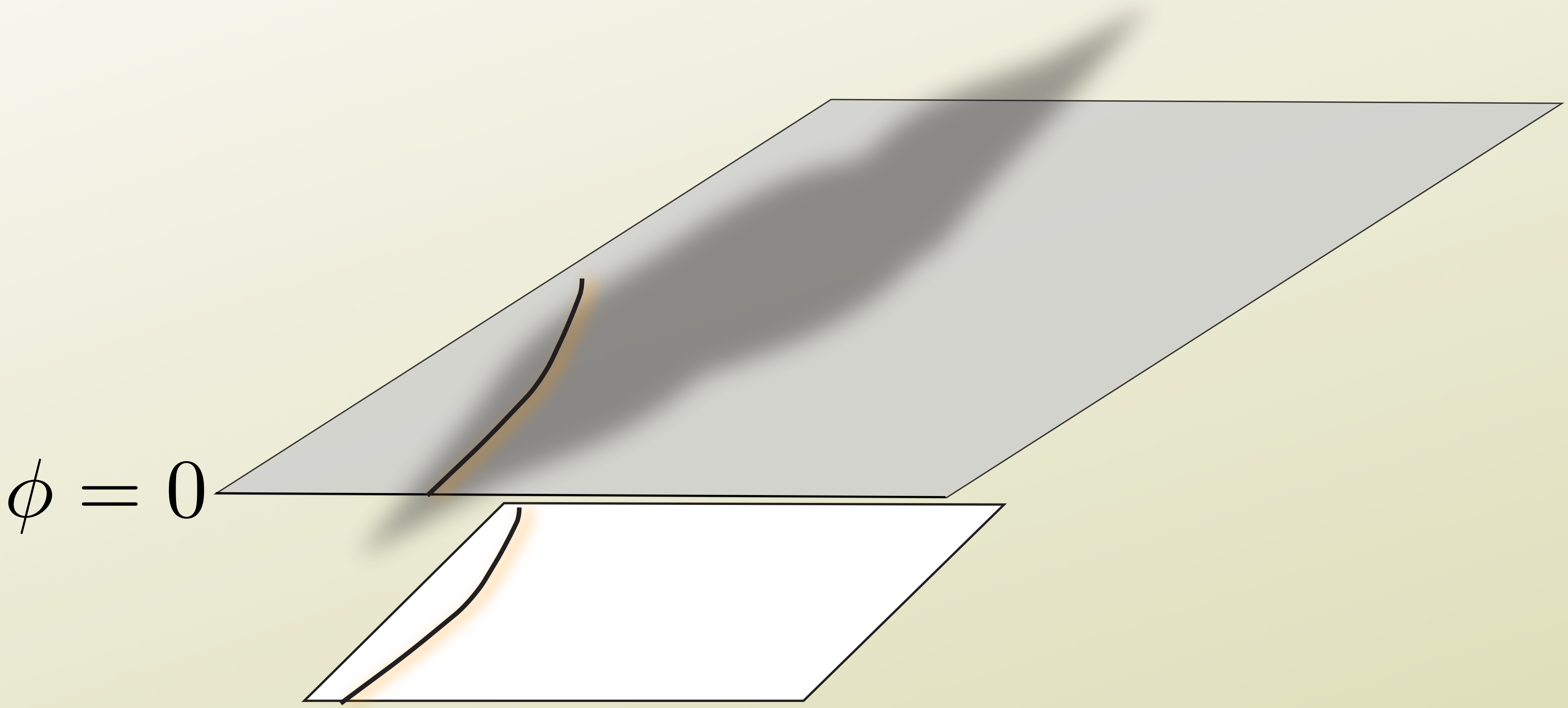

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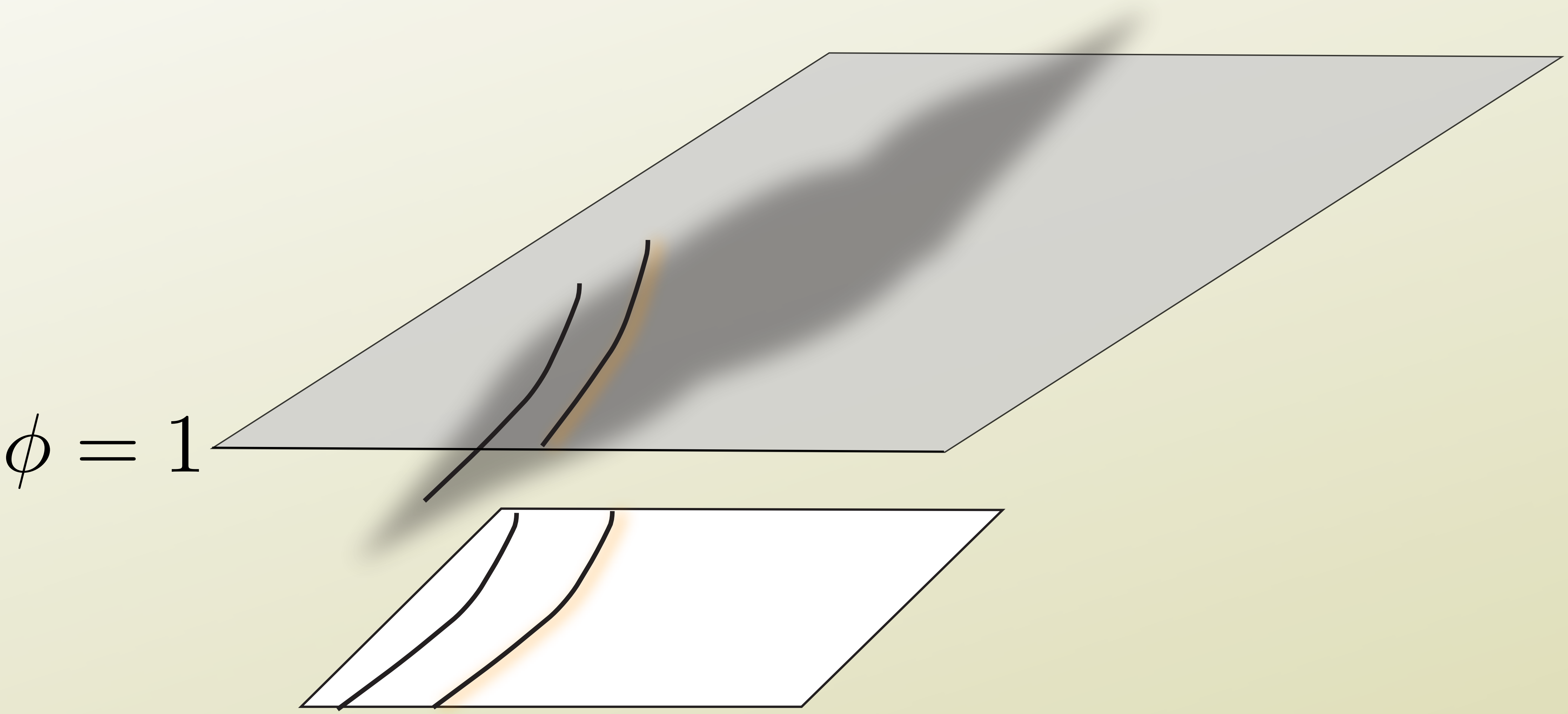

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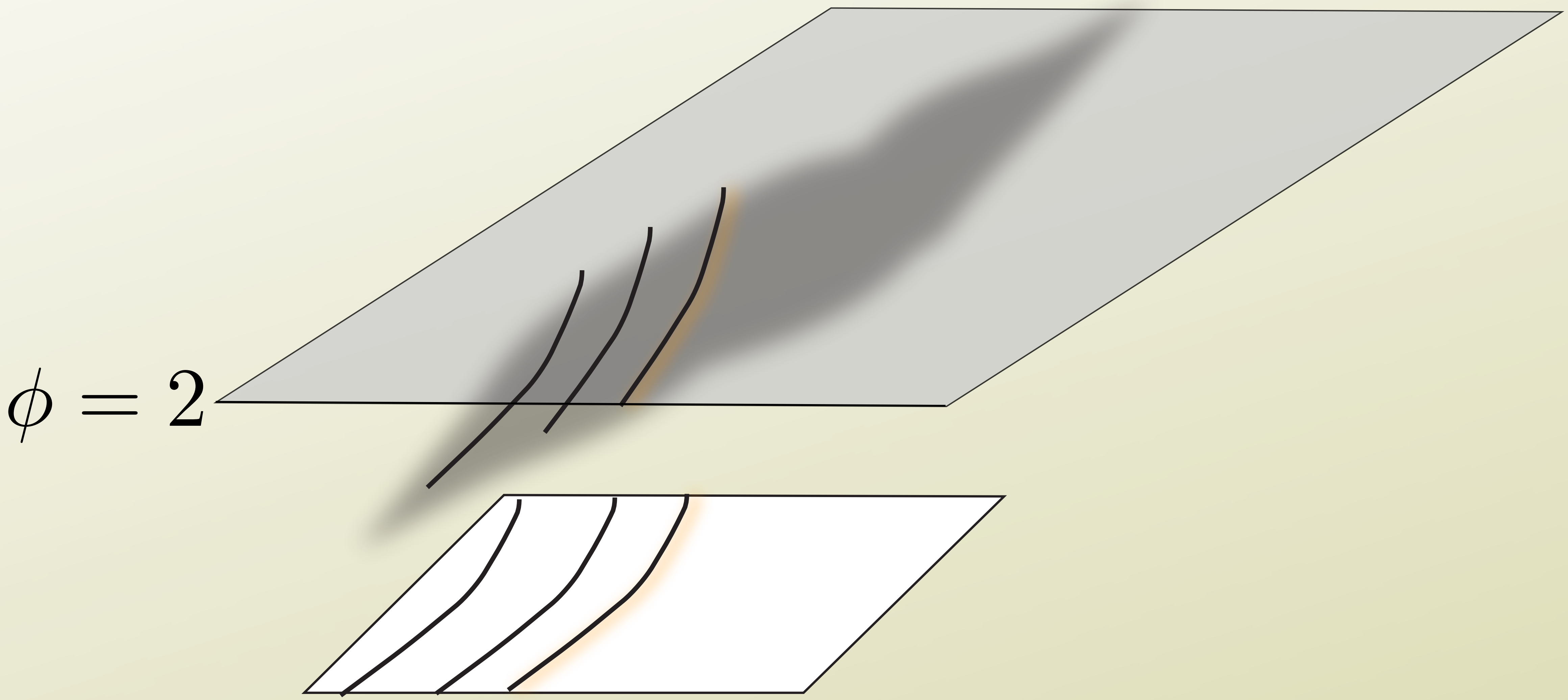

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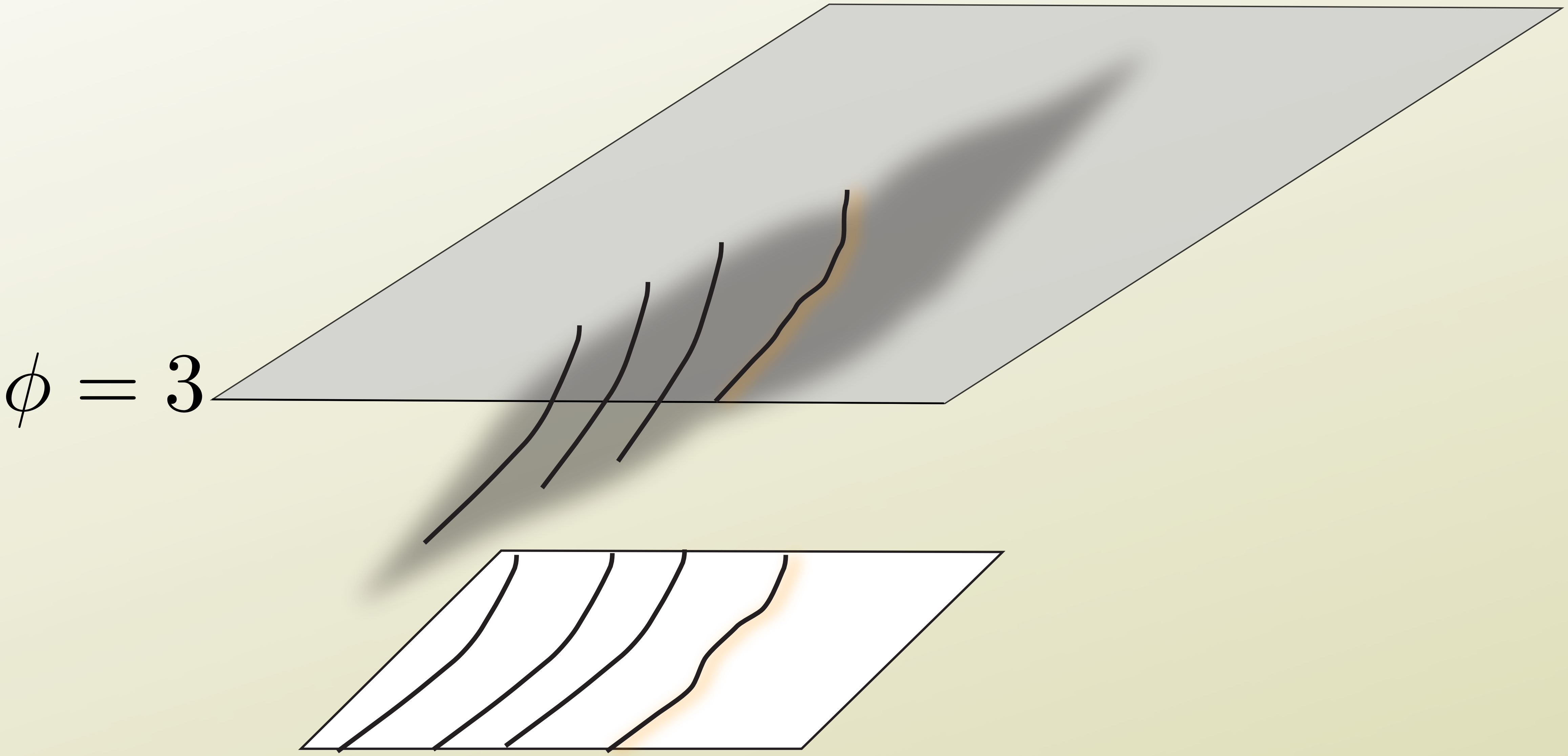

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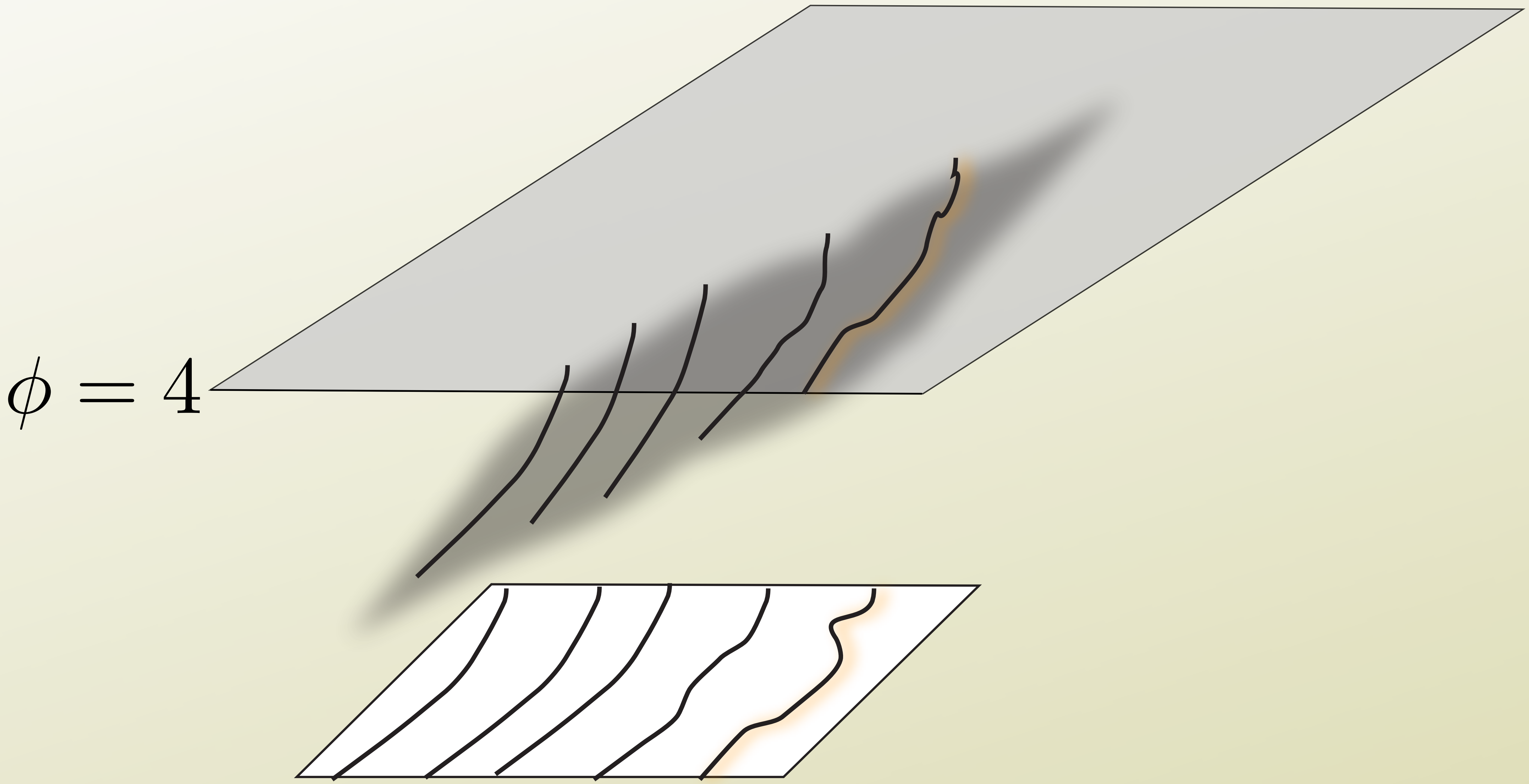

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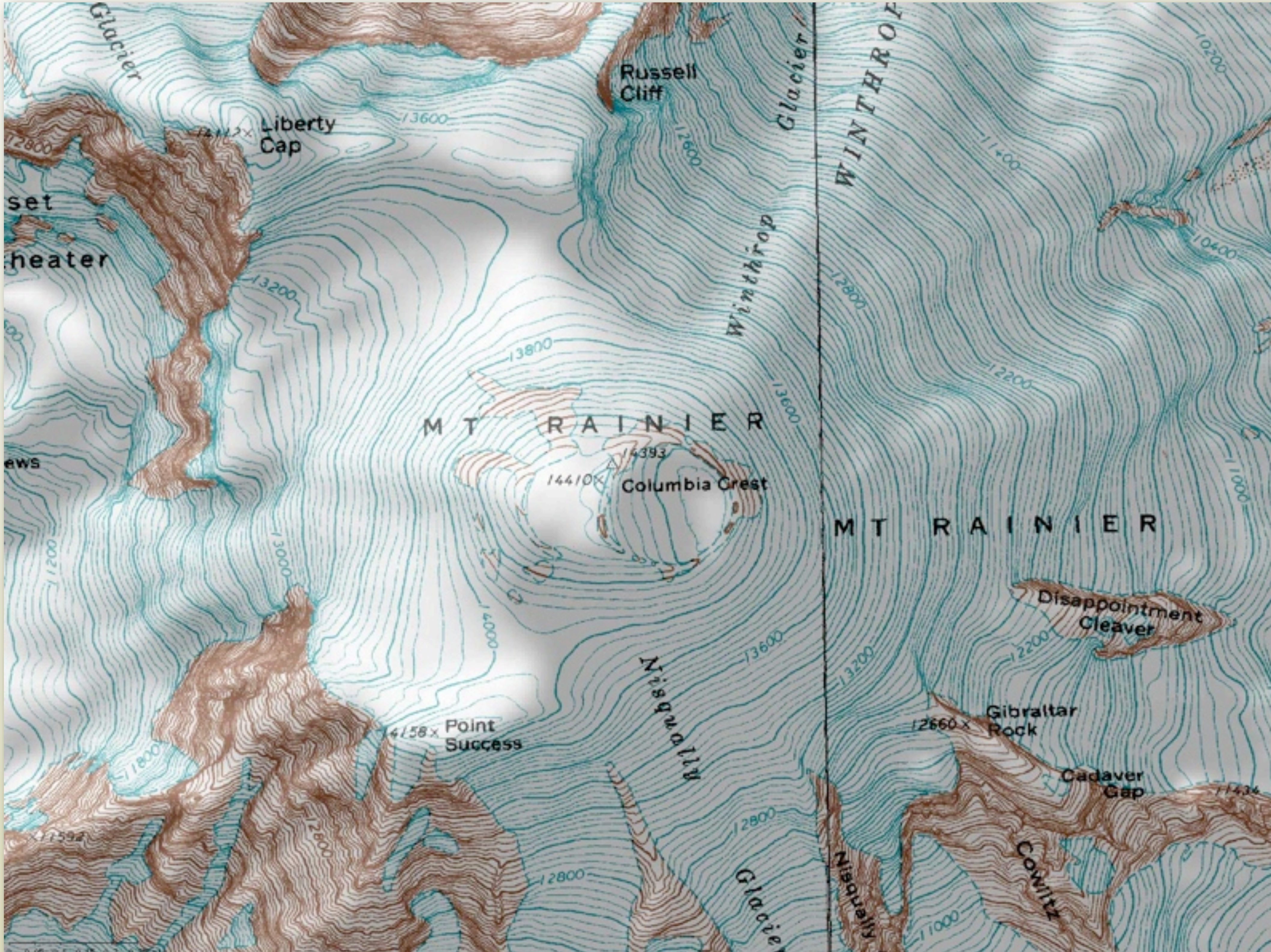
Contour Maps: Smectic Disclinations



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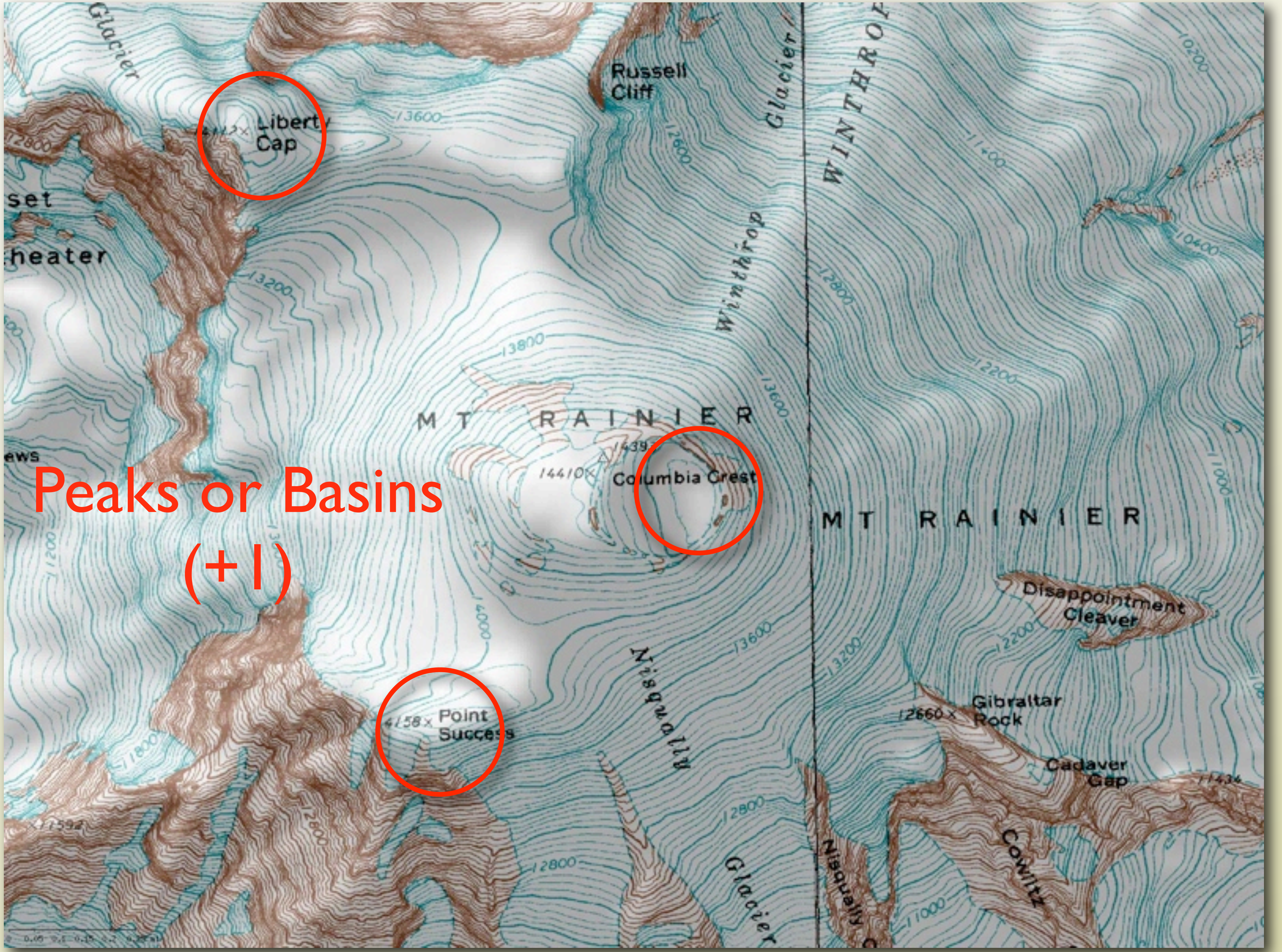

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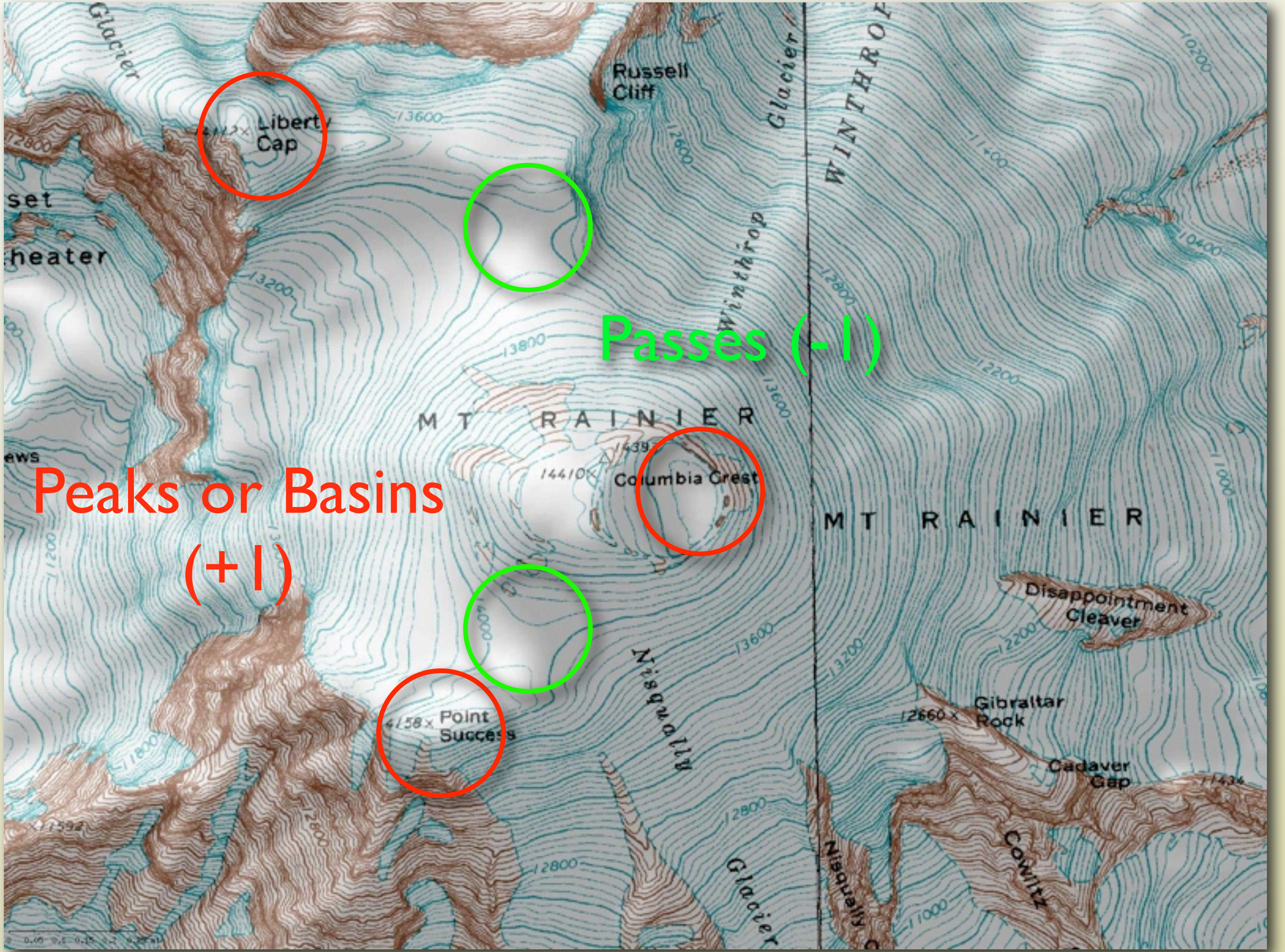

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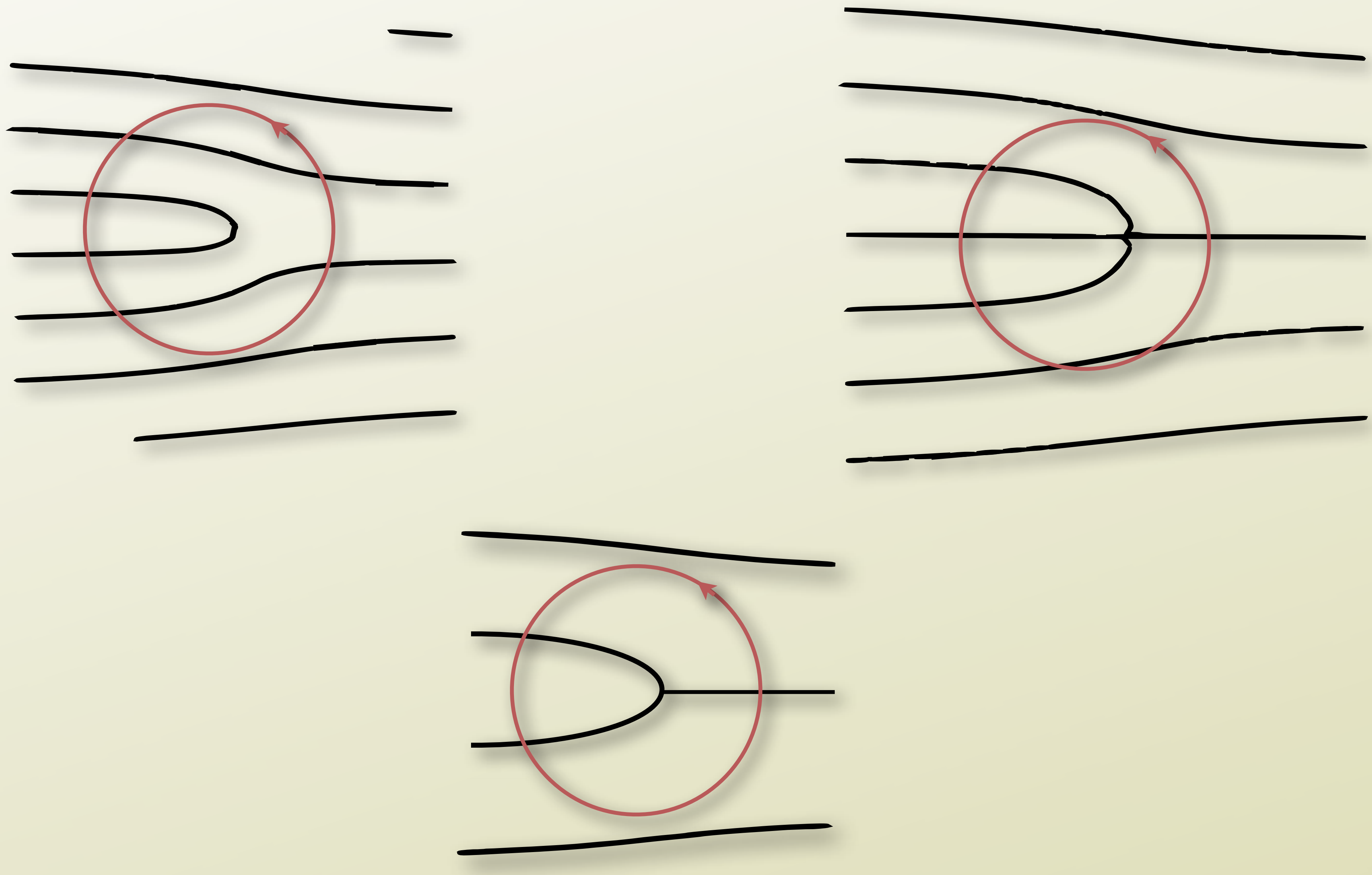
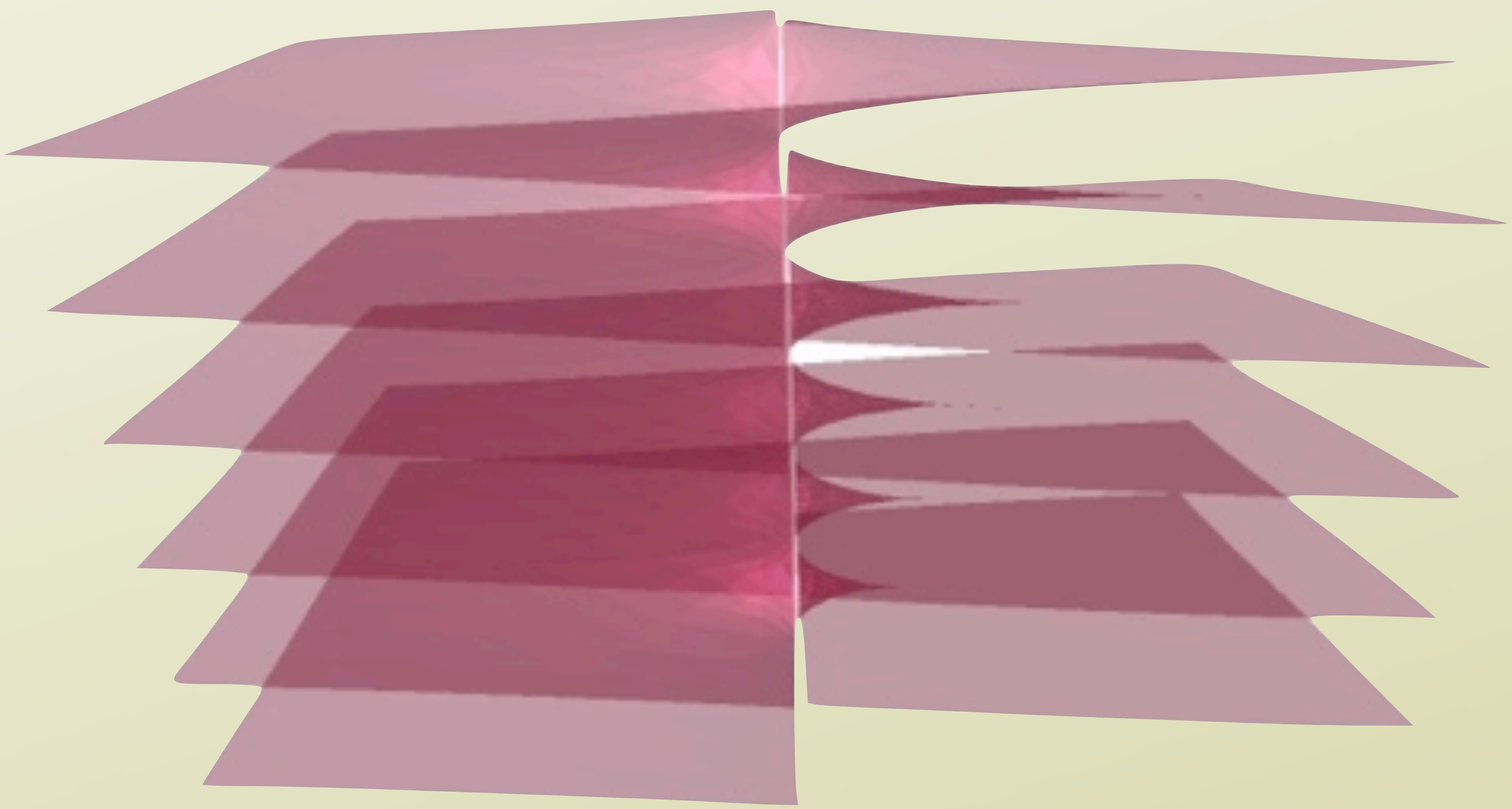
EDGE DISLOCATIONS IN TWO DIMENSIONS

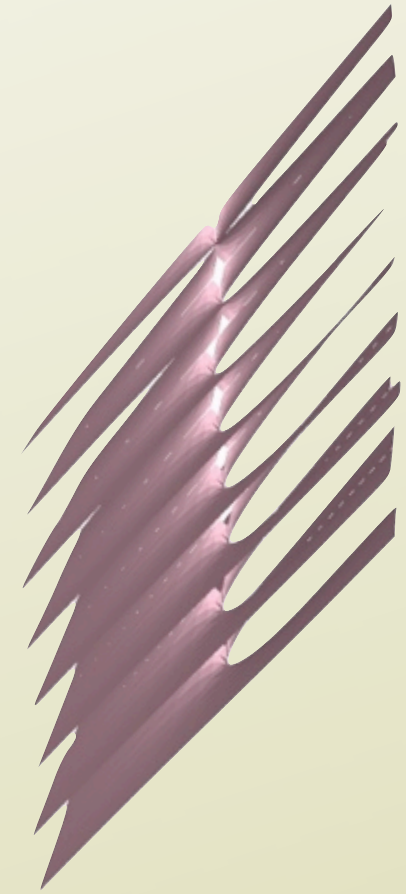


Maps from $\mathbb{R}^2\backslash\{0\} \to S^1$

+2 Dislocation

Dislocation is a helicoid!





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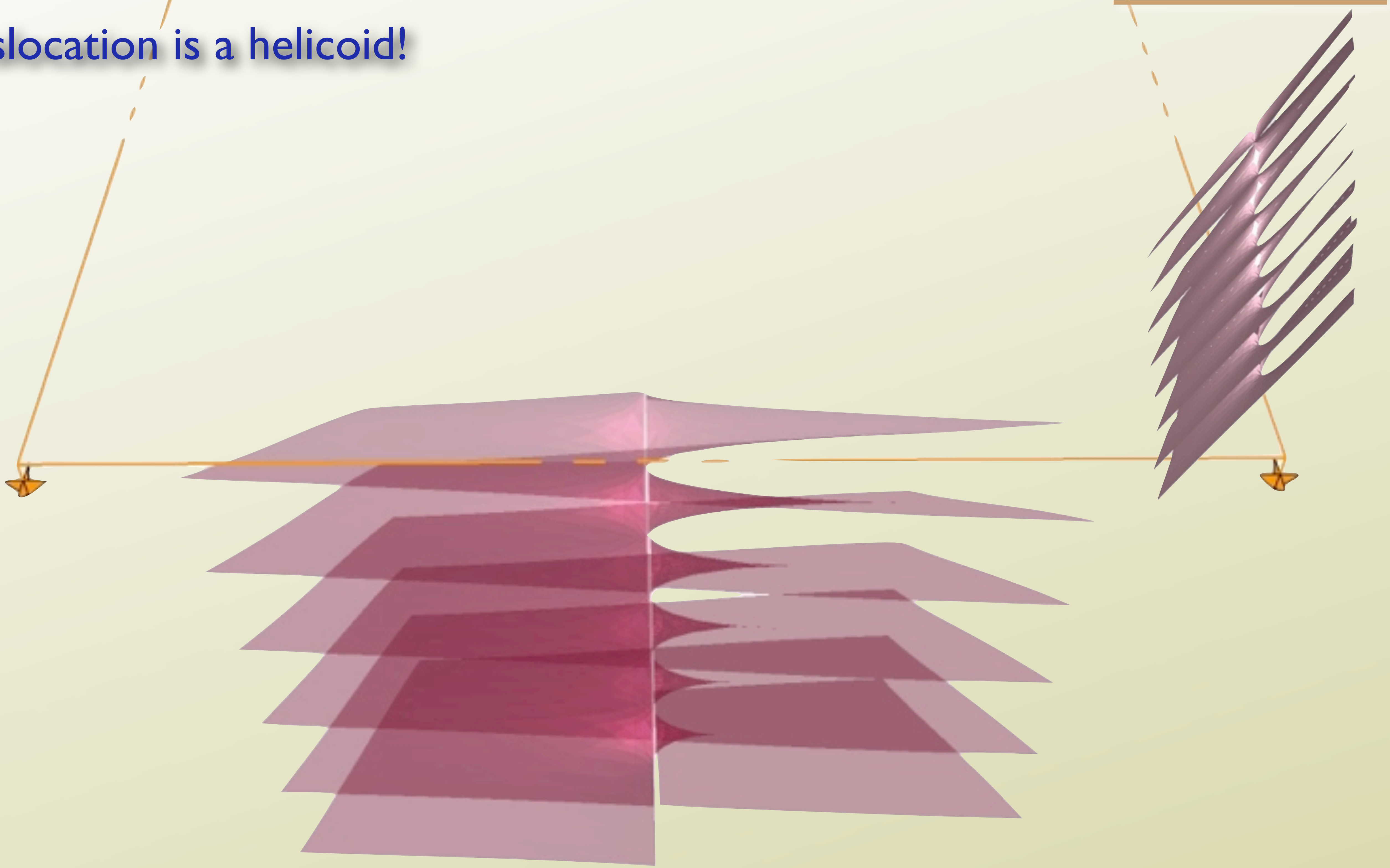

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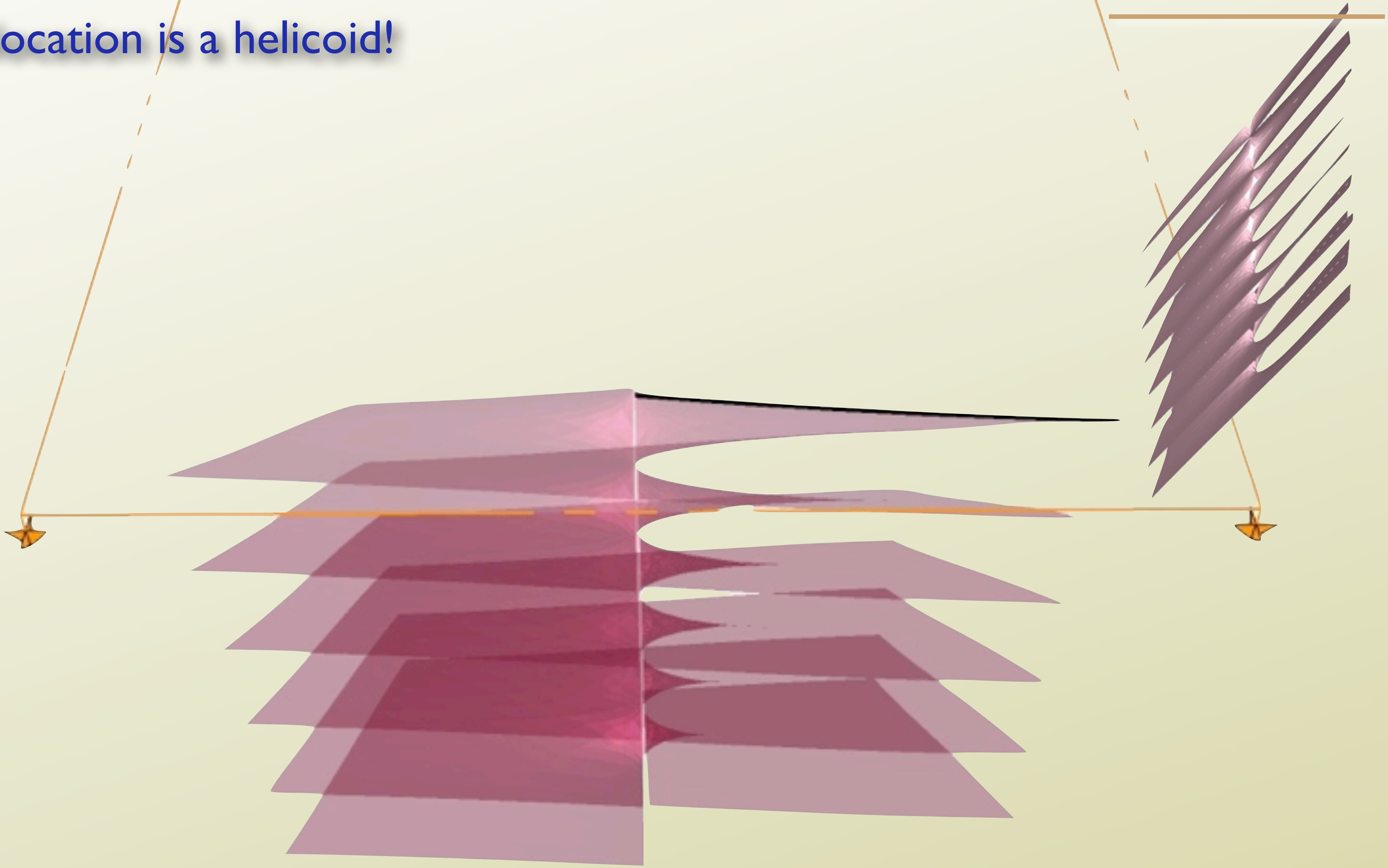

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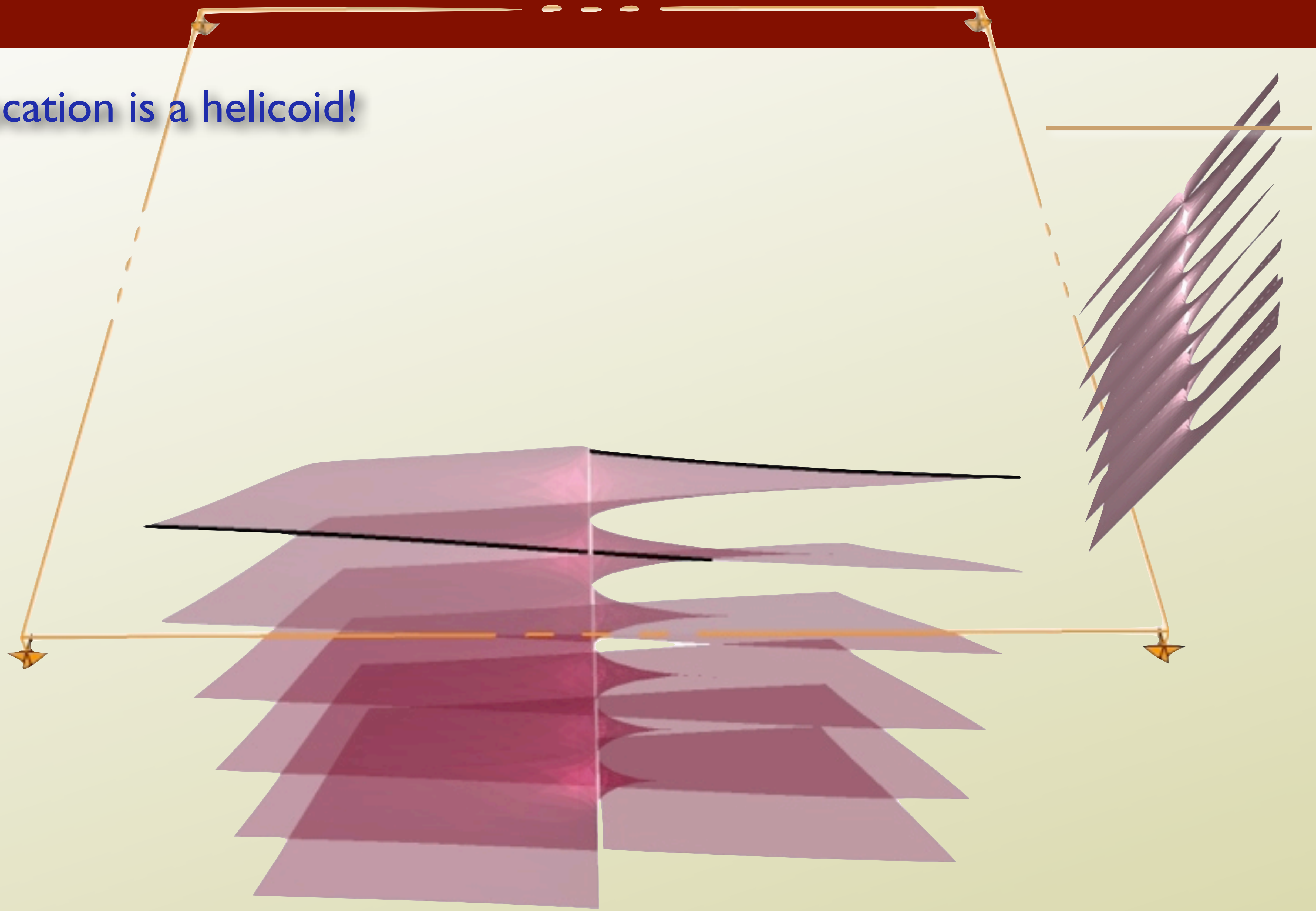

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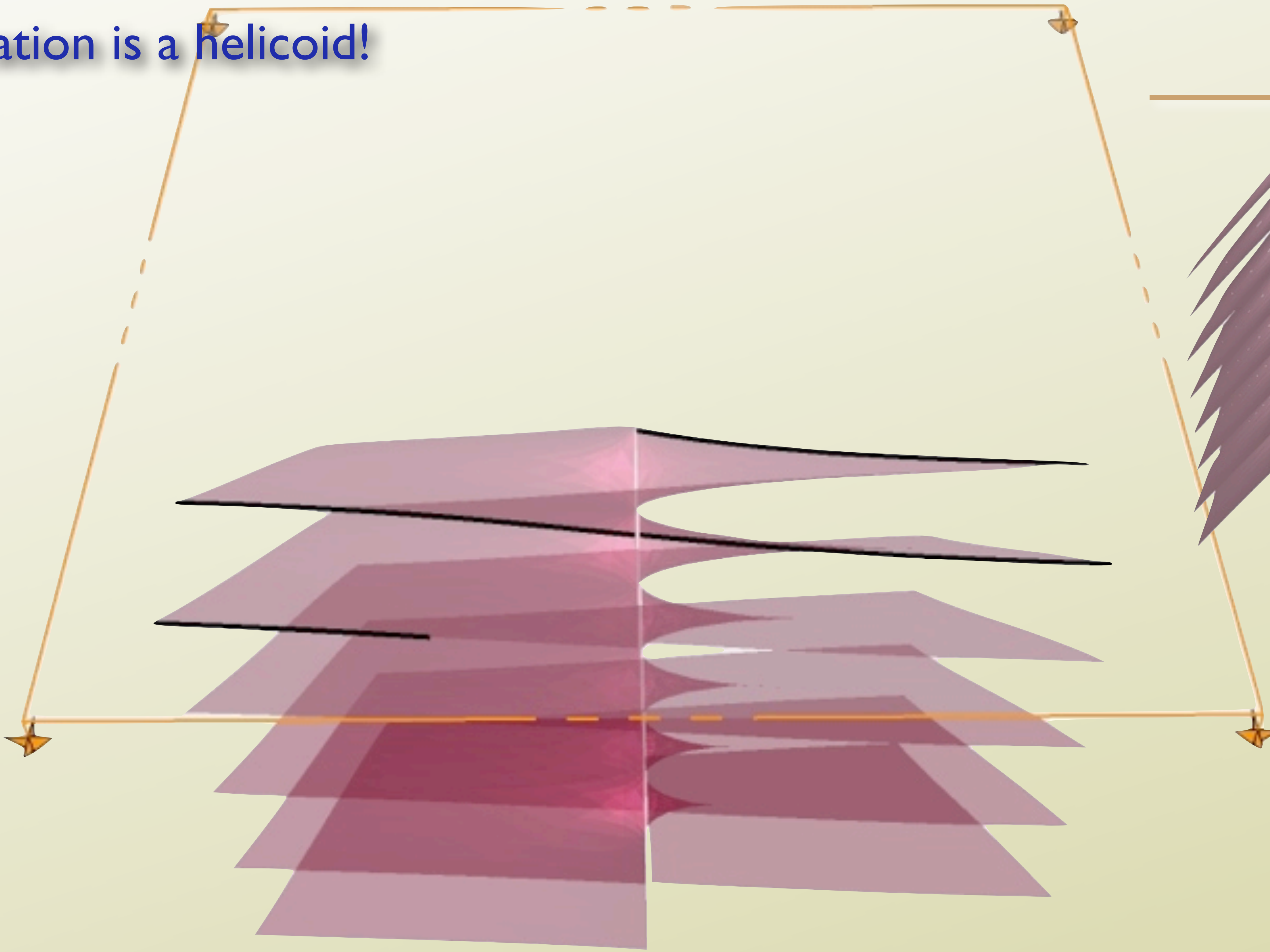

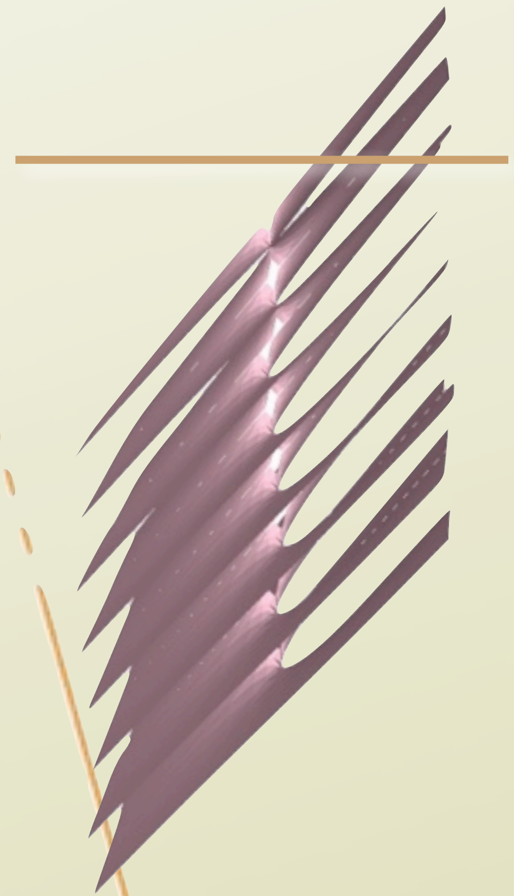

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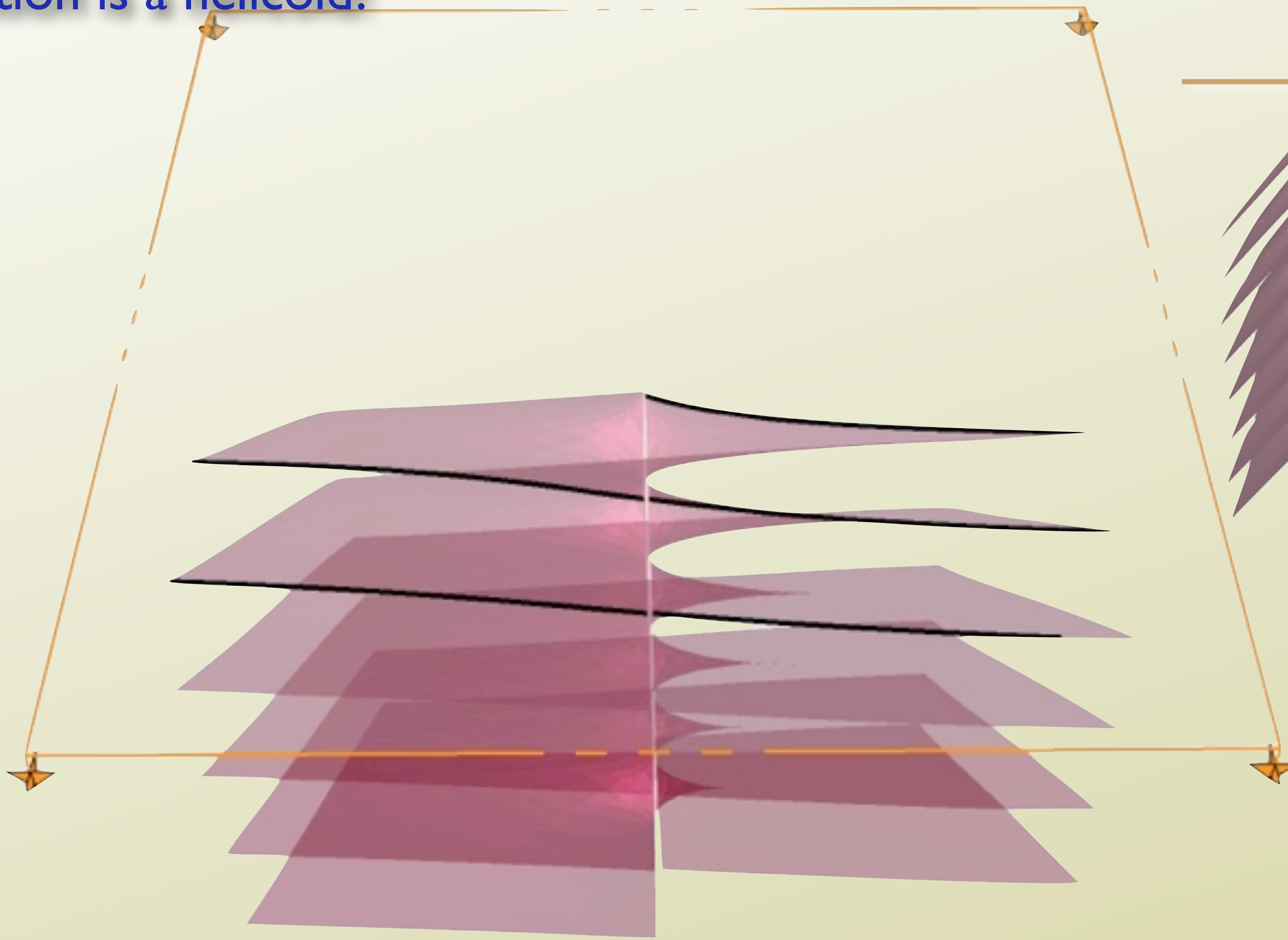

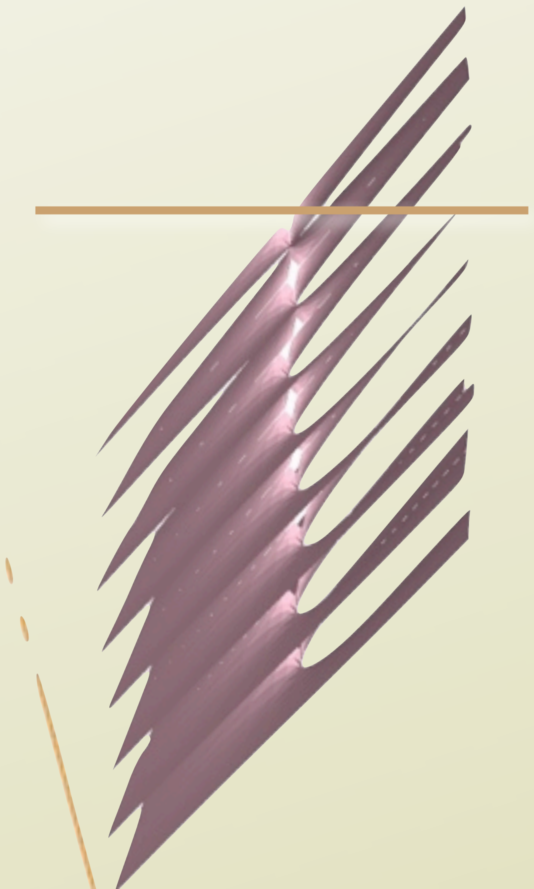

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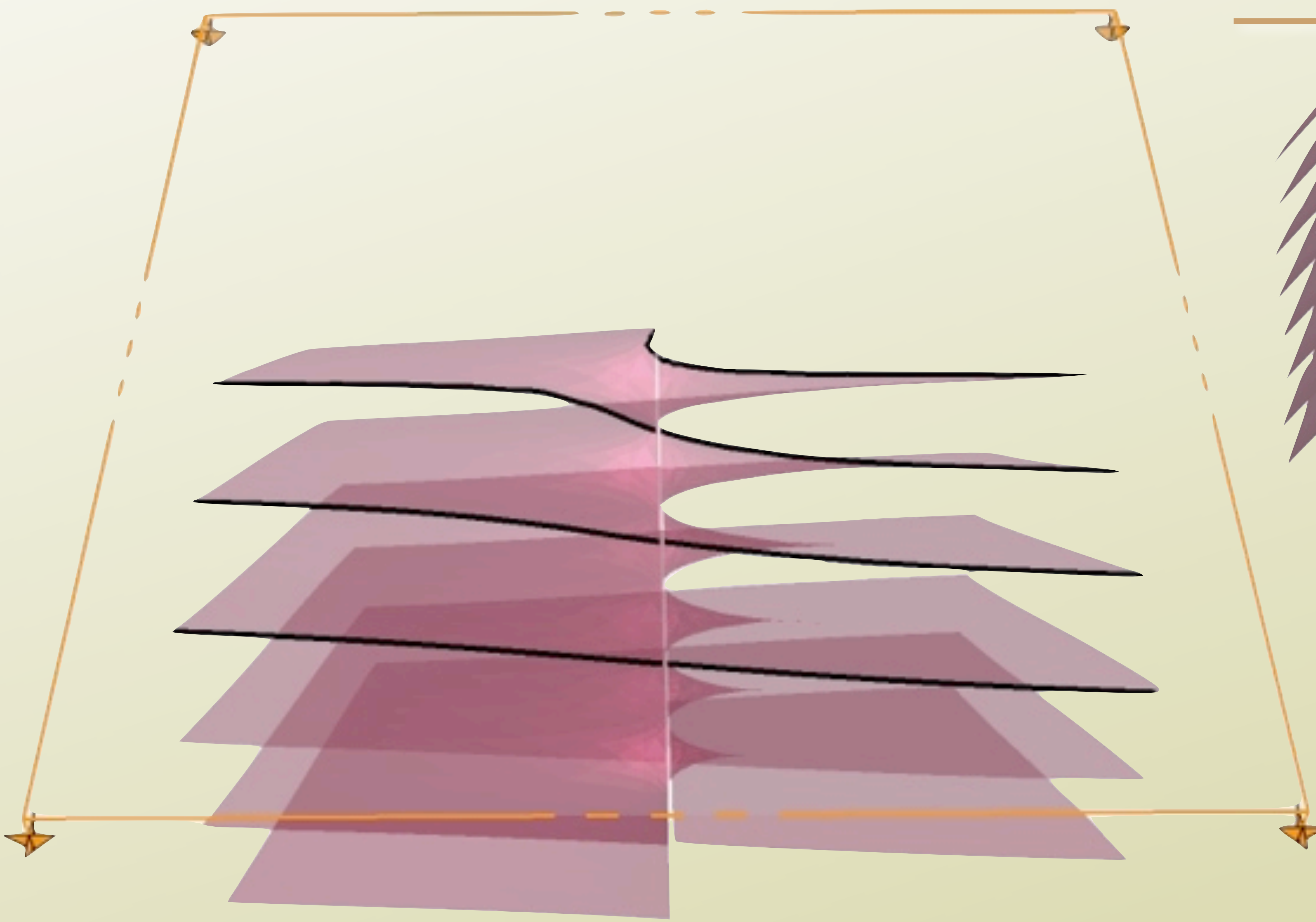

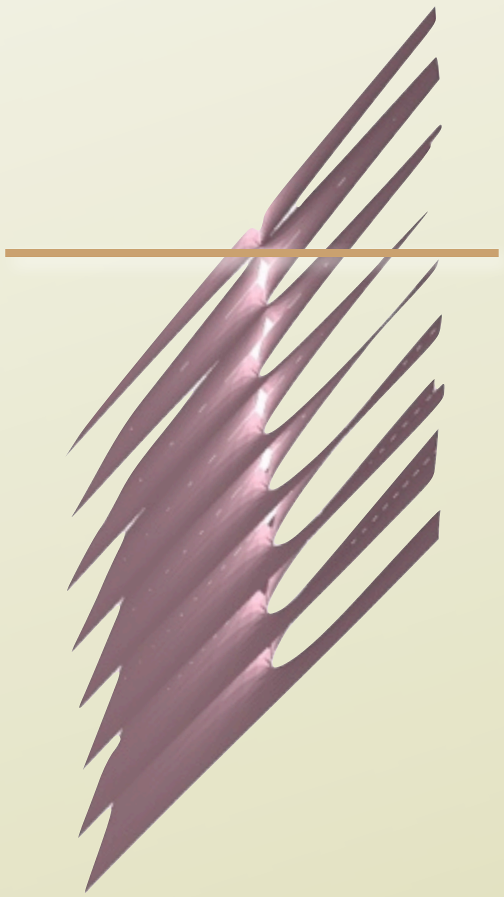

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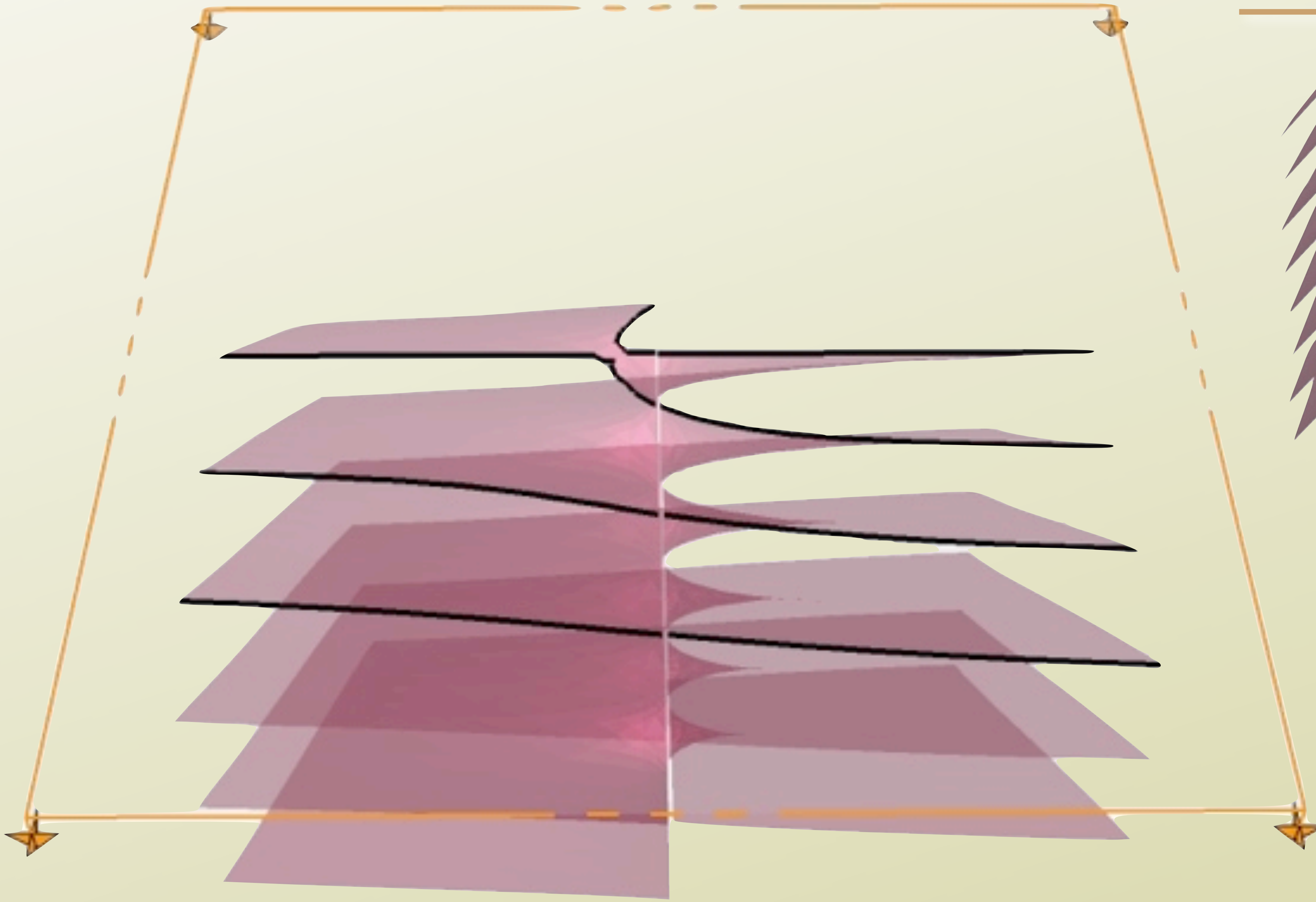

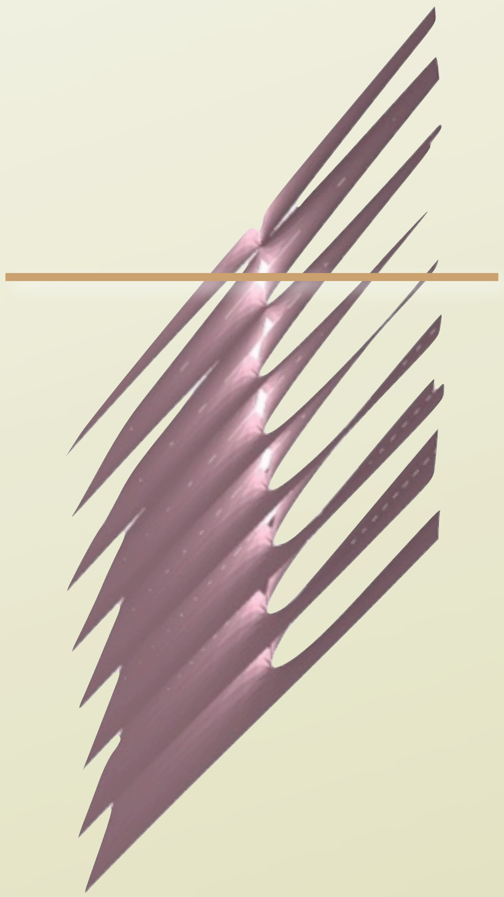

Penn

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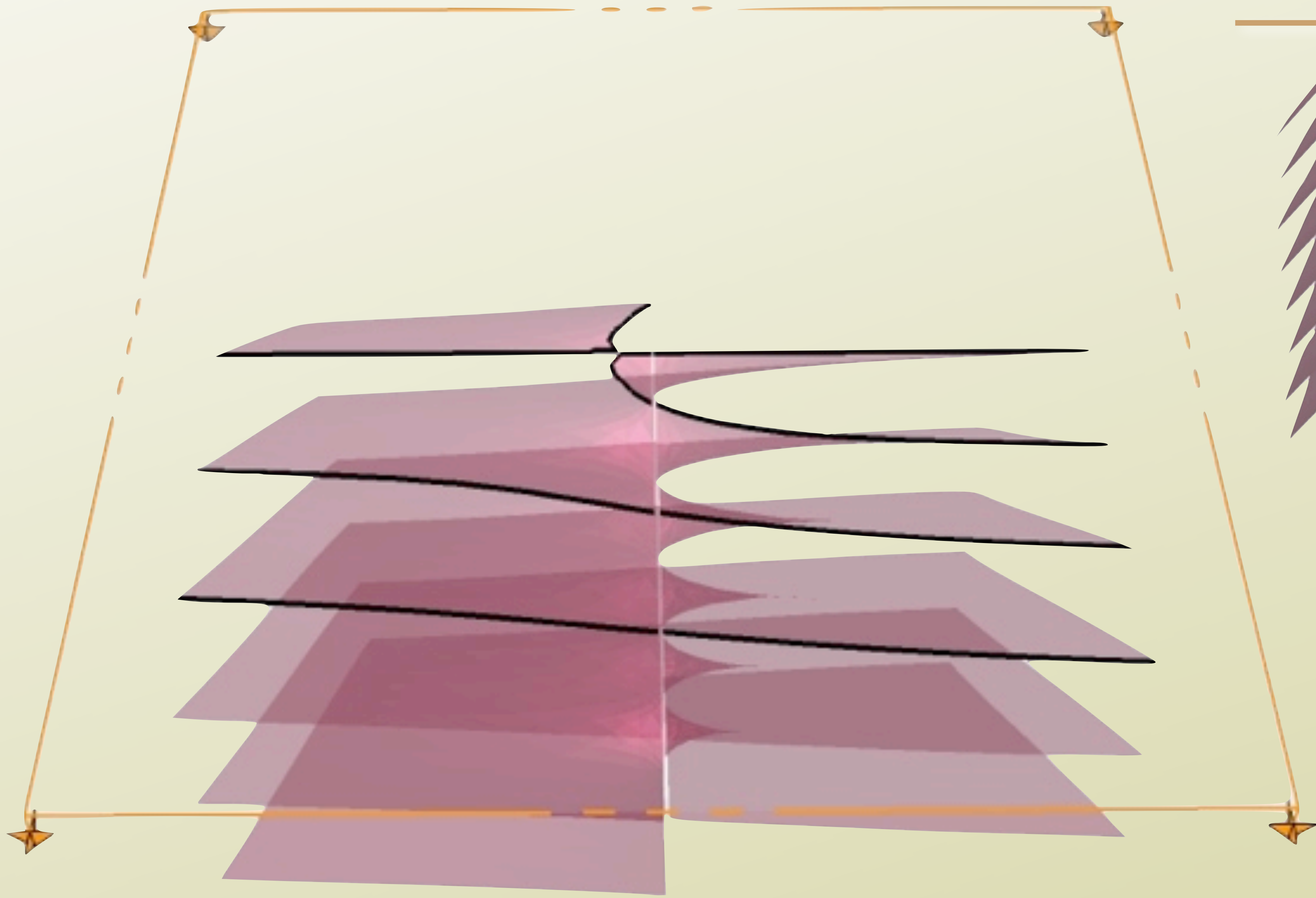

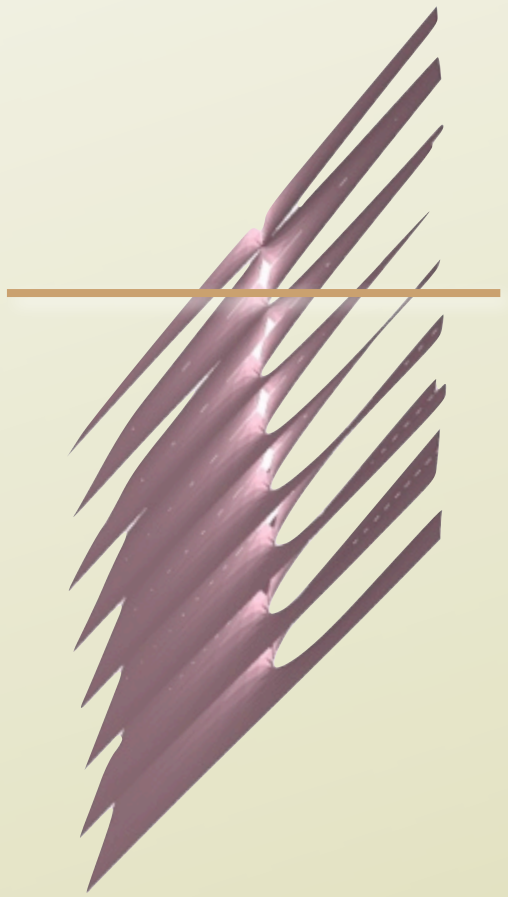

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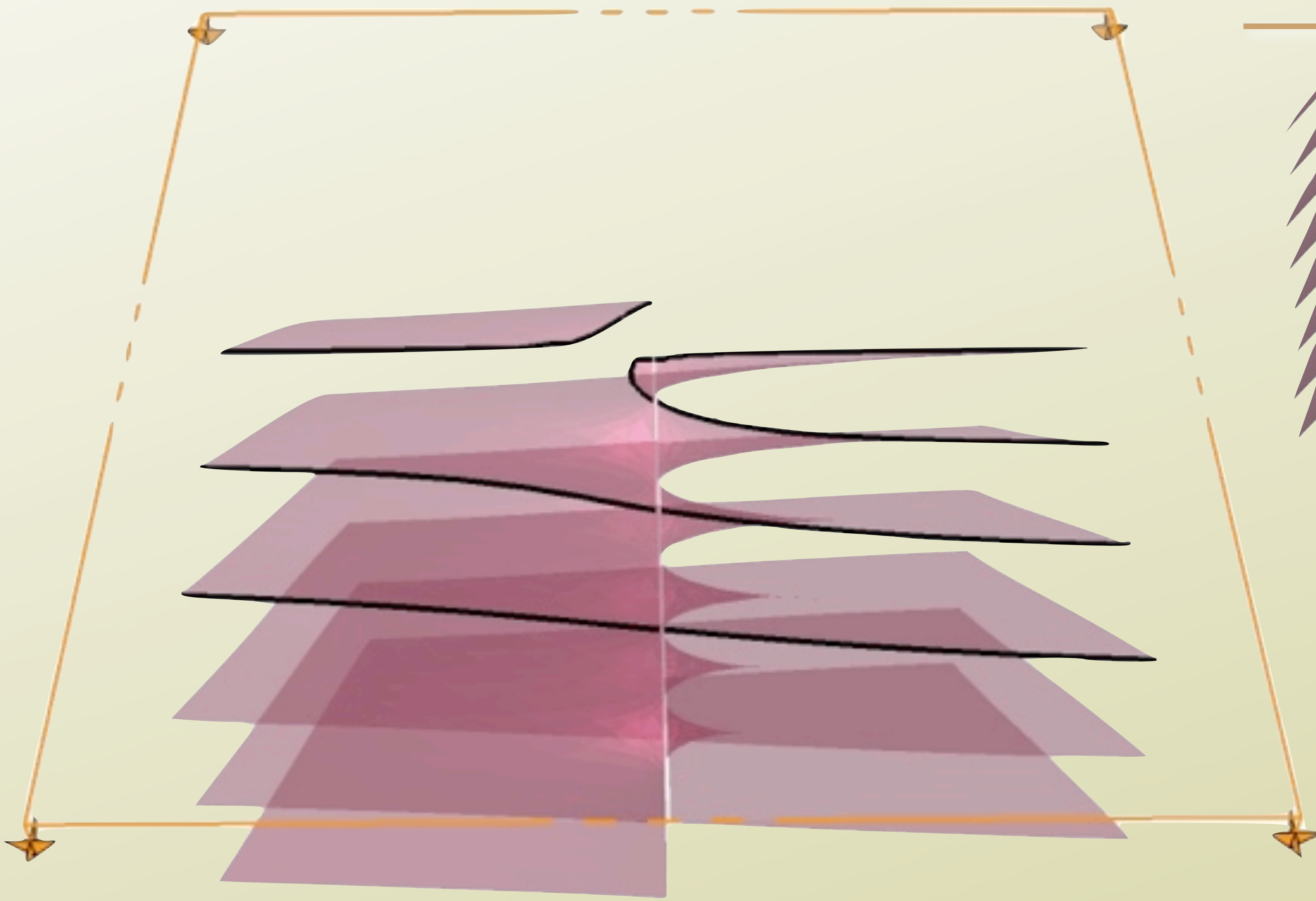

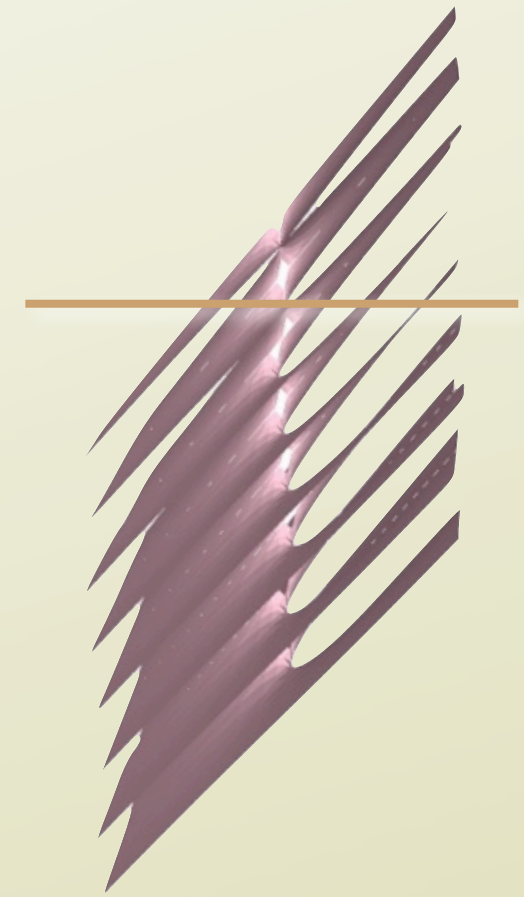

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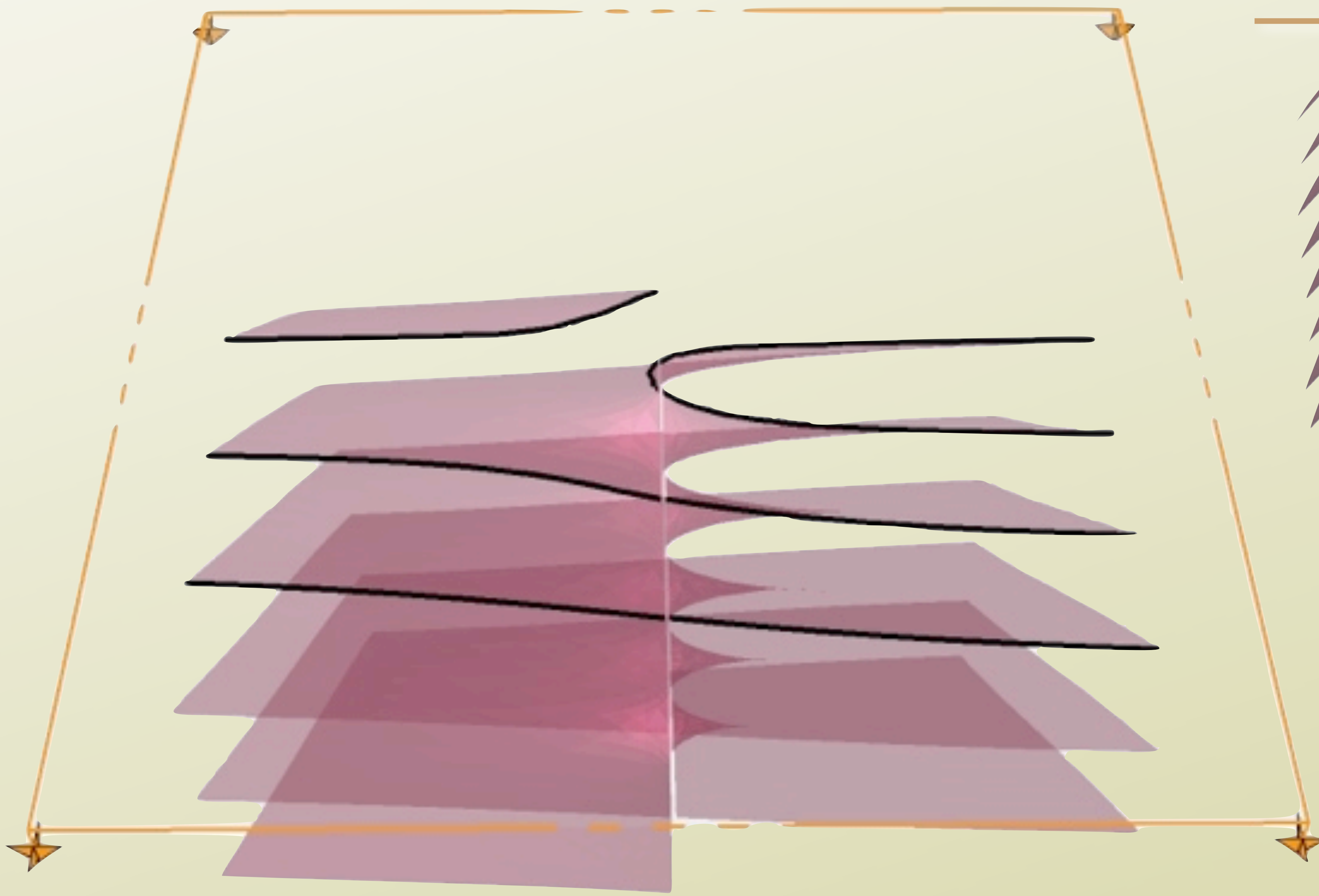

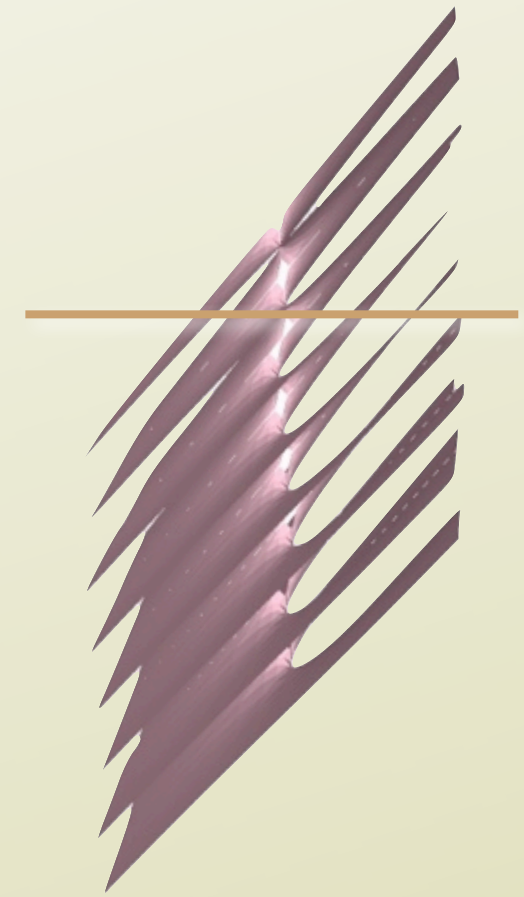

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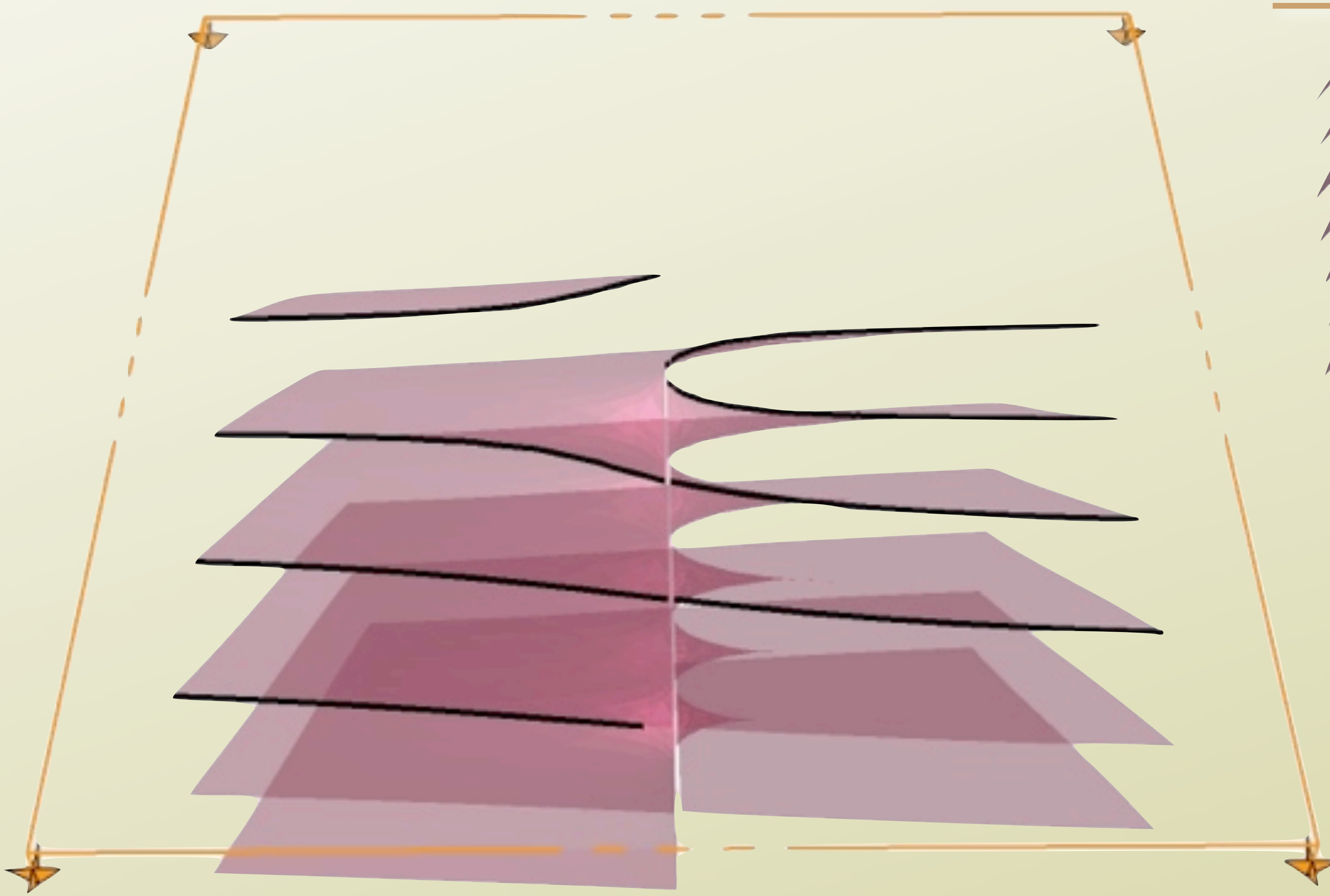

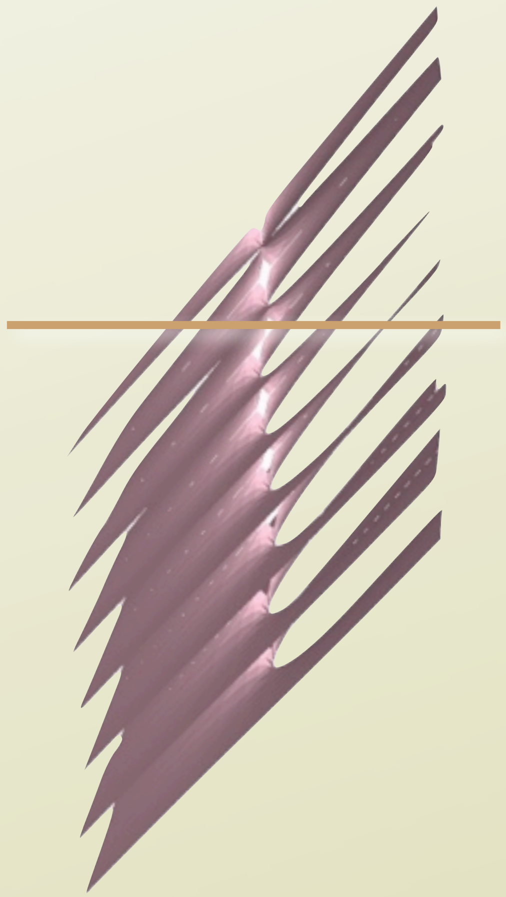

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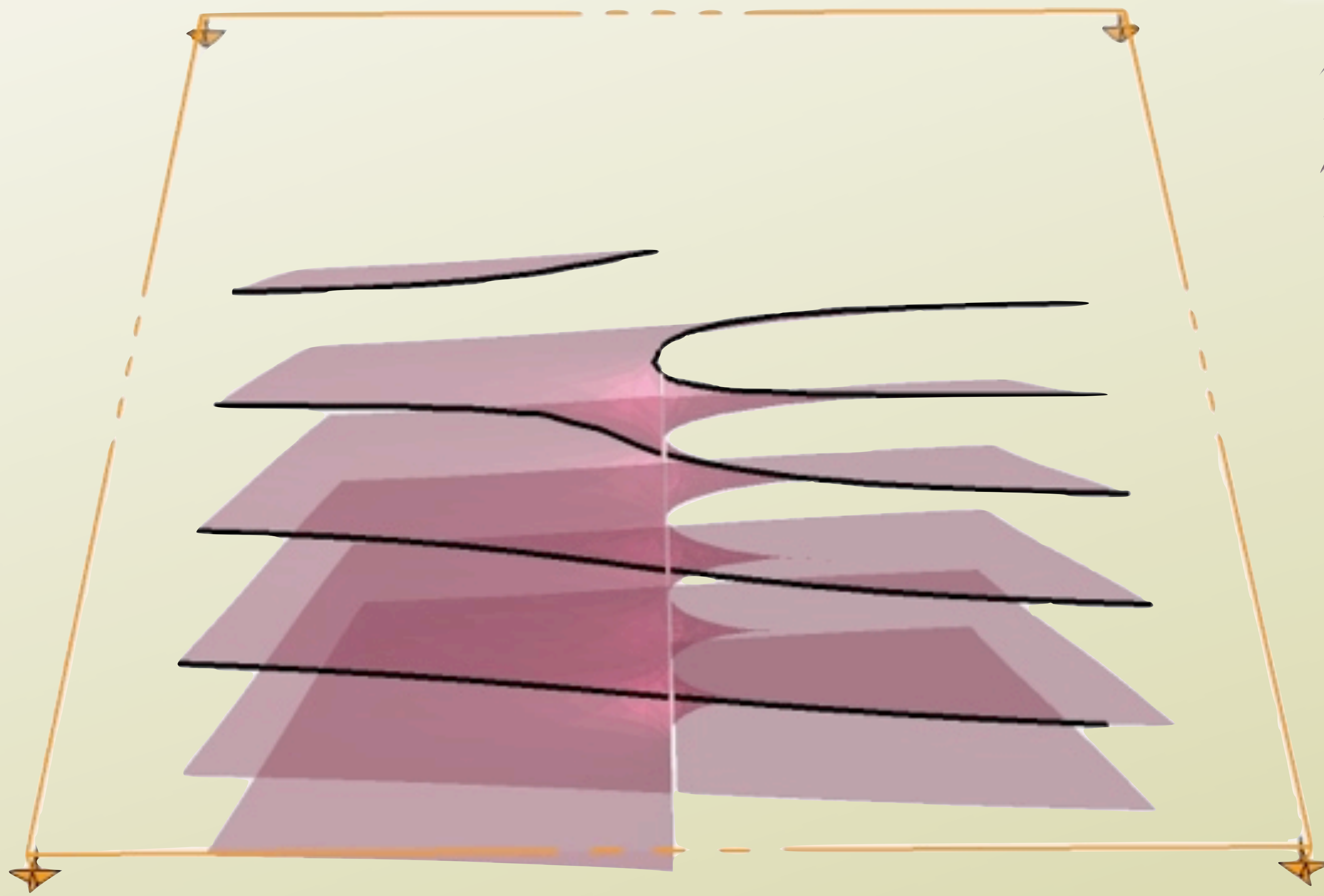

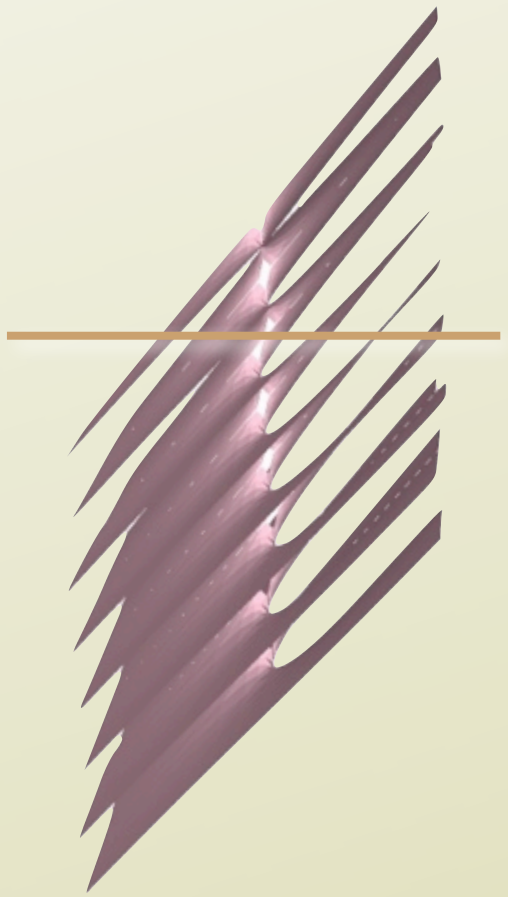

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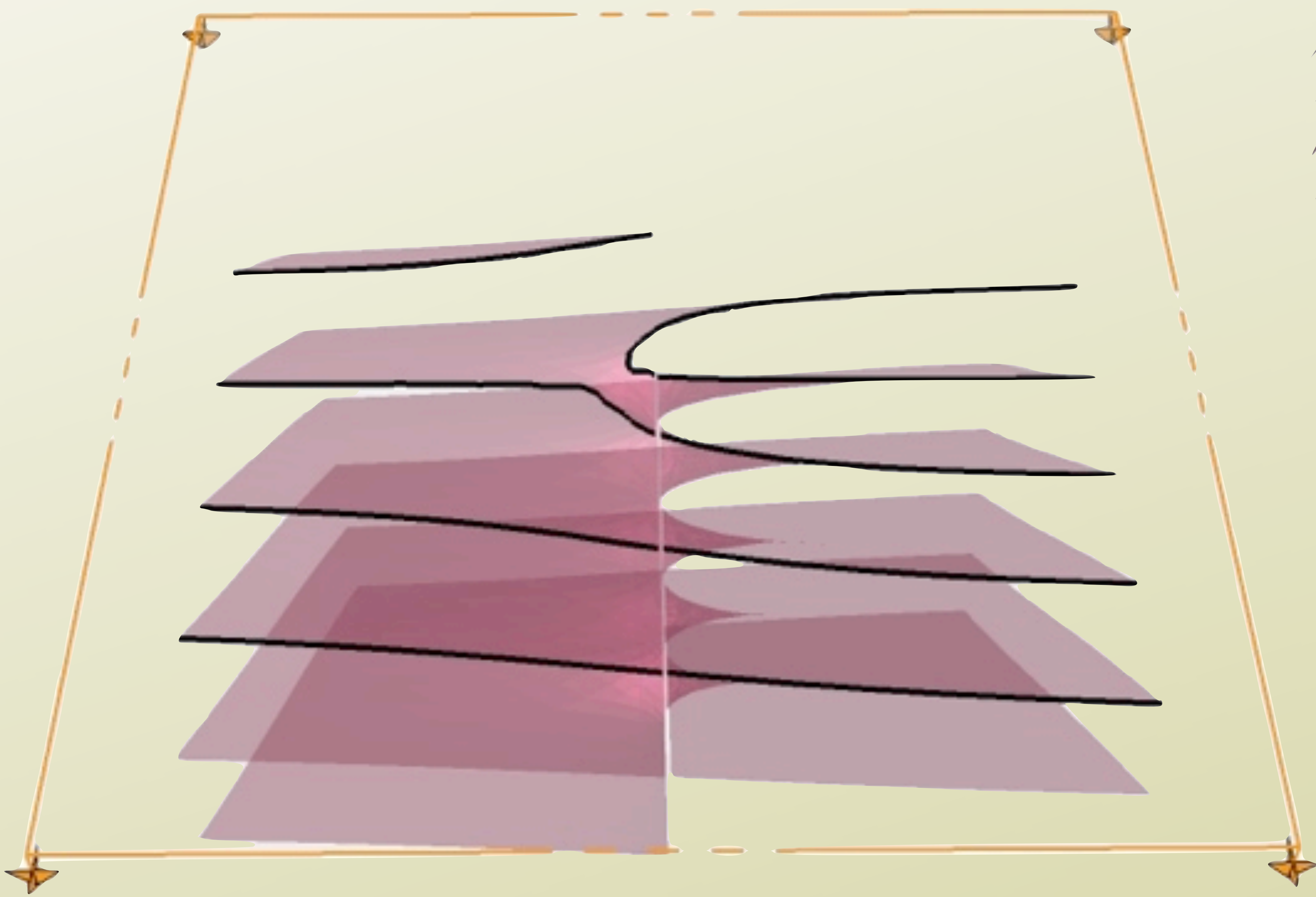

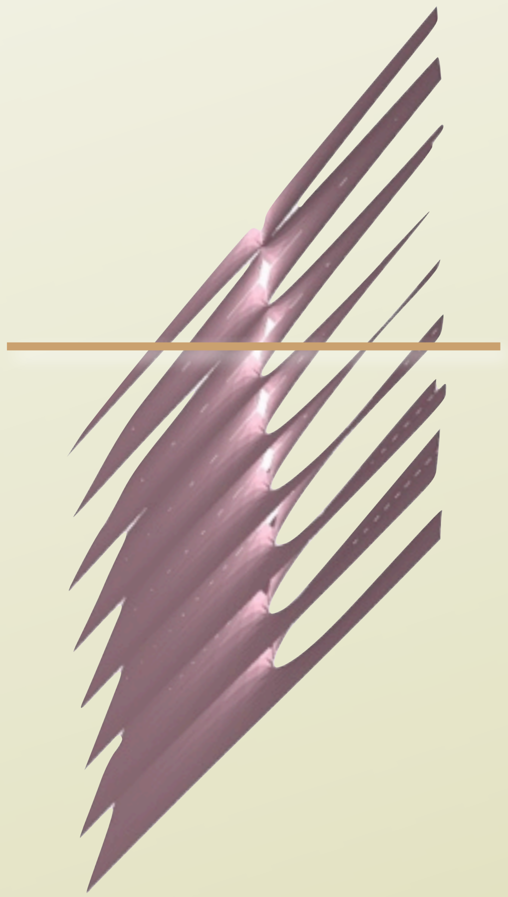

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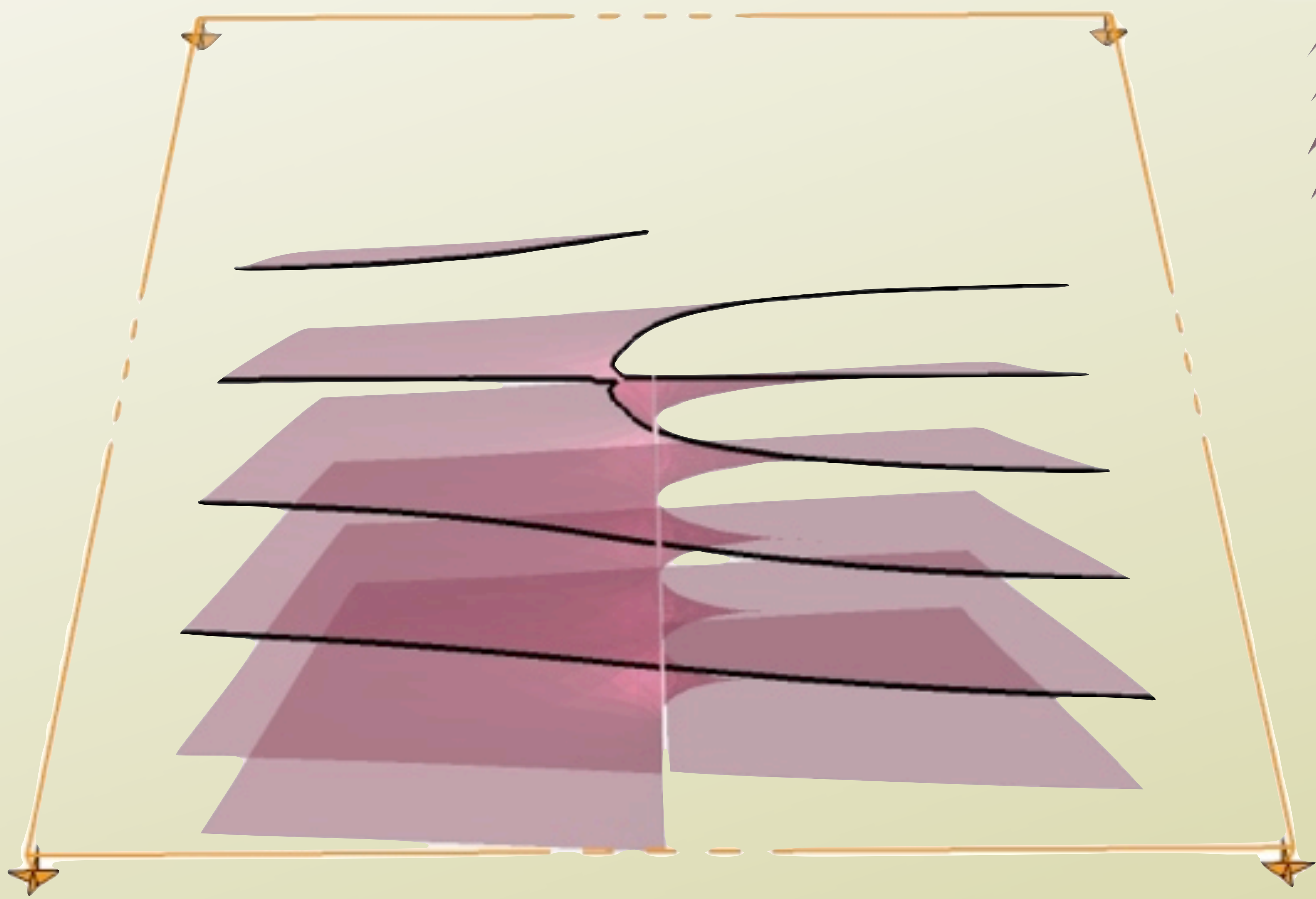

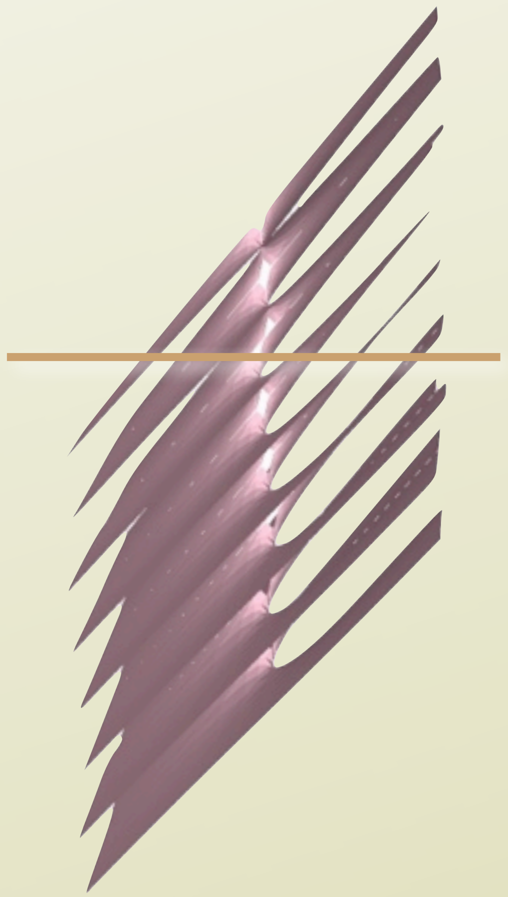

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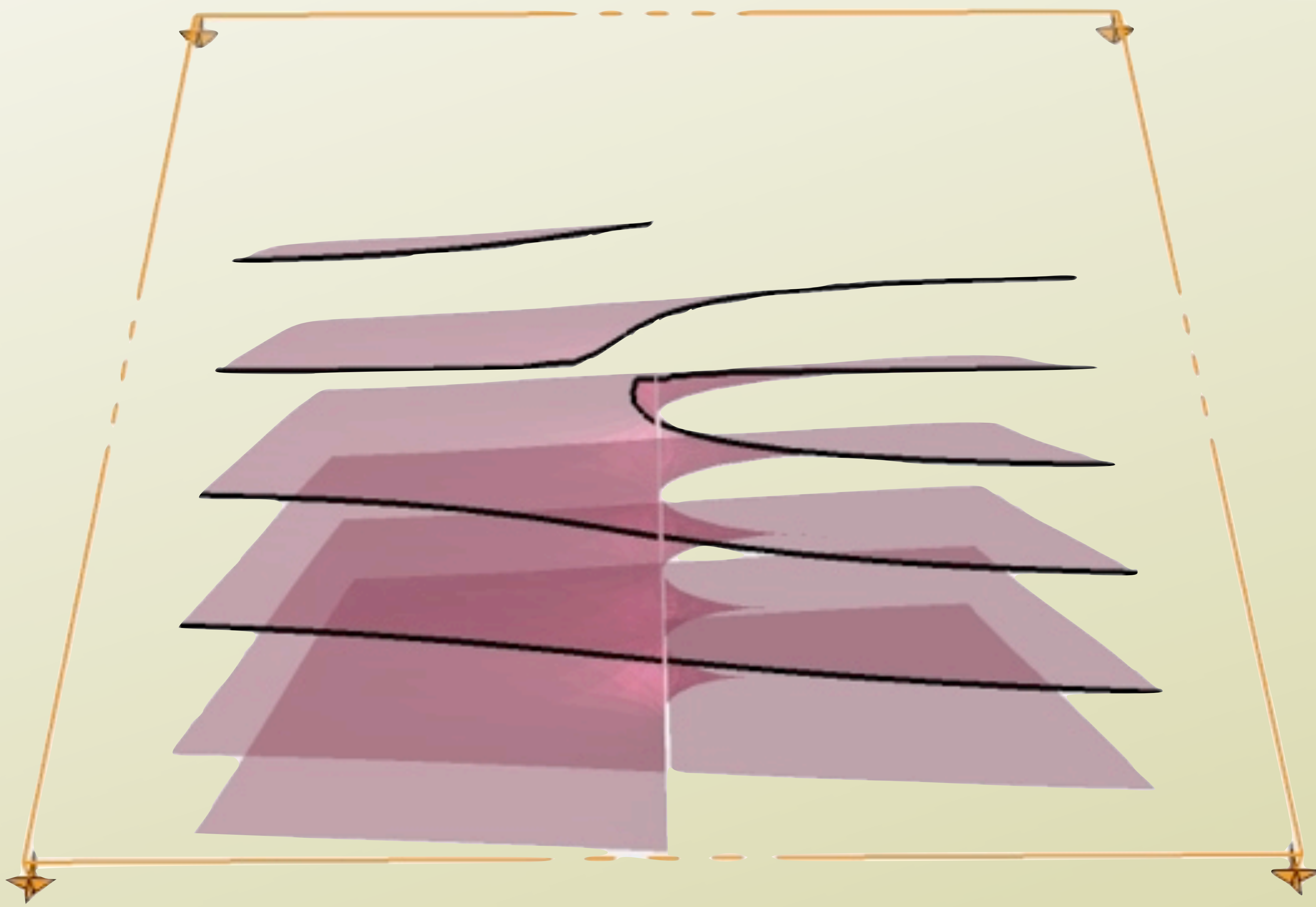

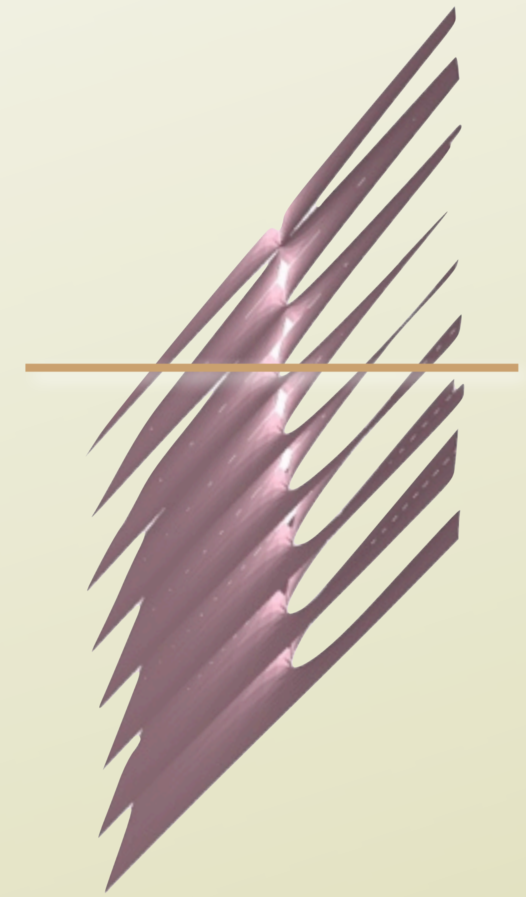

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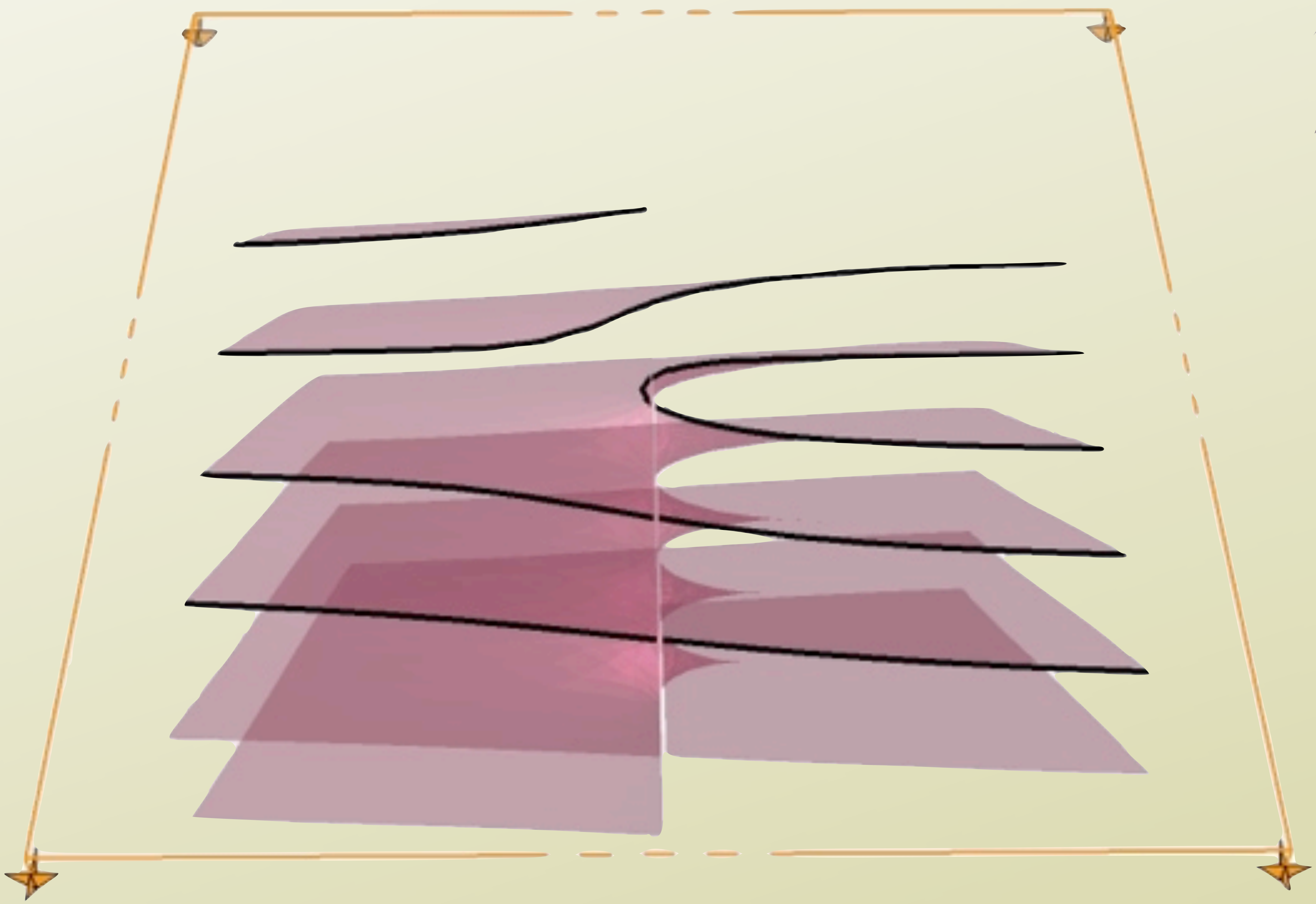

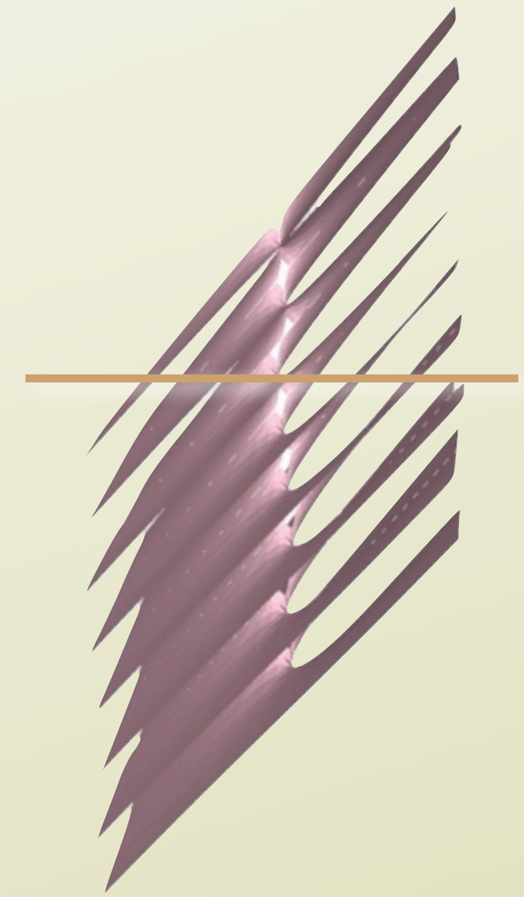

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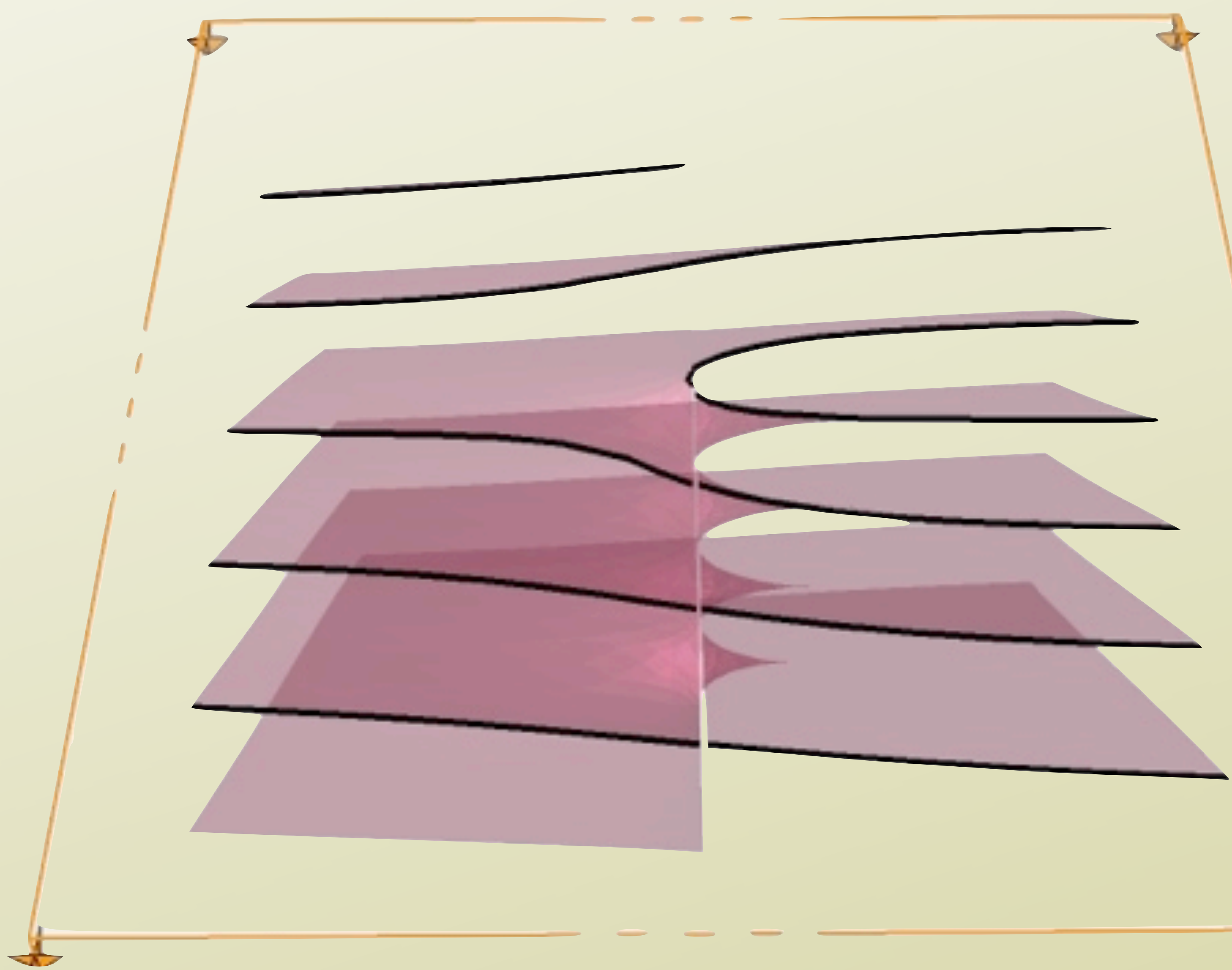
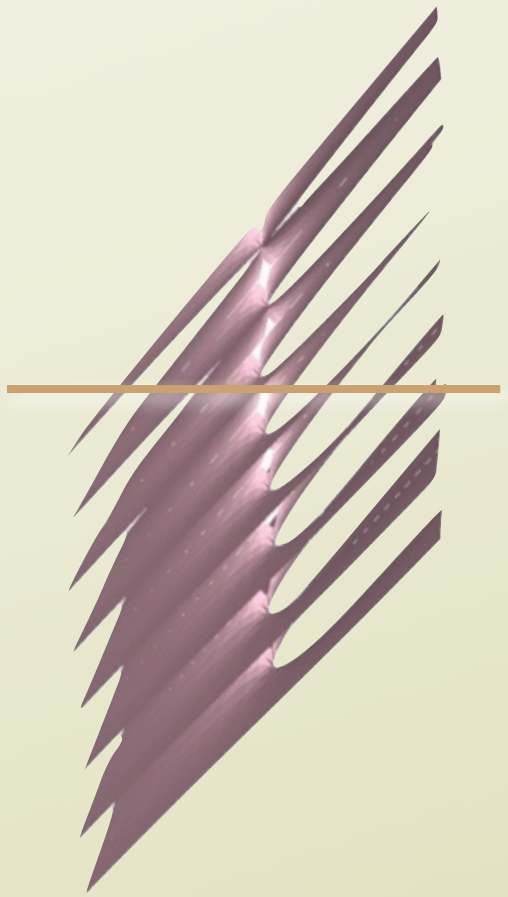




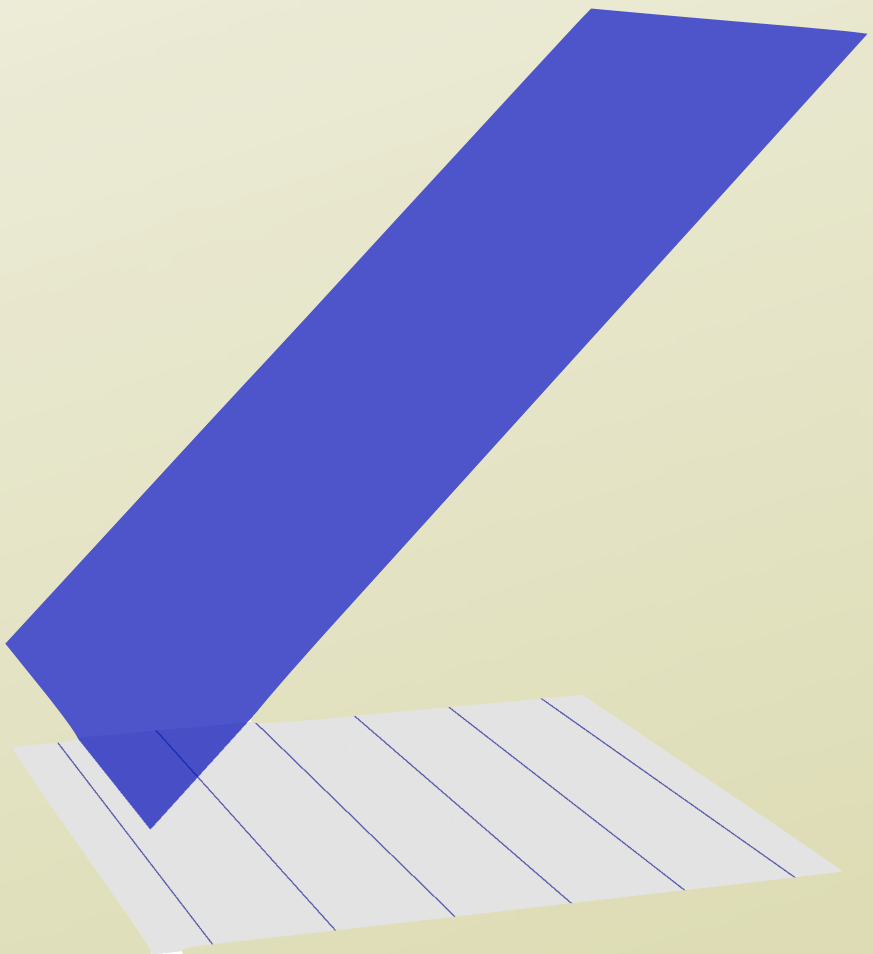
Smectic Symmetries: Layer or Layers?

density wave: $\rho \propto \cos\left(\frac{2\pi\phi}{a}\right)$

Phase is periodic ... $\phi \sim \phi + a$

... and unoriented $\phi \sim -\phi$

$\Rightarrow \quad \phi \in S^1/\mathbb{Z}_2$



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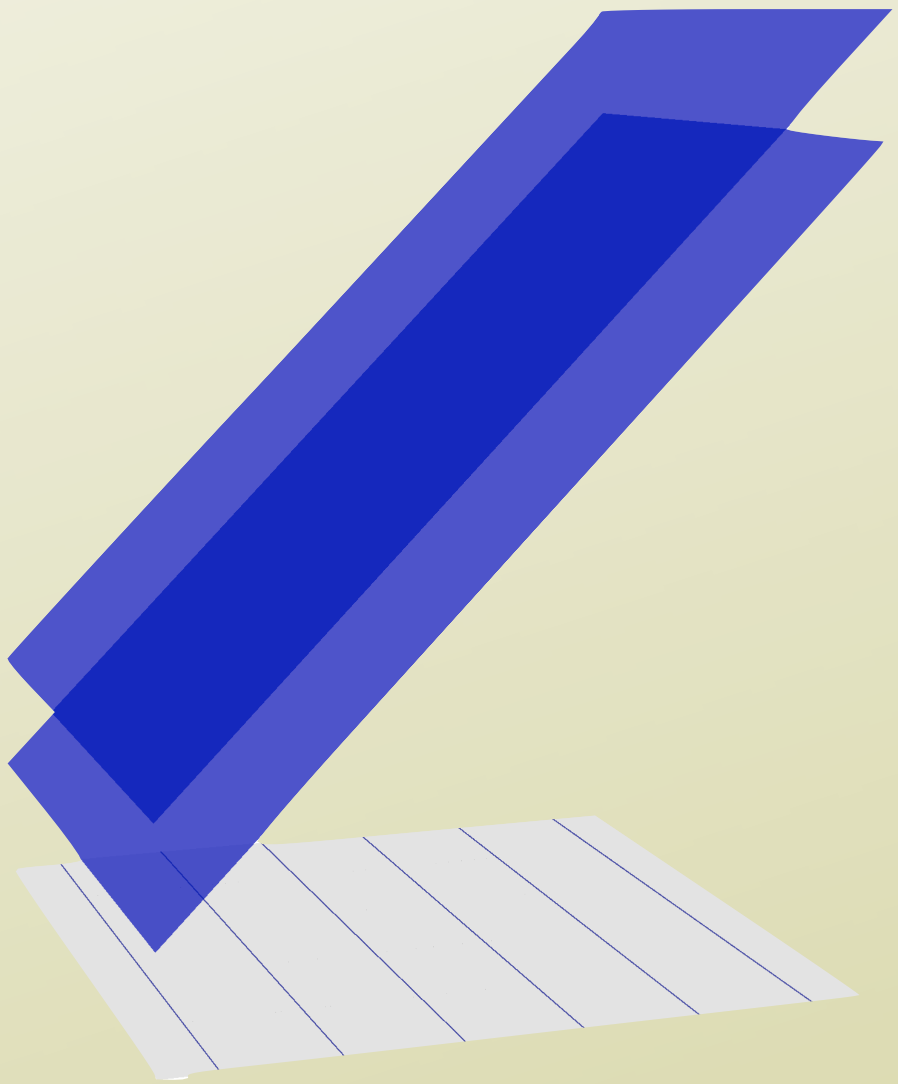

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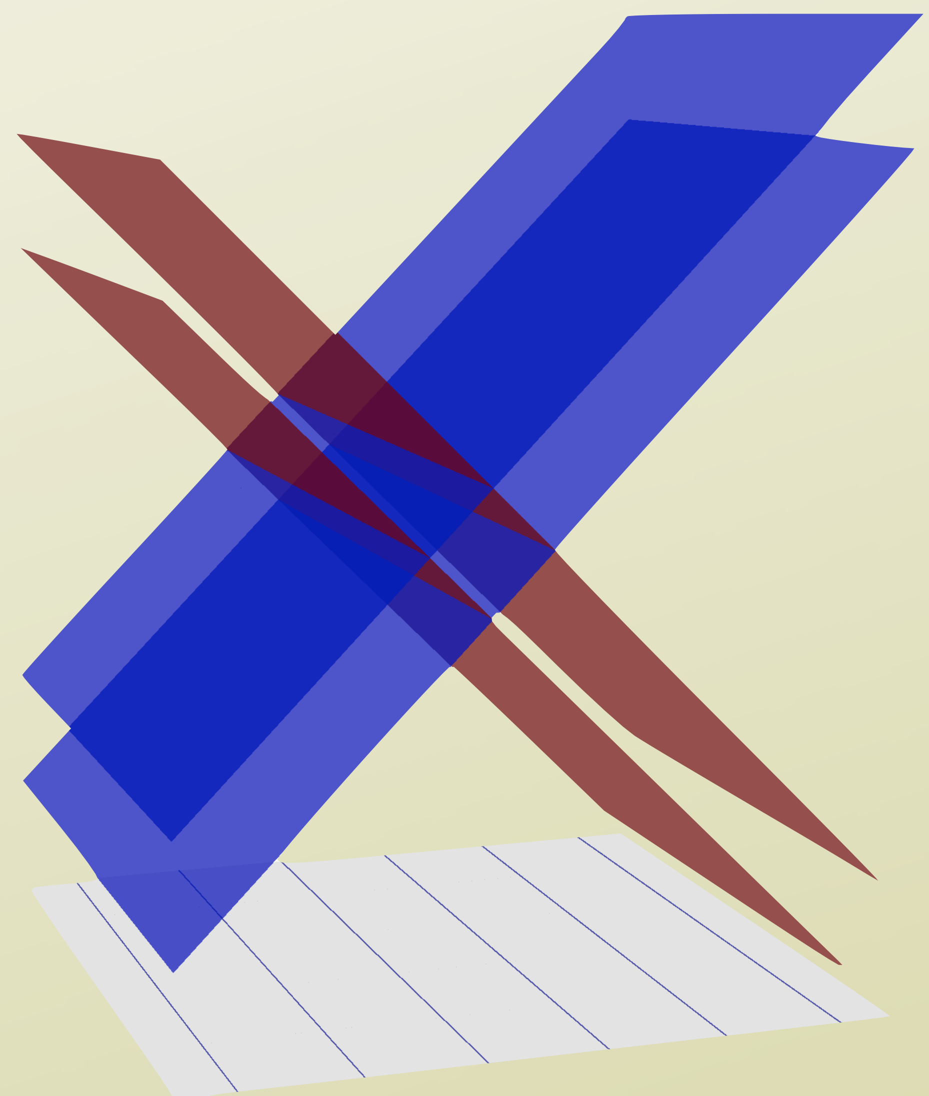

- sheets cross at the fixed points of these point symmetries
- only slices at these heights yield consistent smectics
- critical points are constrained to these heights

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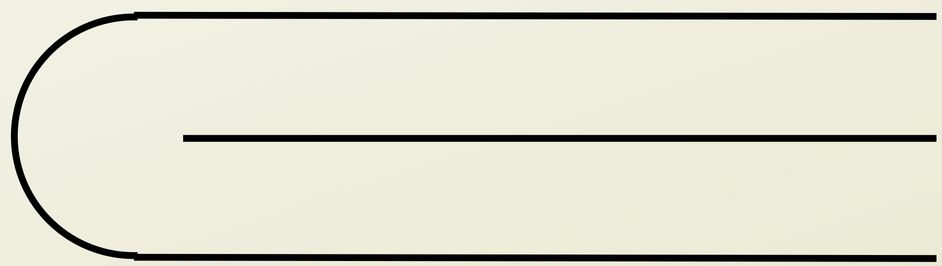
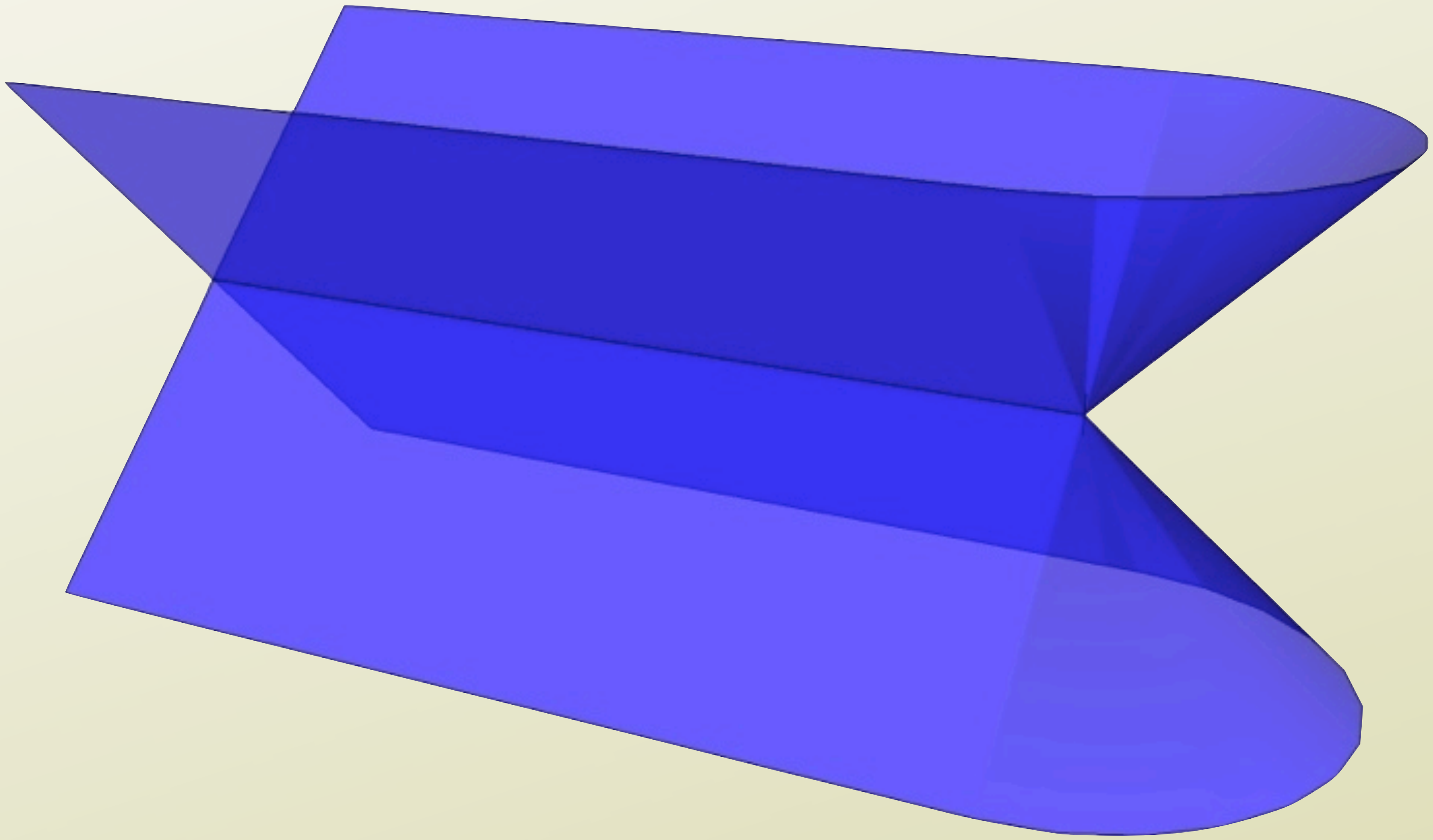
+1/2 DISCLINATION





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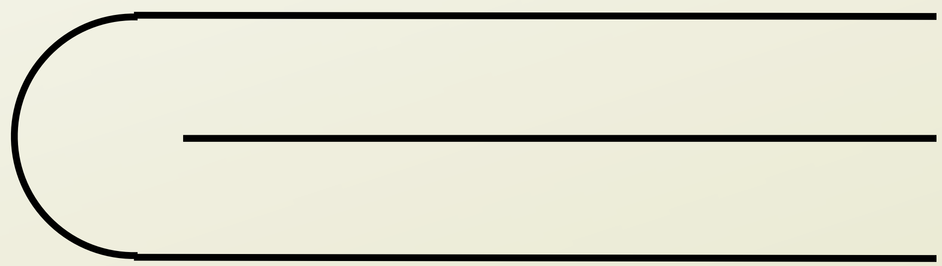

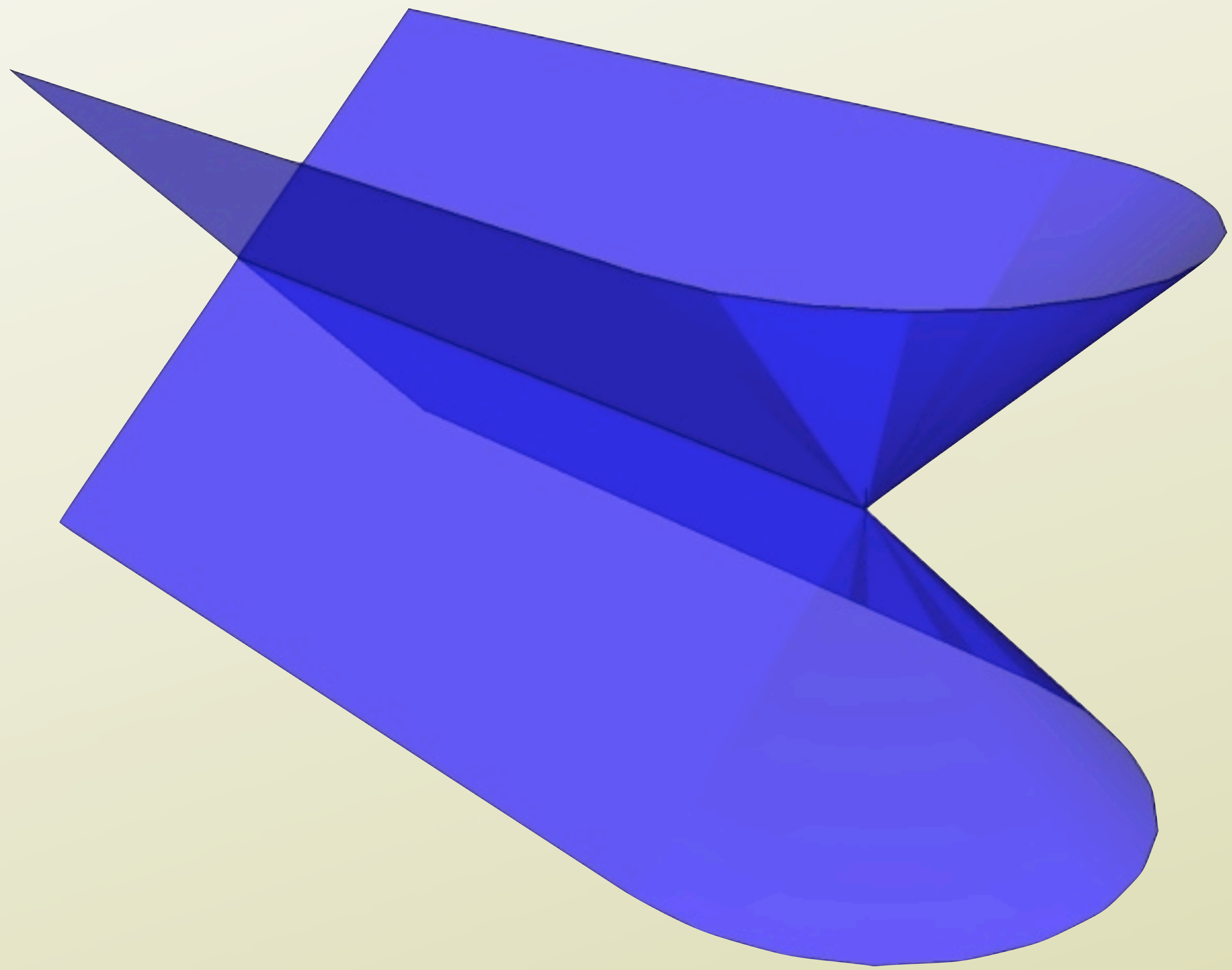

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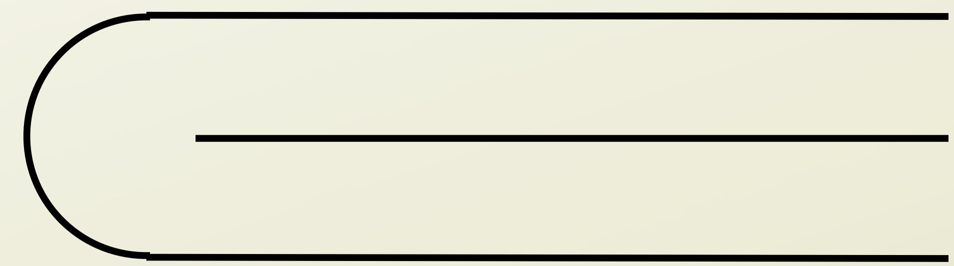

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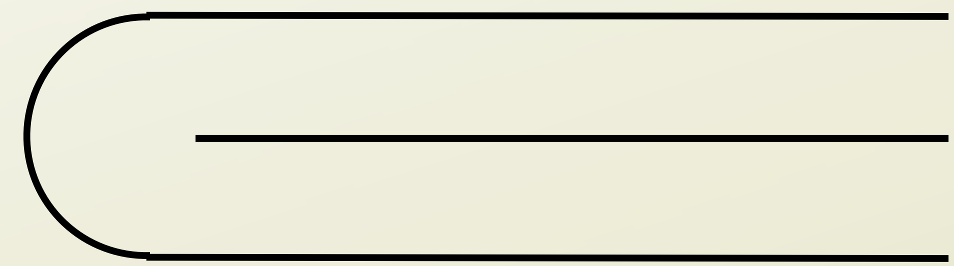

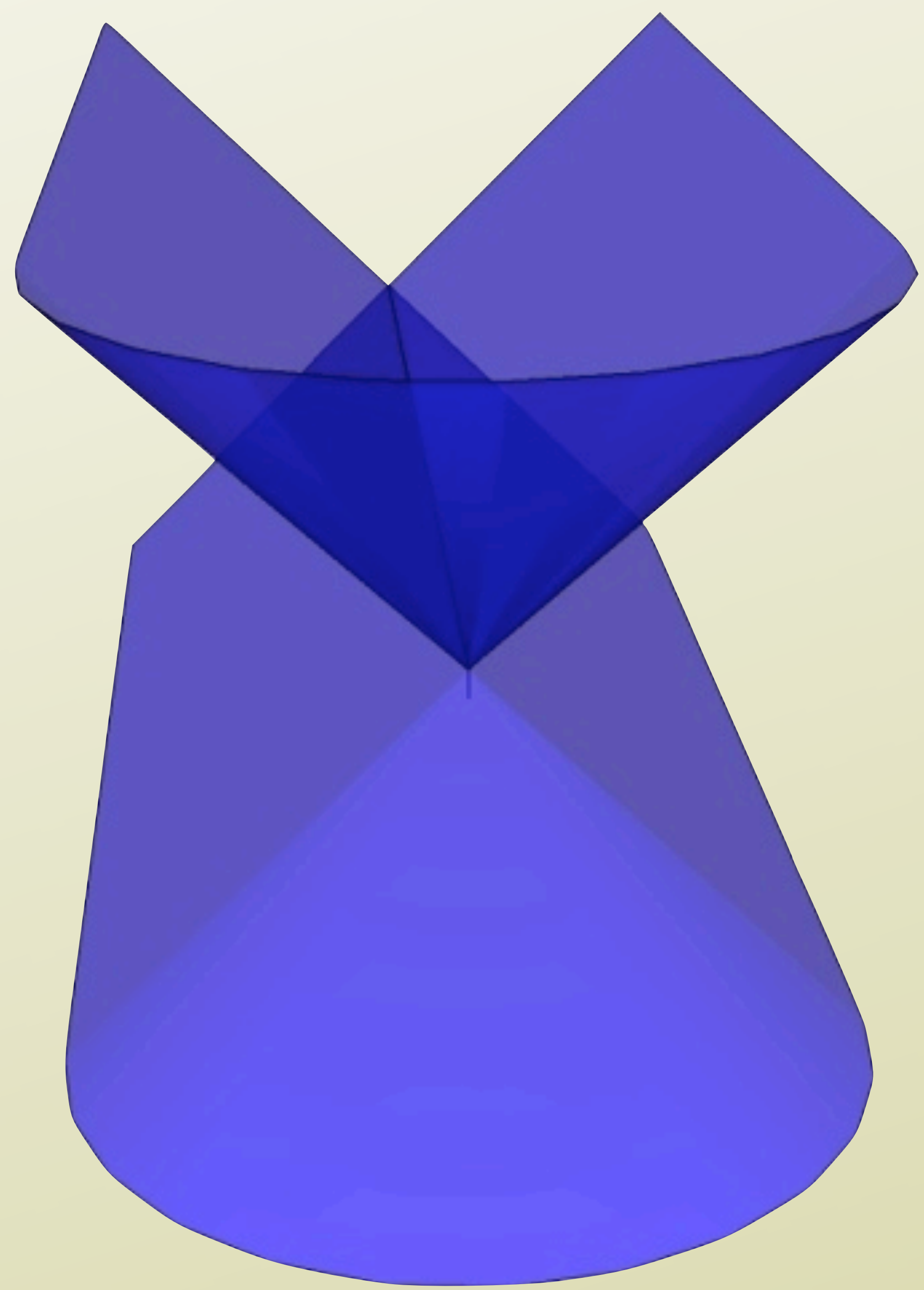

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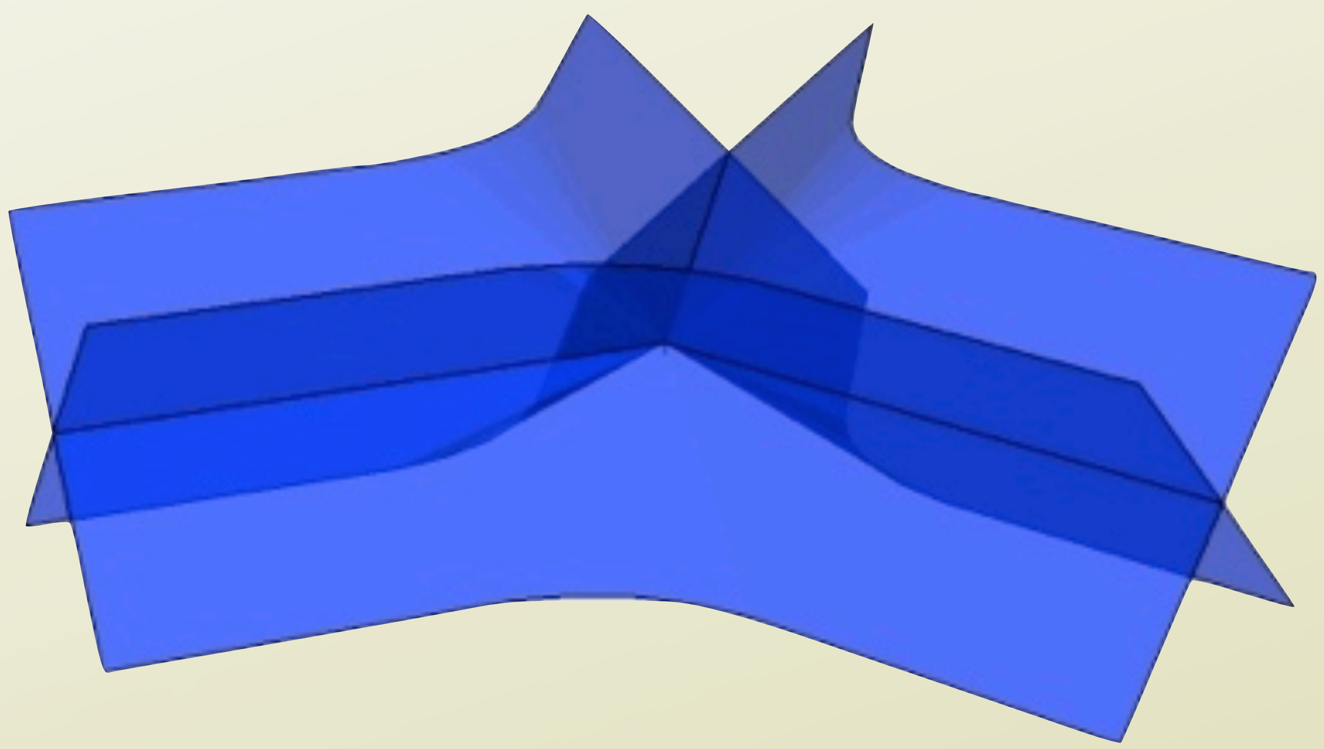

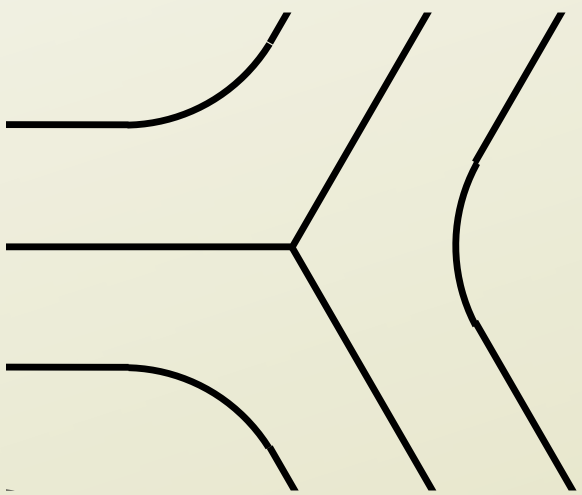

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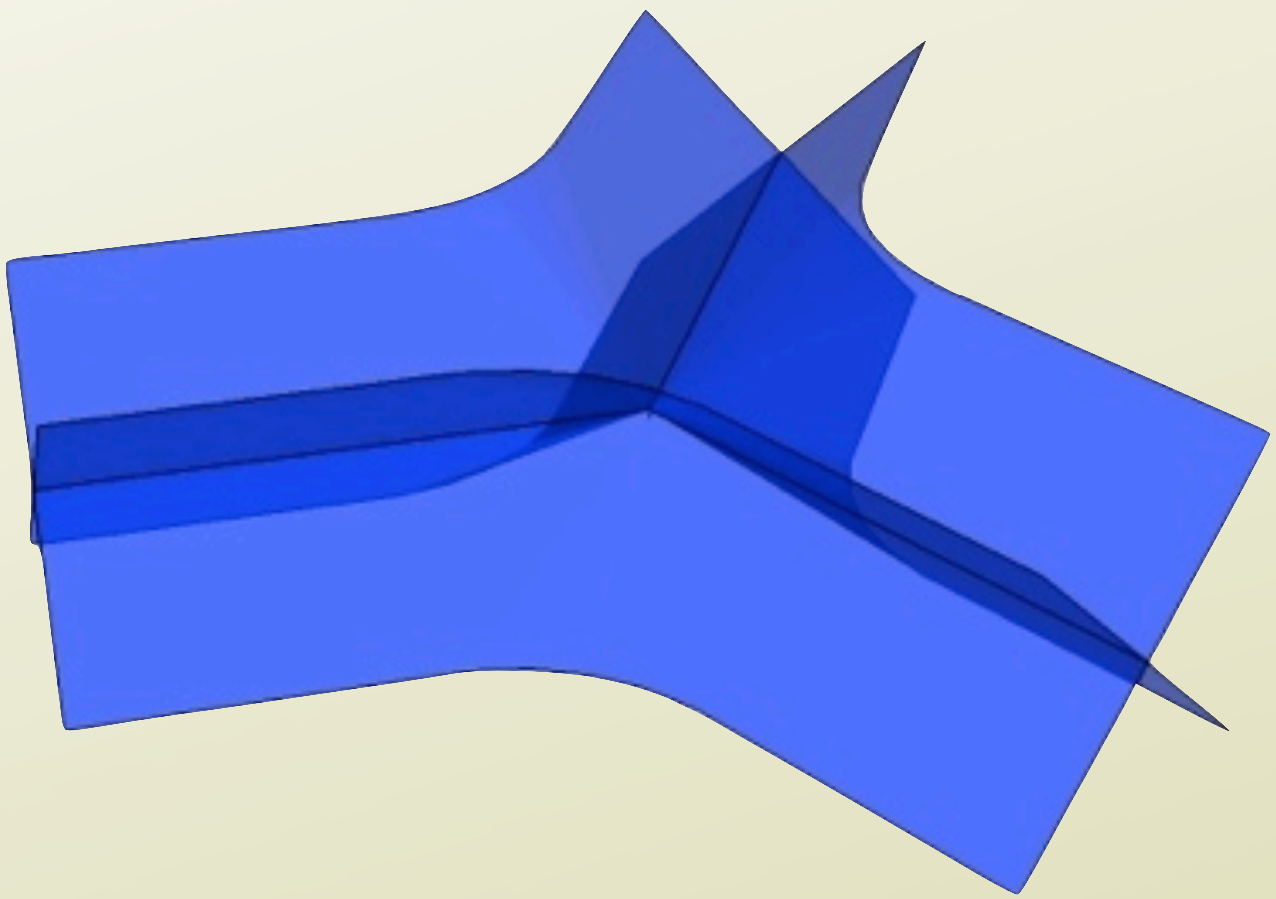

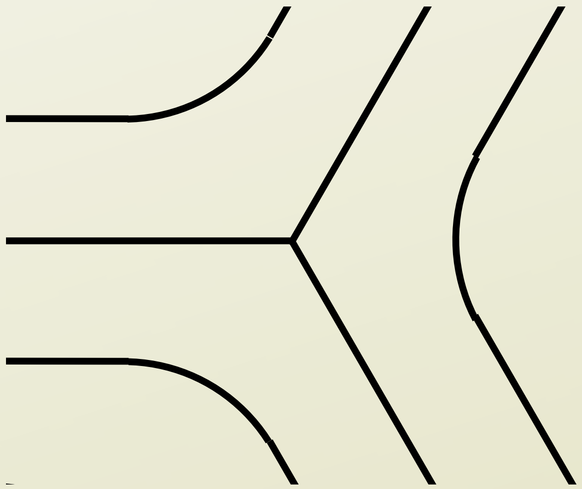

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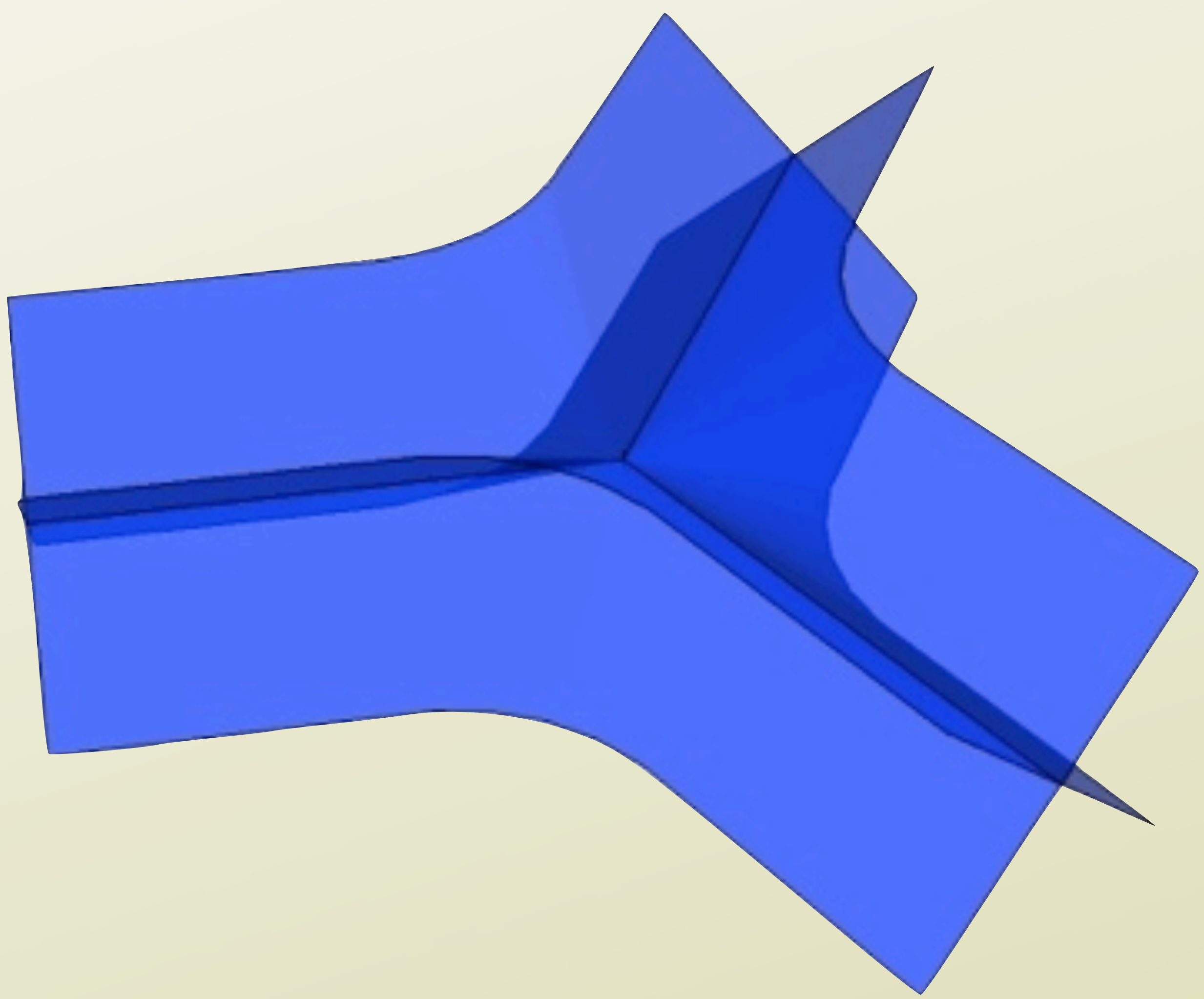

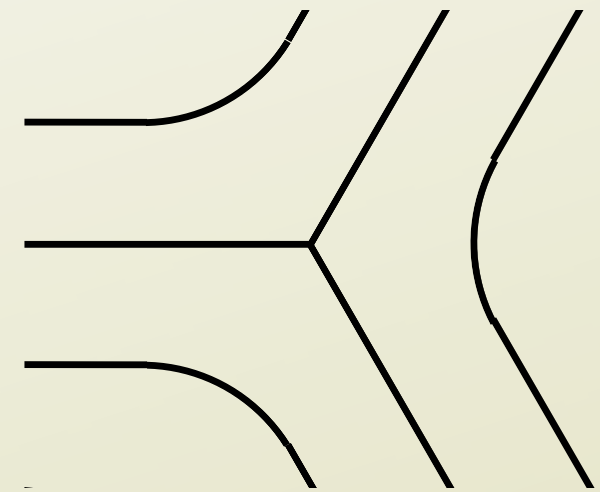

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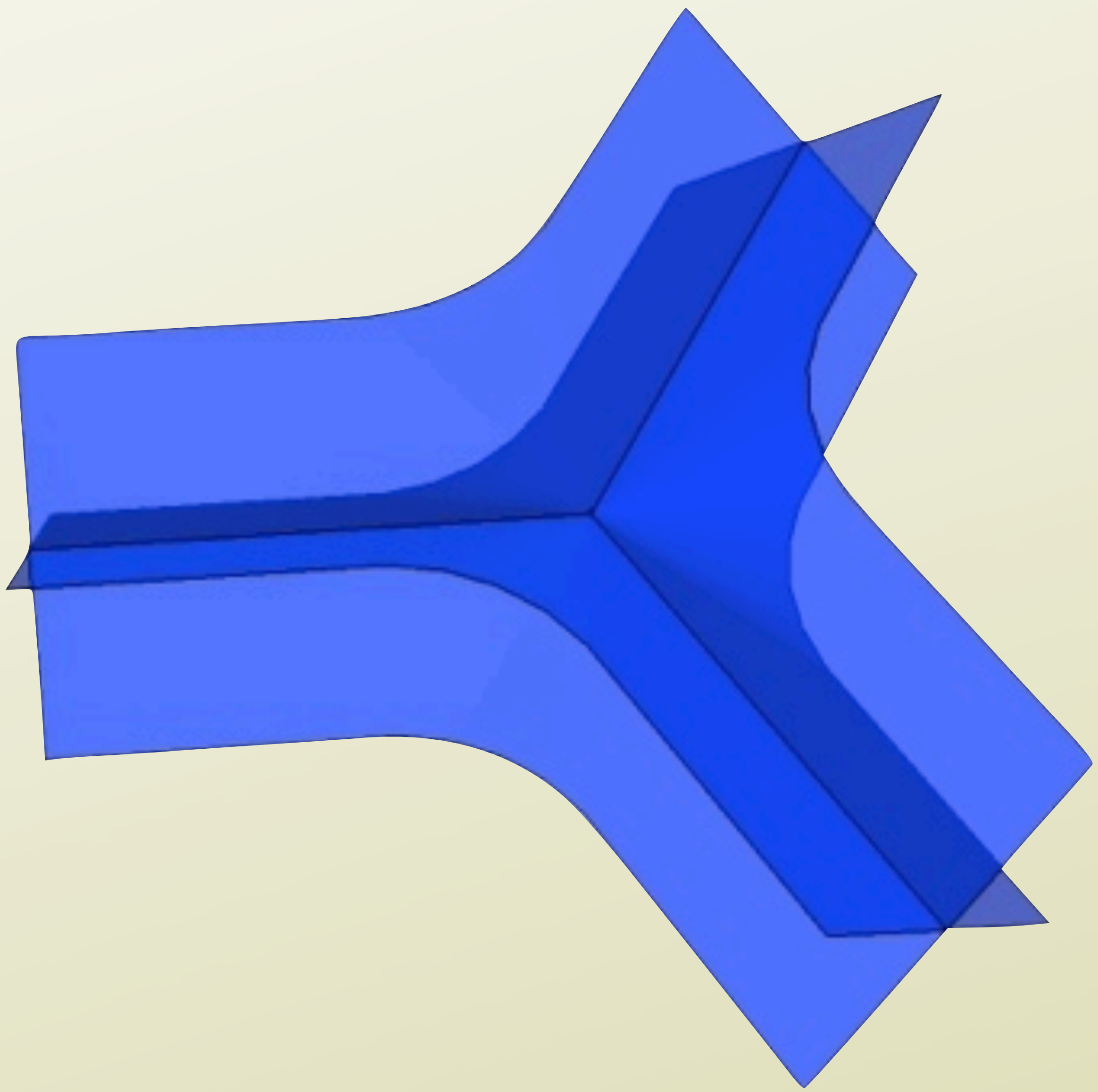

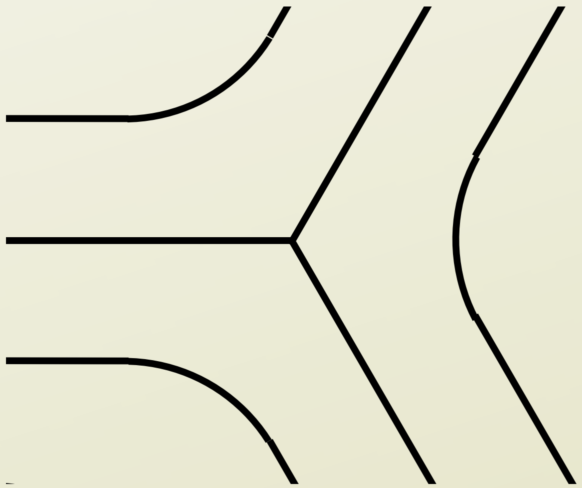

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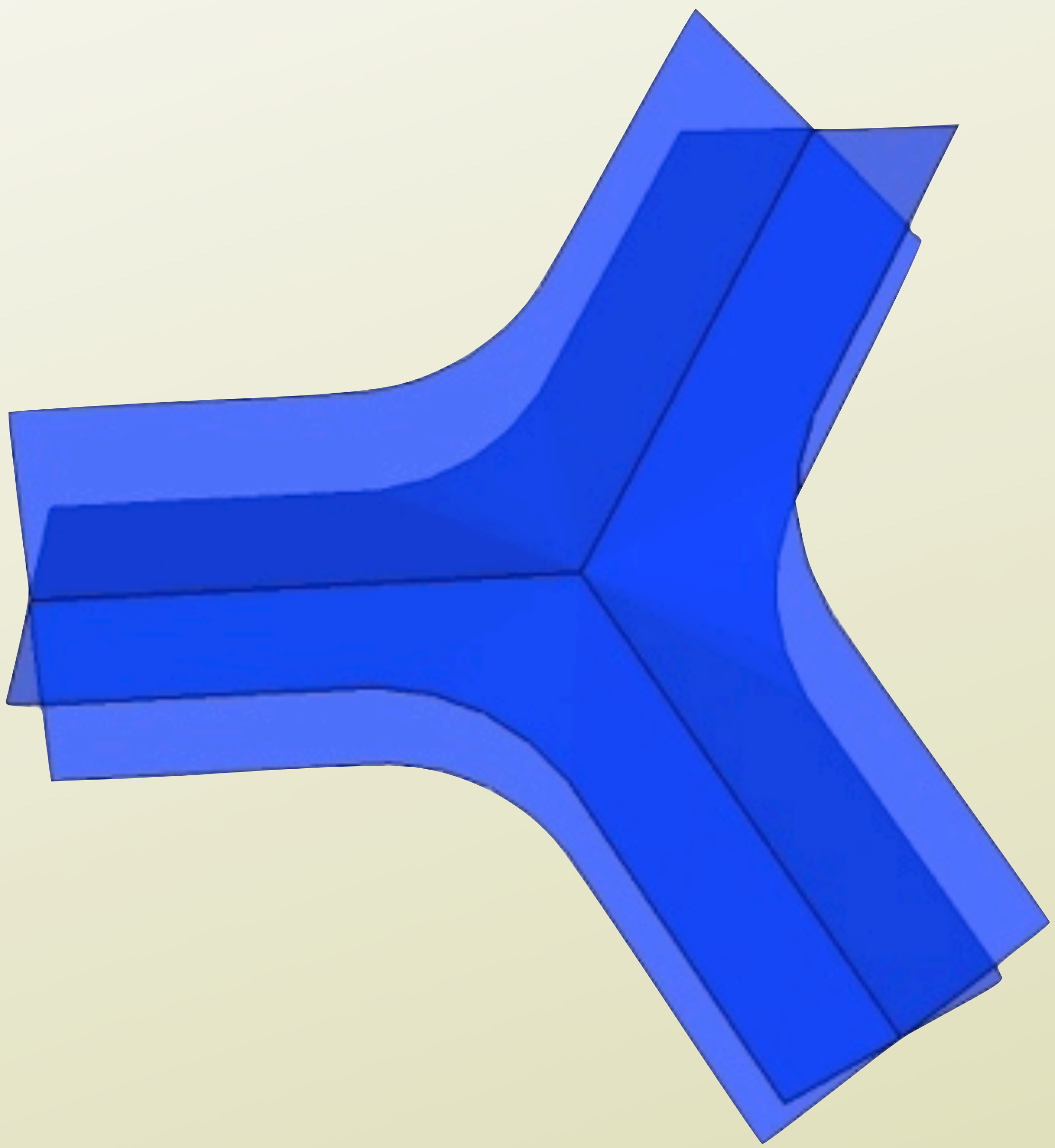

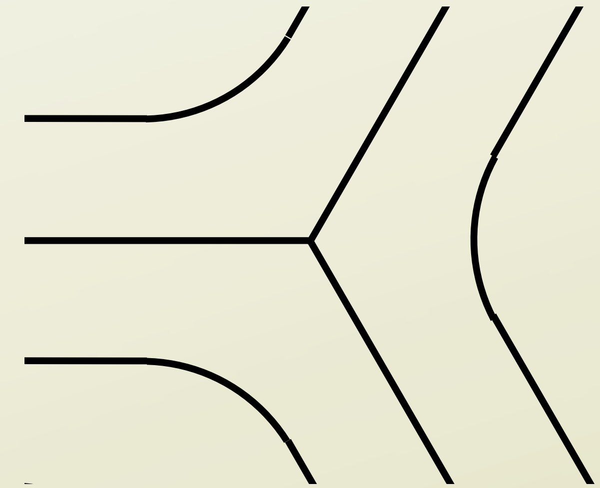

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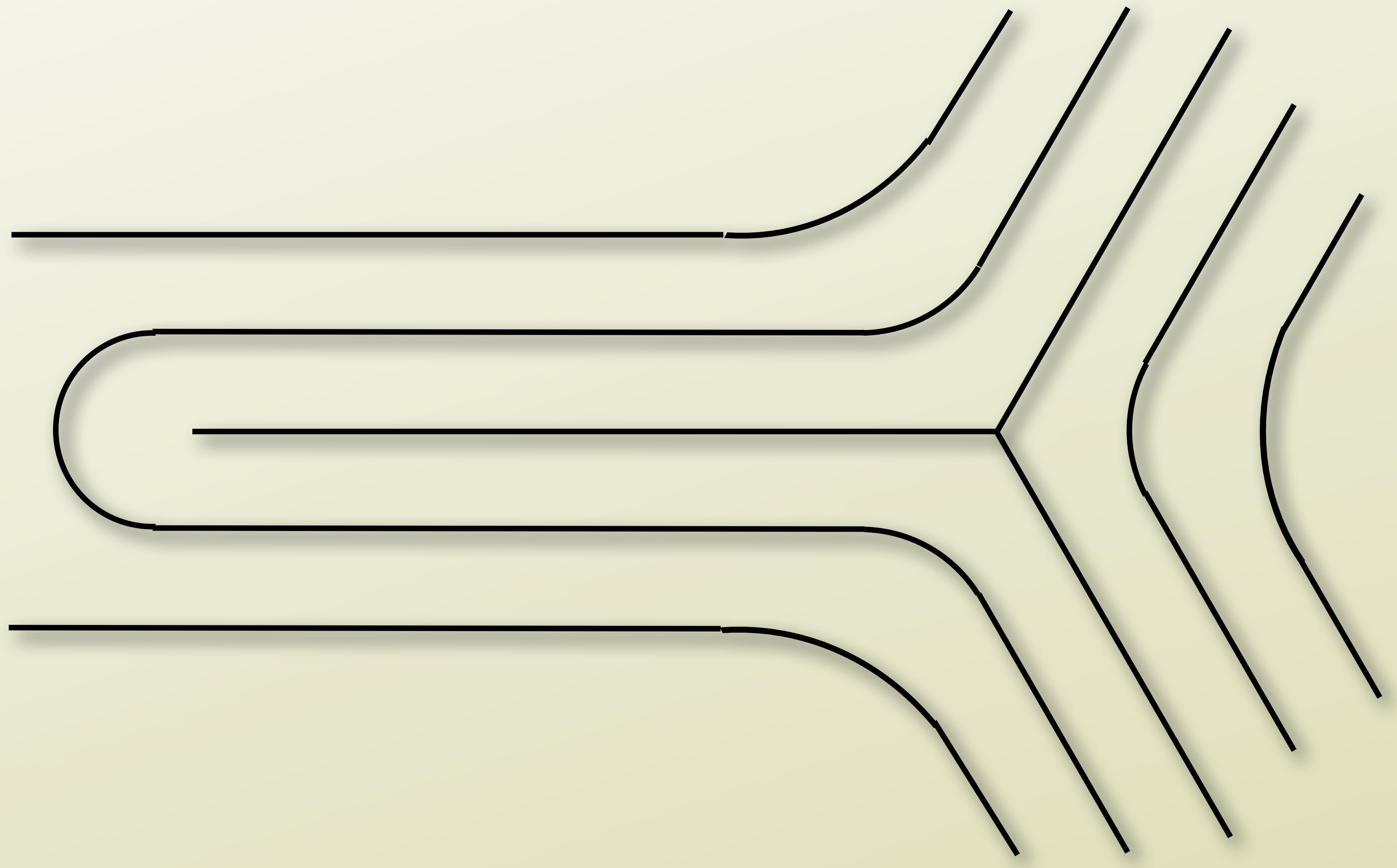

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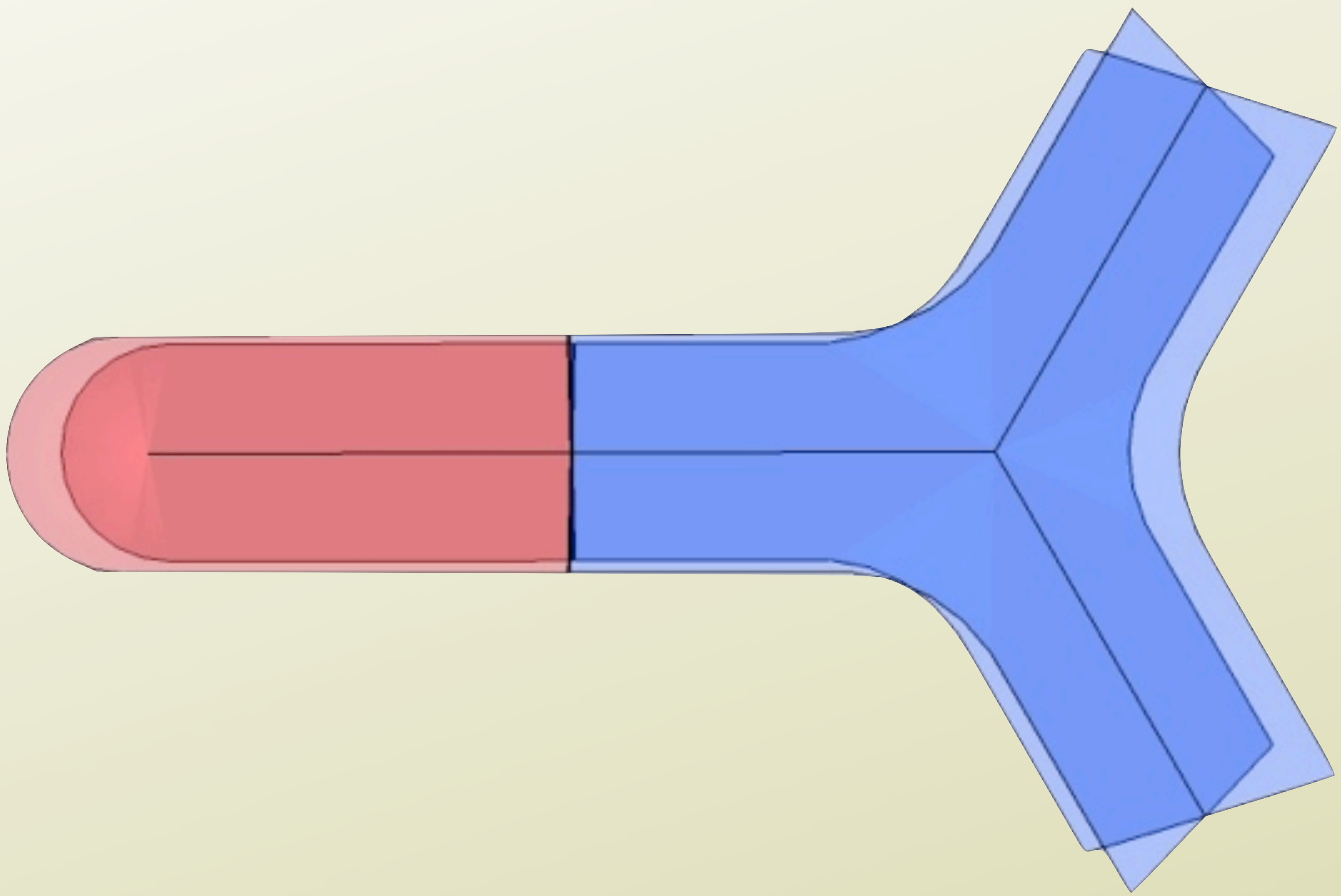

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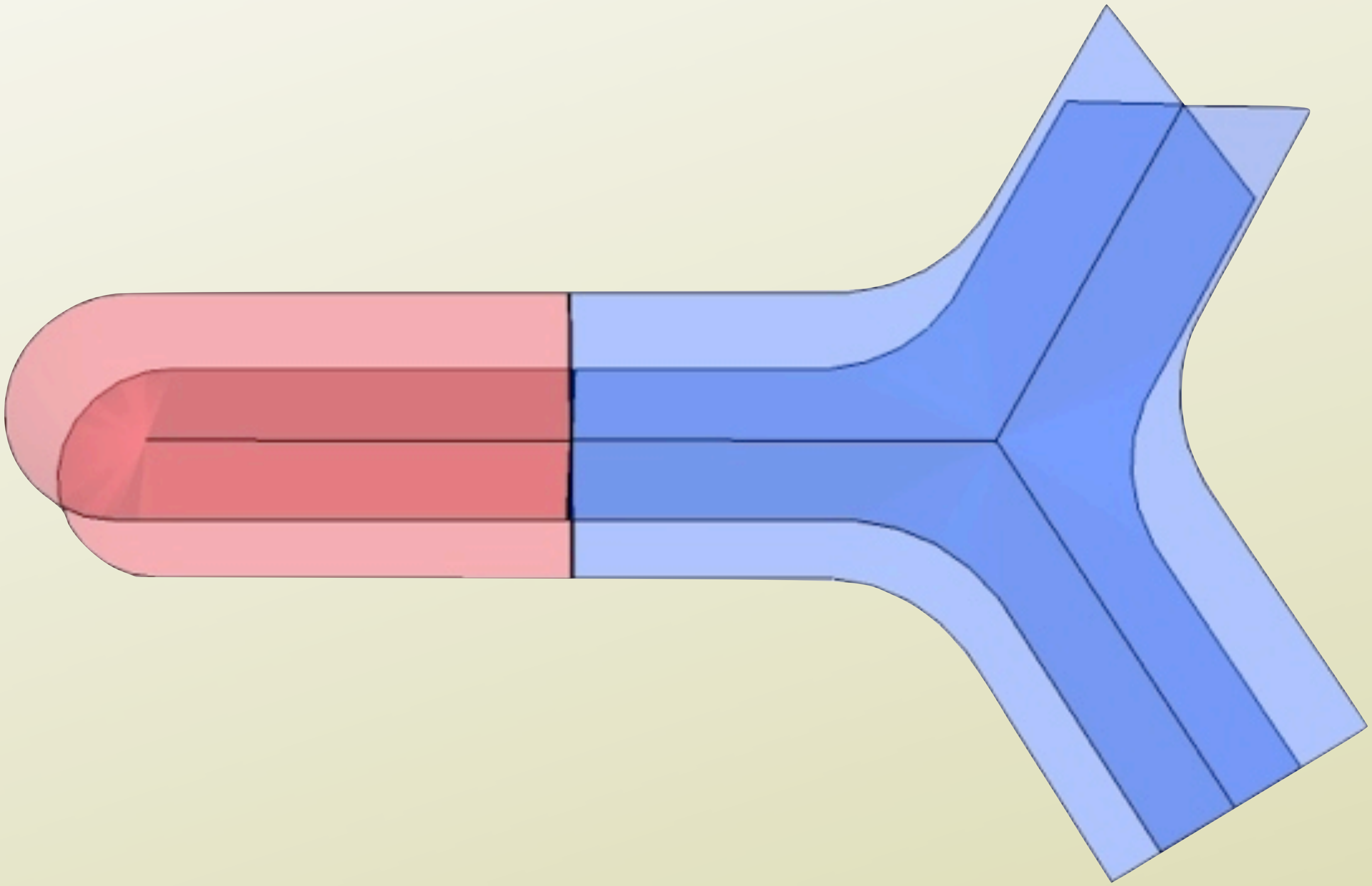

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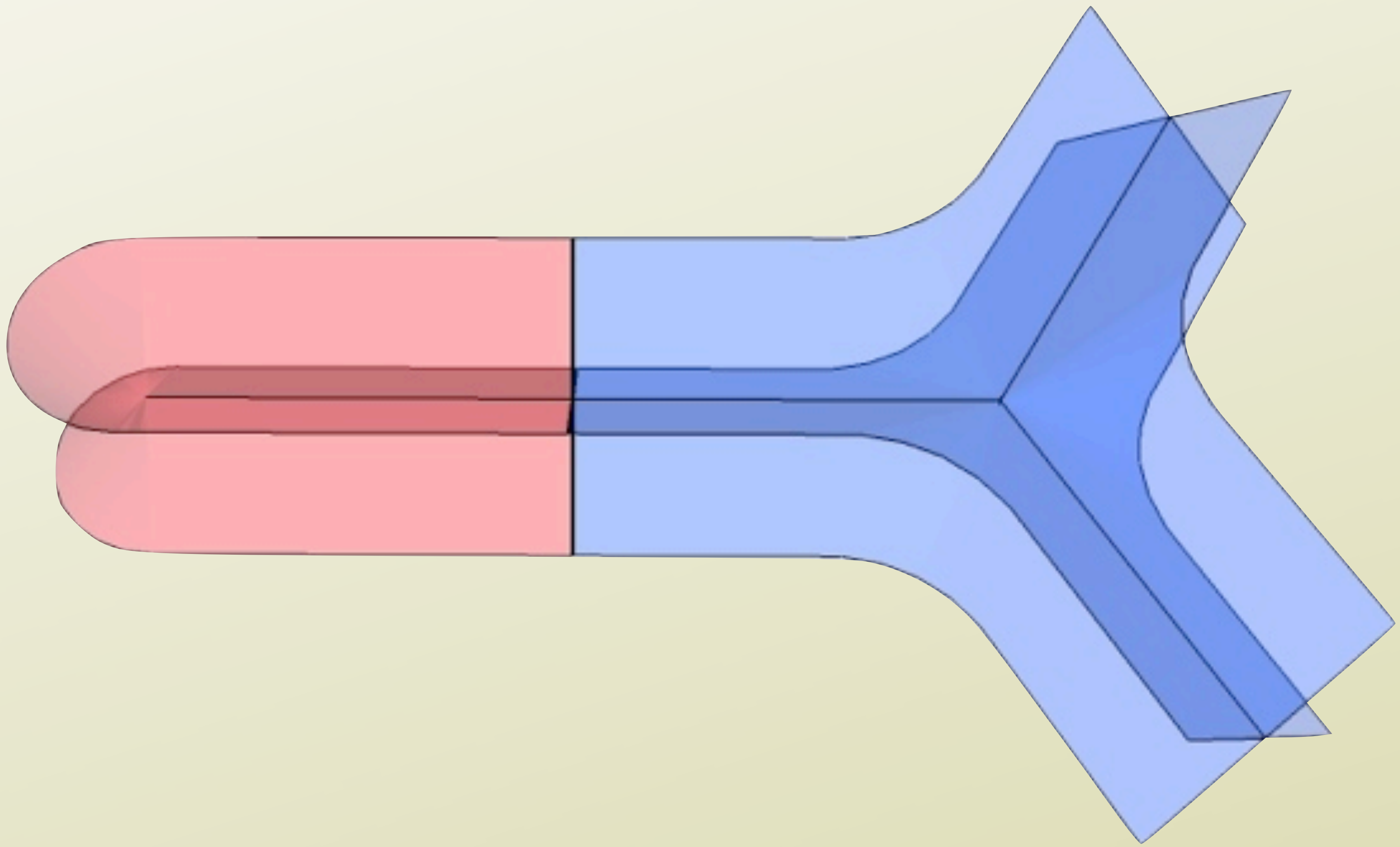

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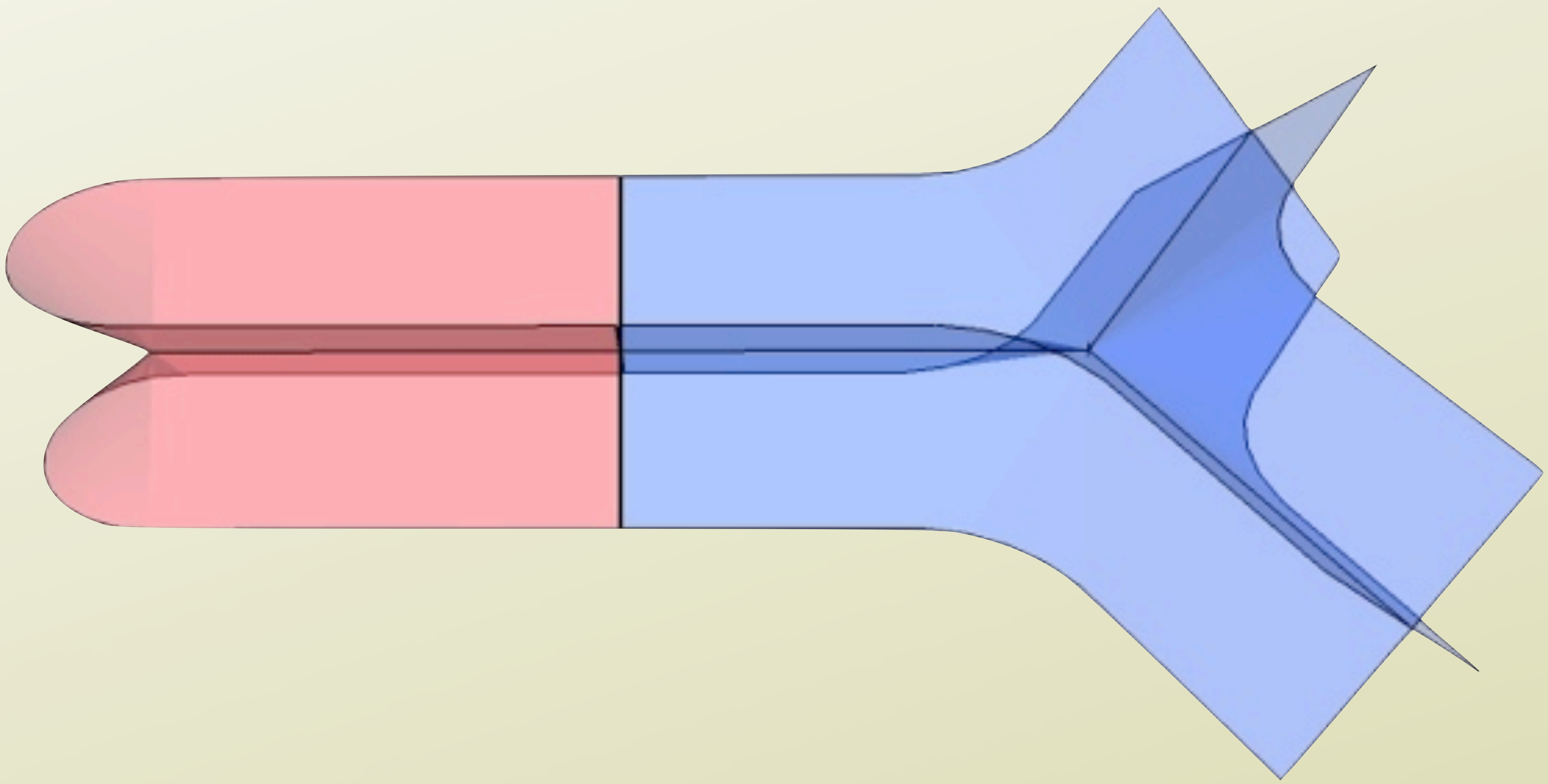

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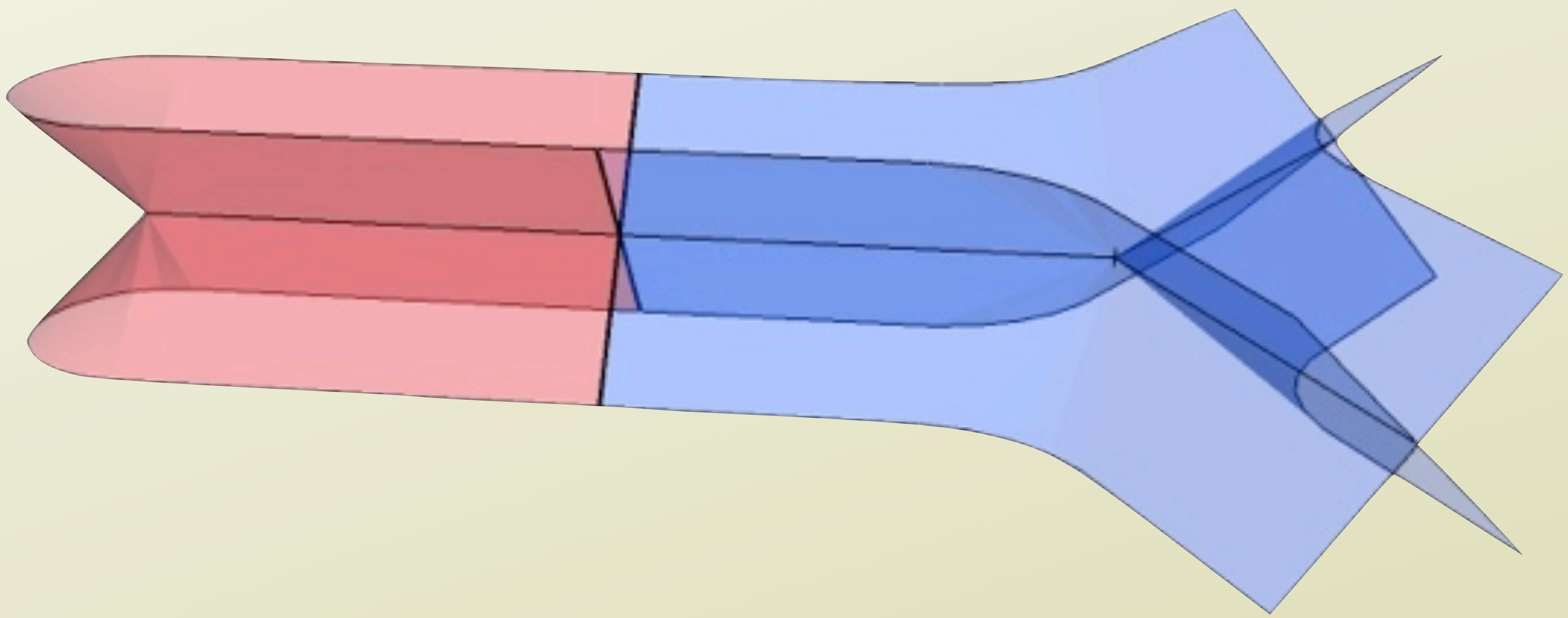
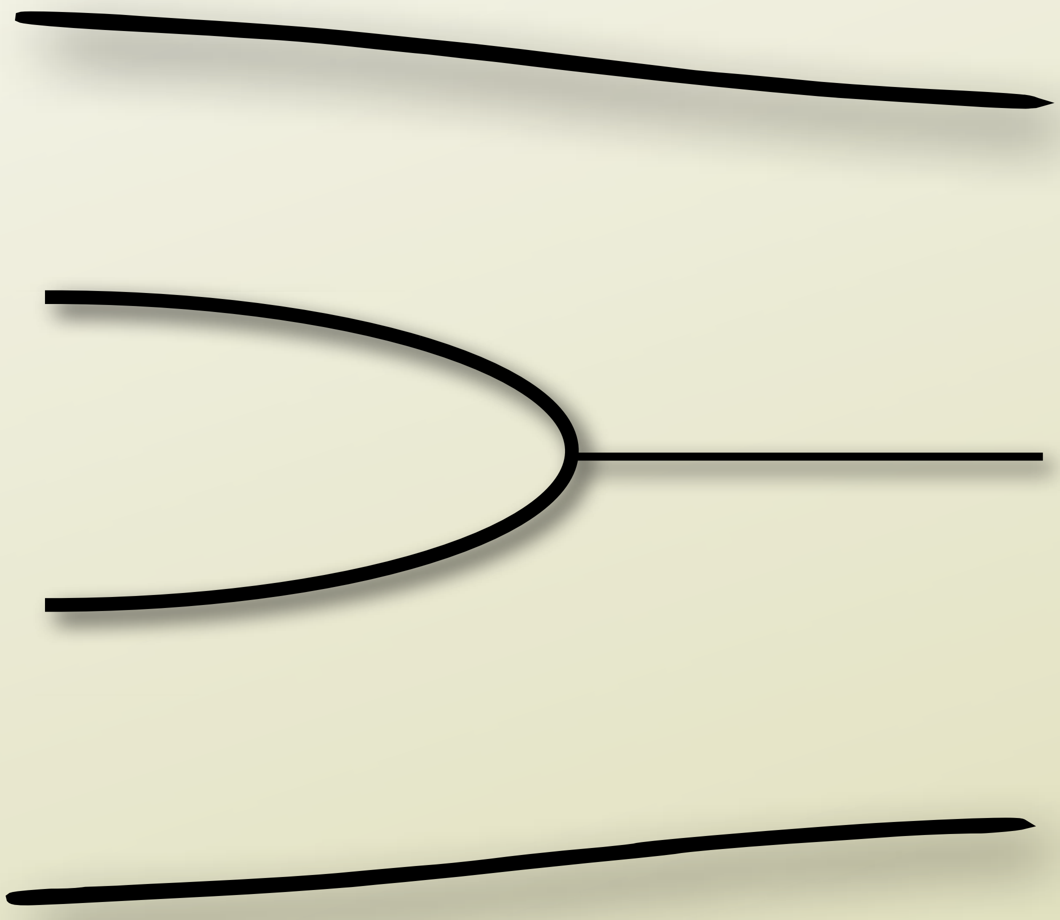


THE DISLOCATION



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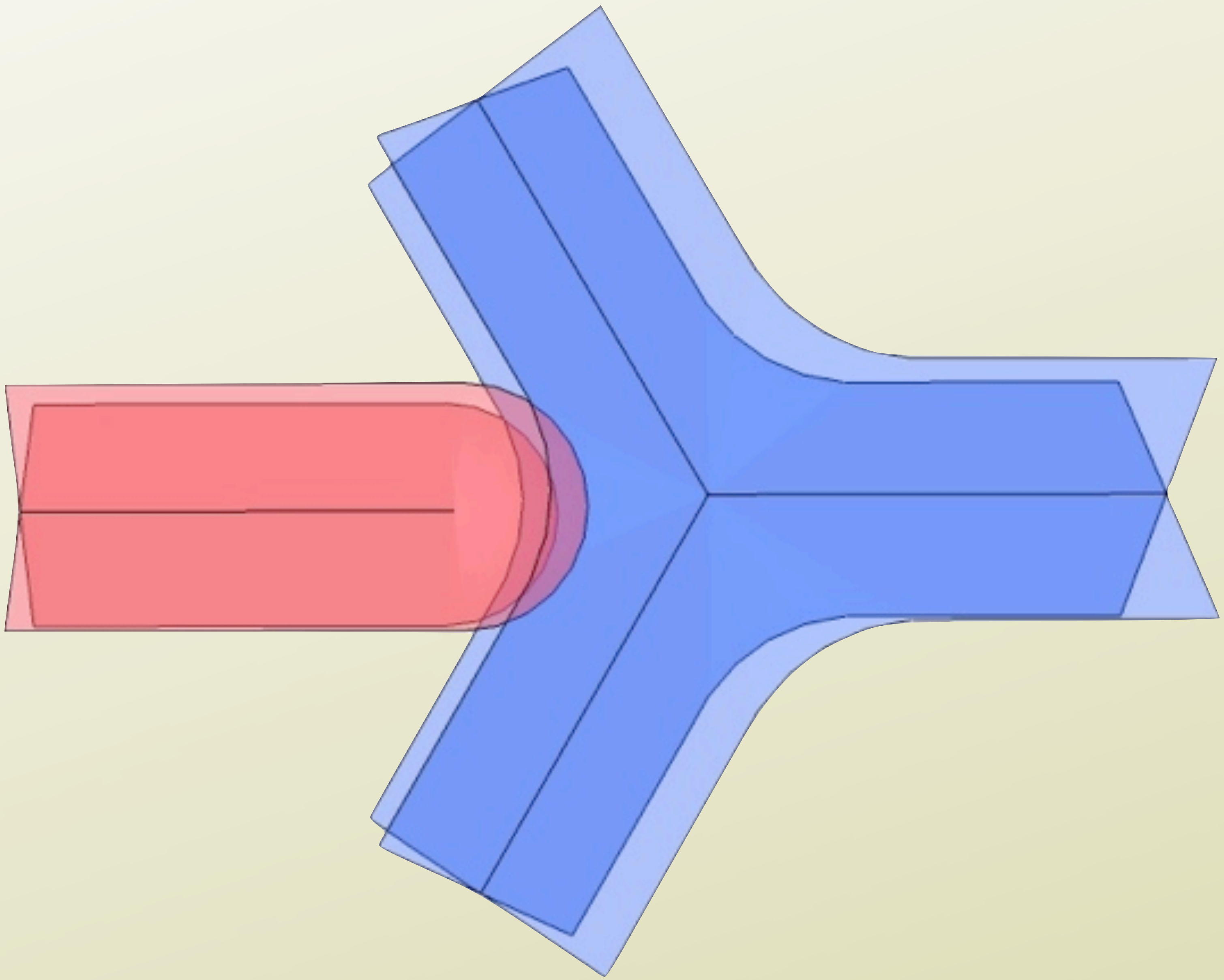

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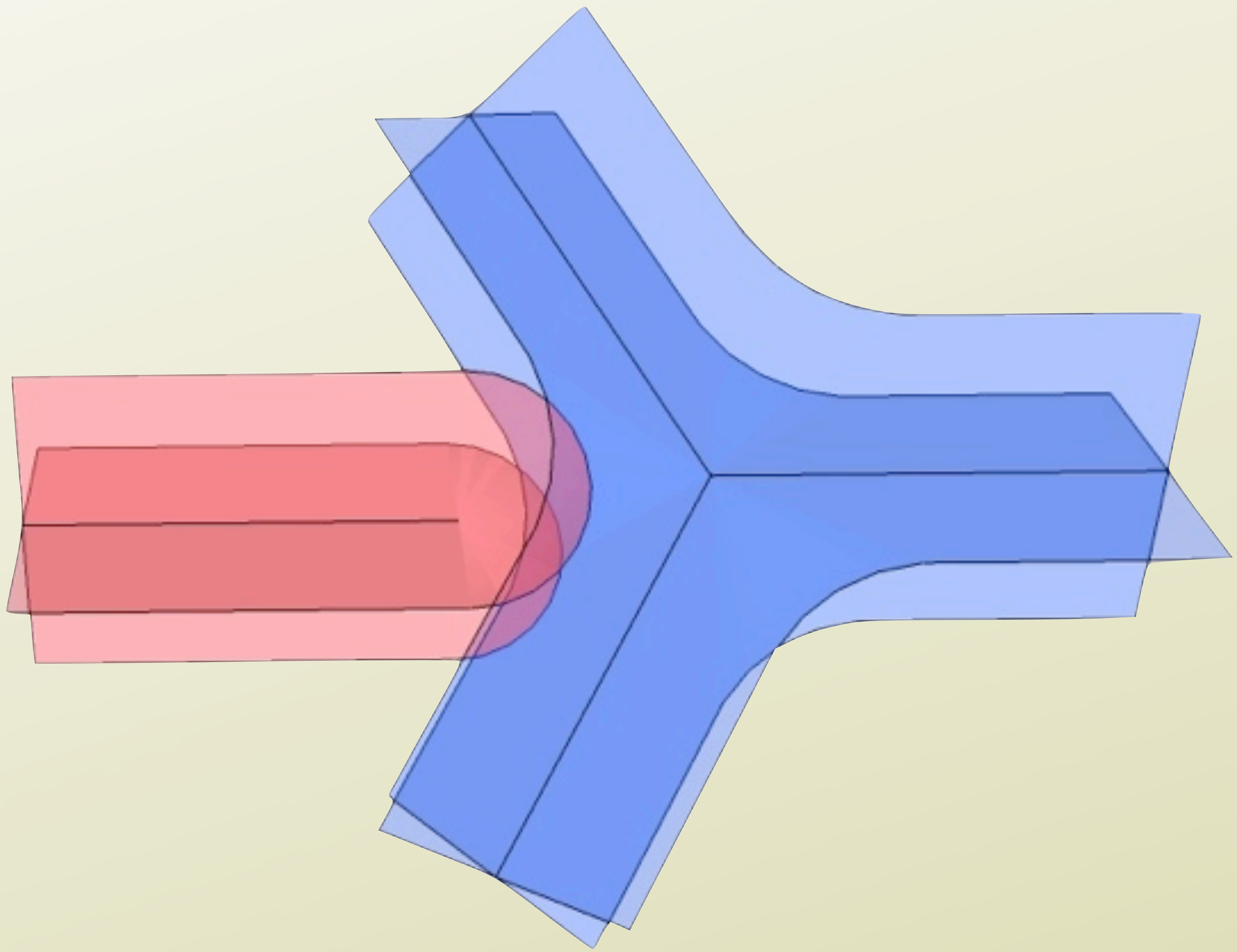

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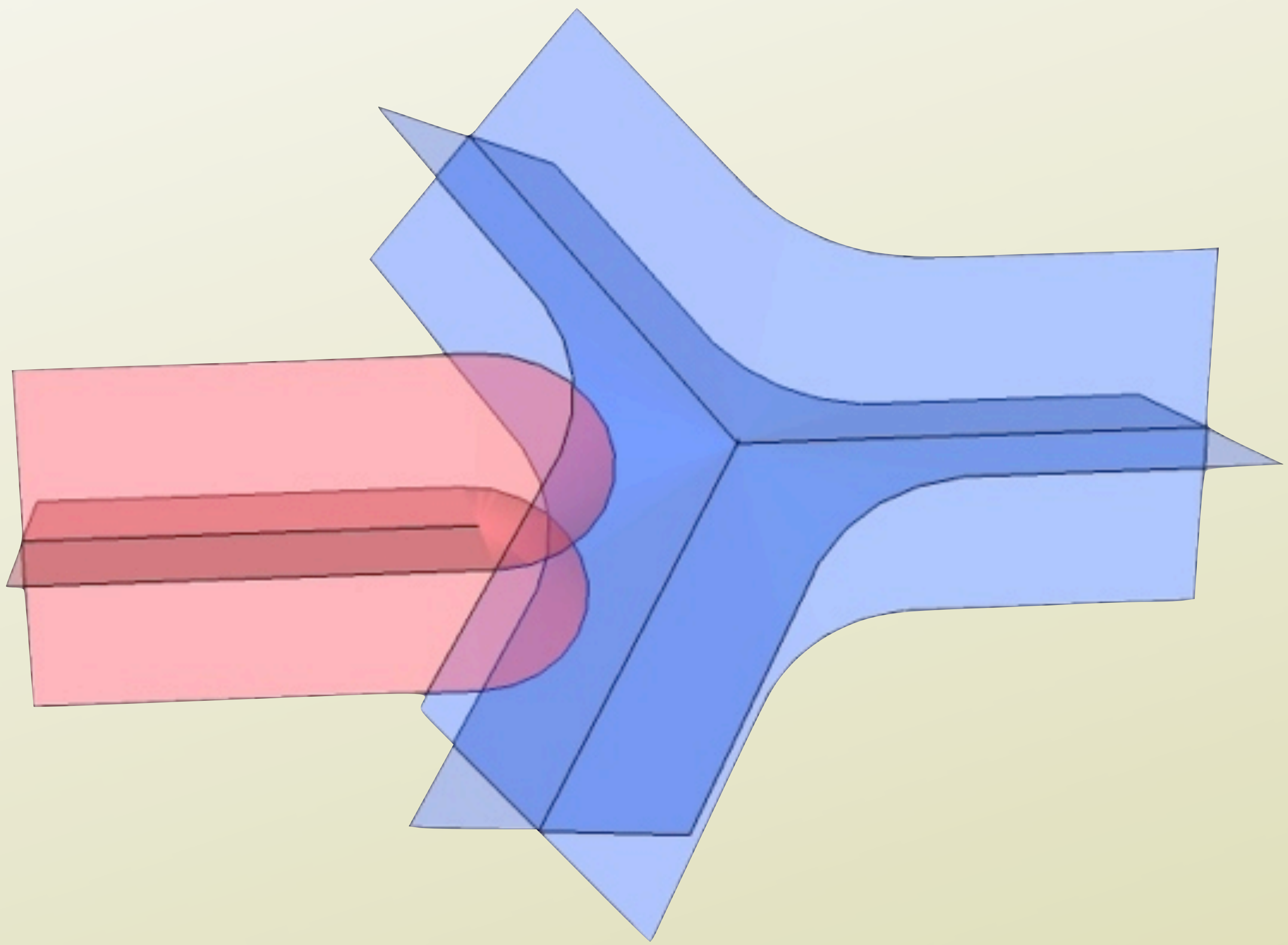

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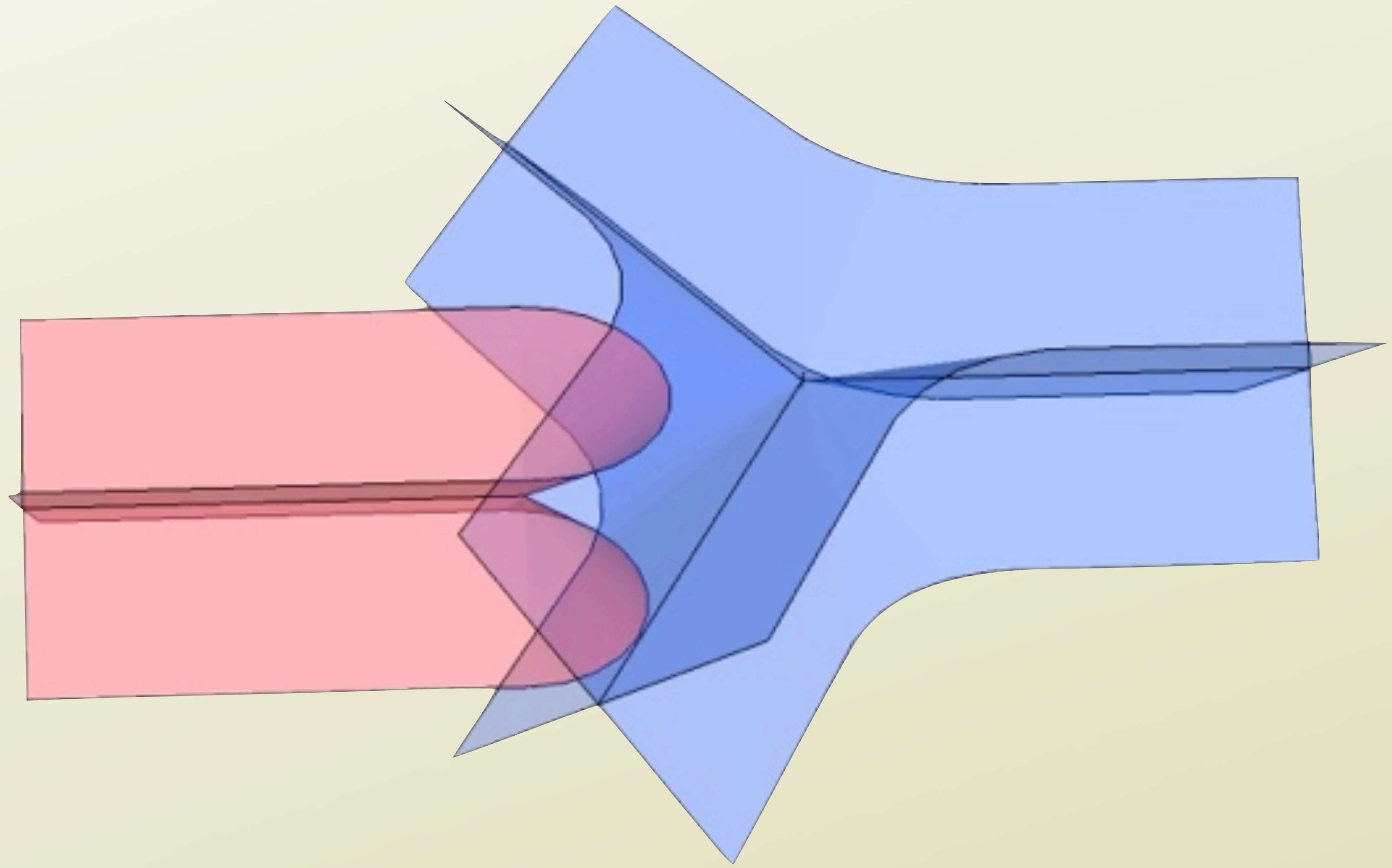

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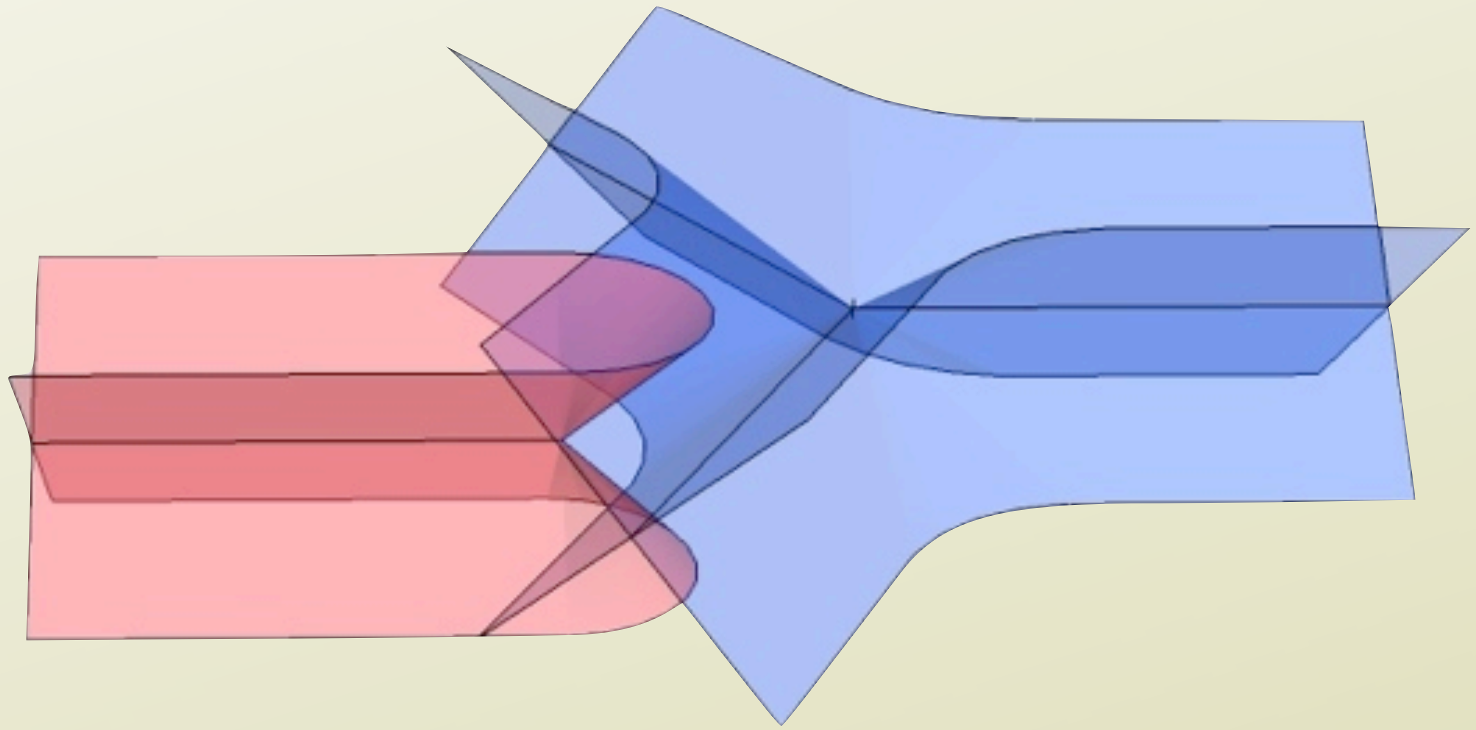

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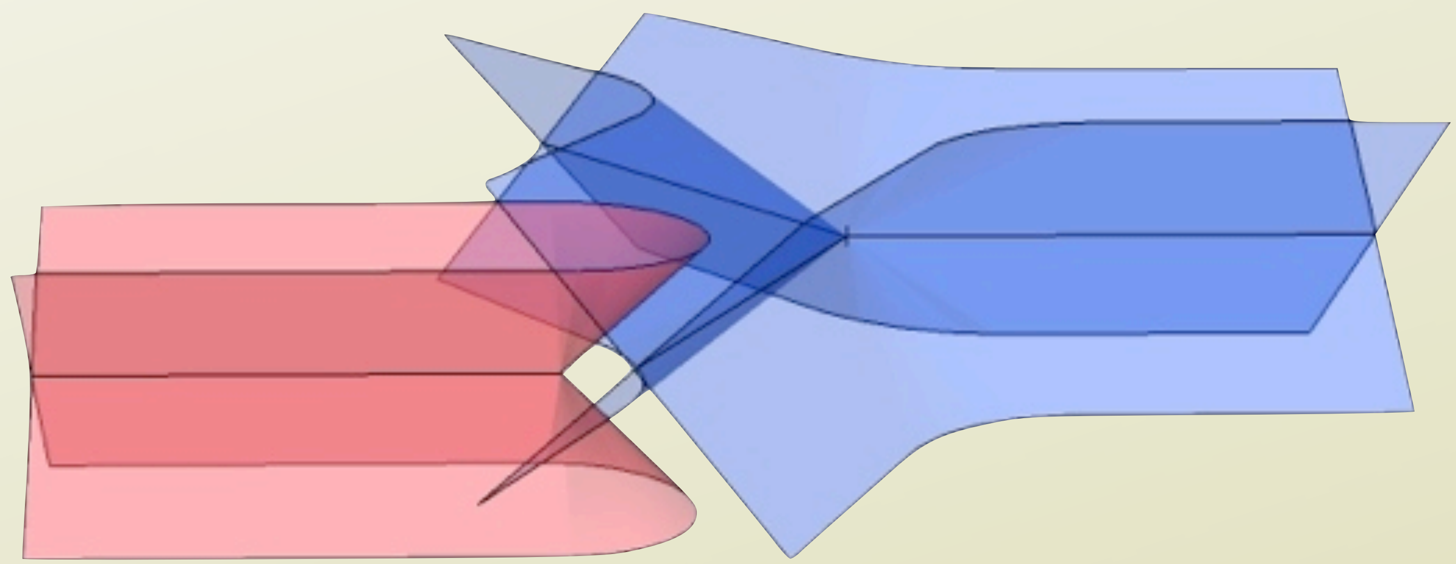

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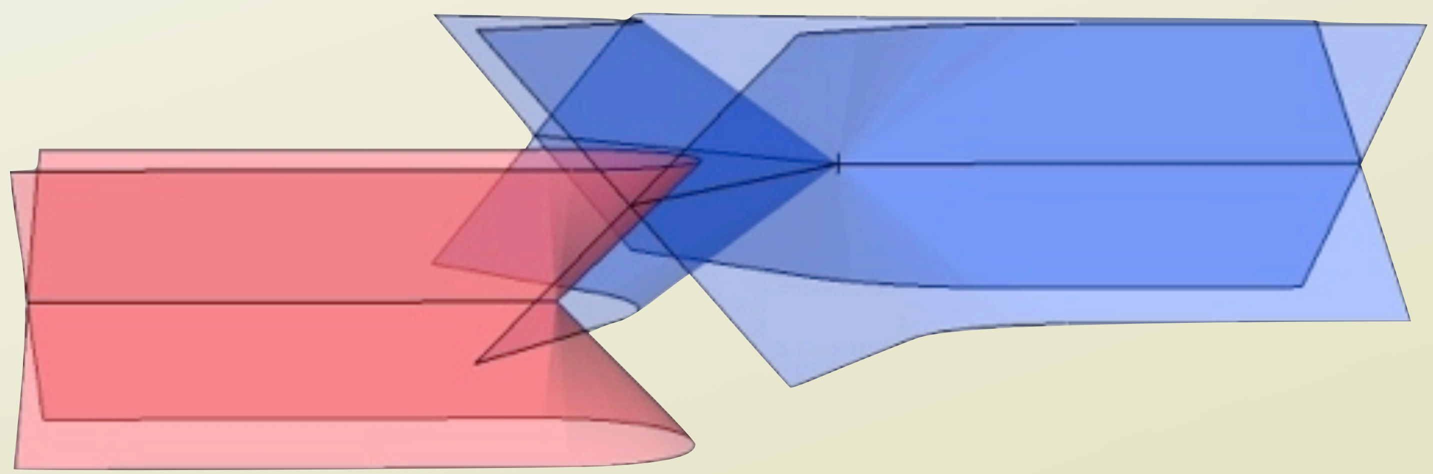

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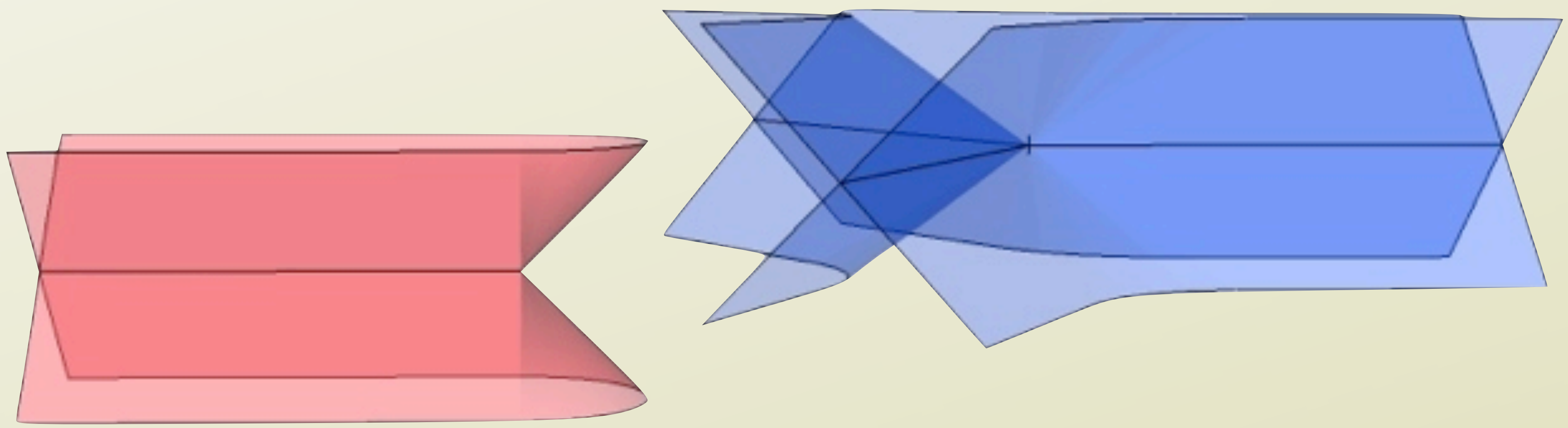
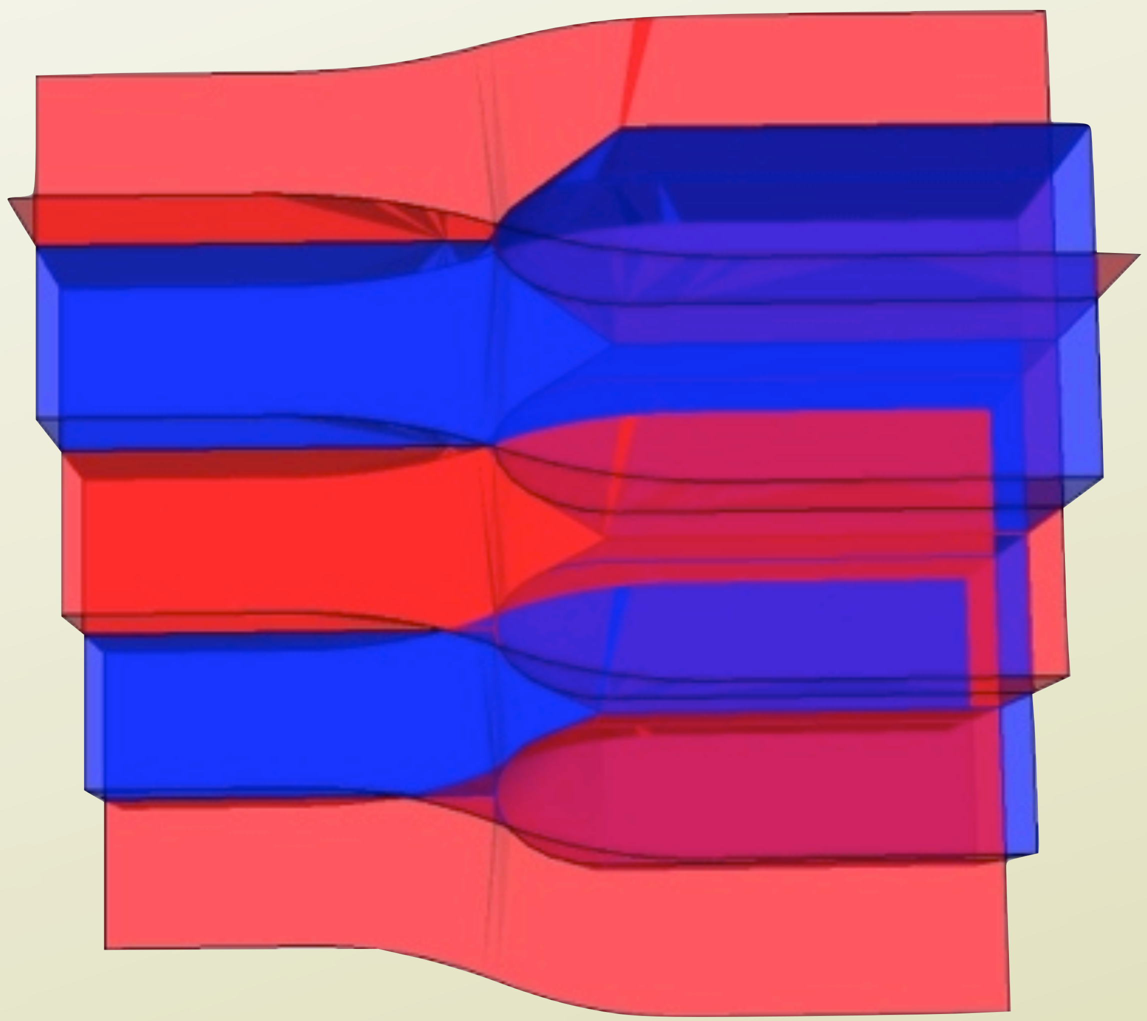


DISCLINATION DIPOLE: +1 DISLOCATION



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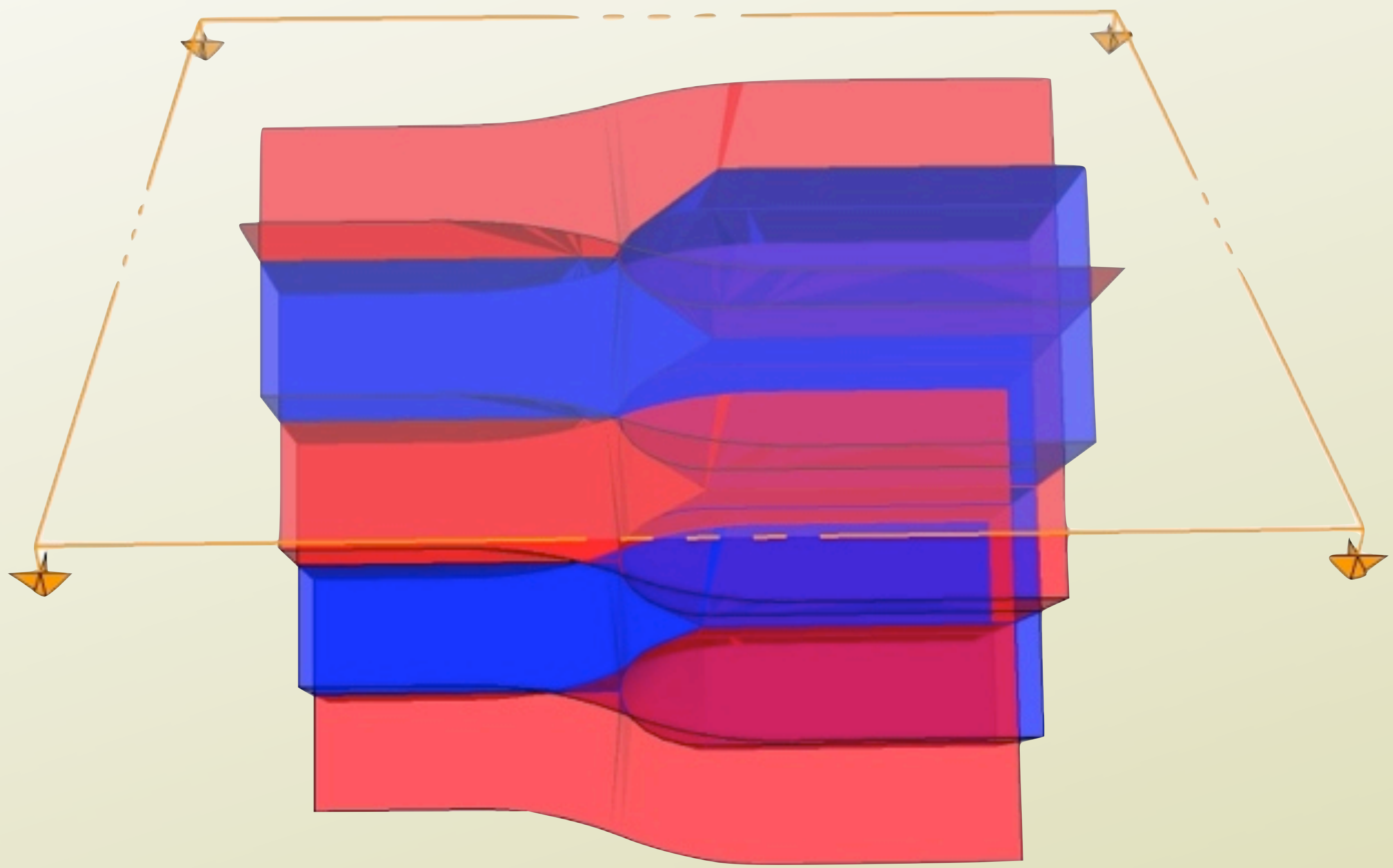

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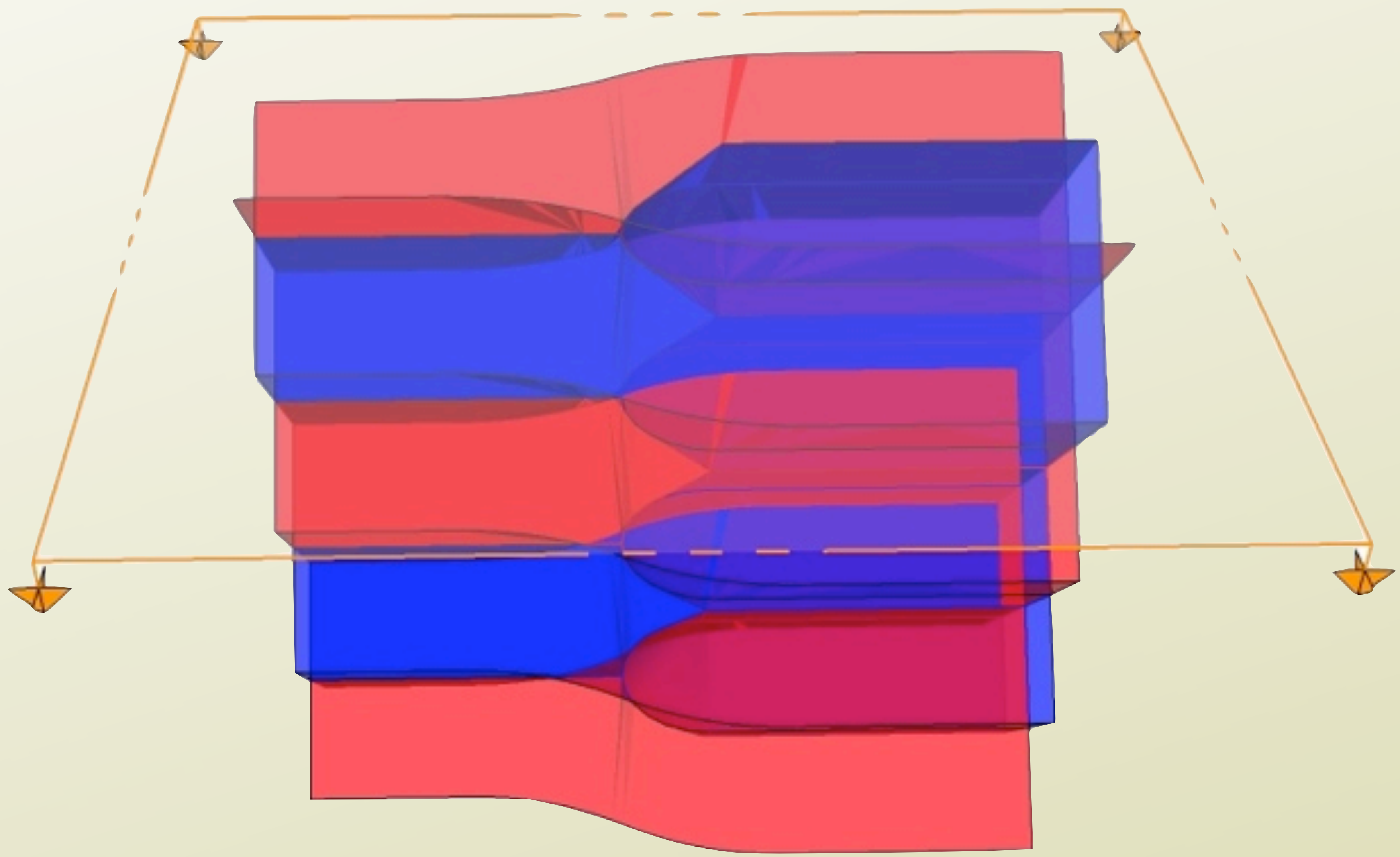

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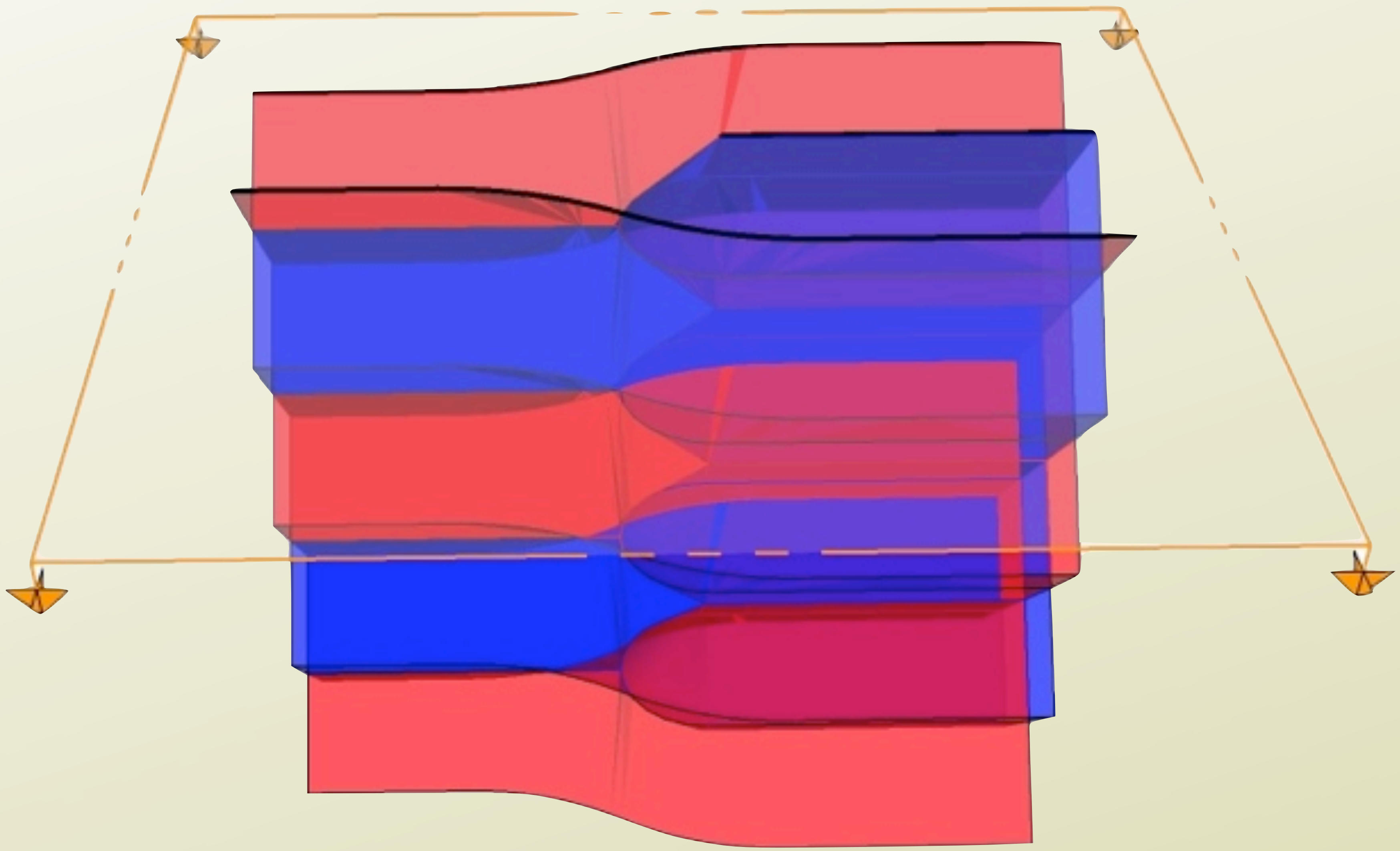

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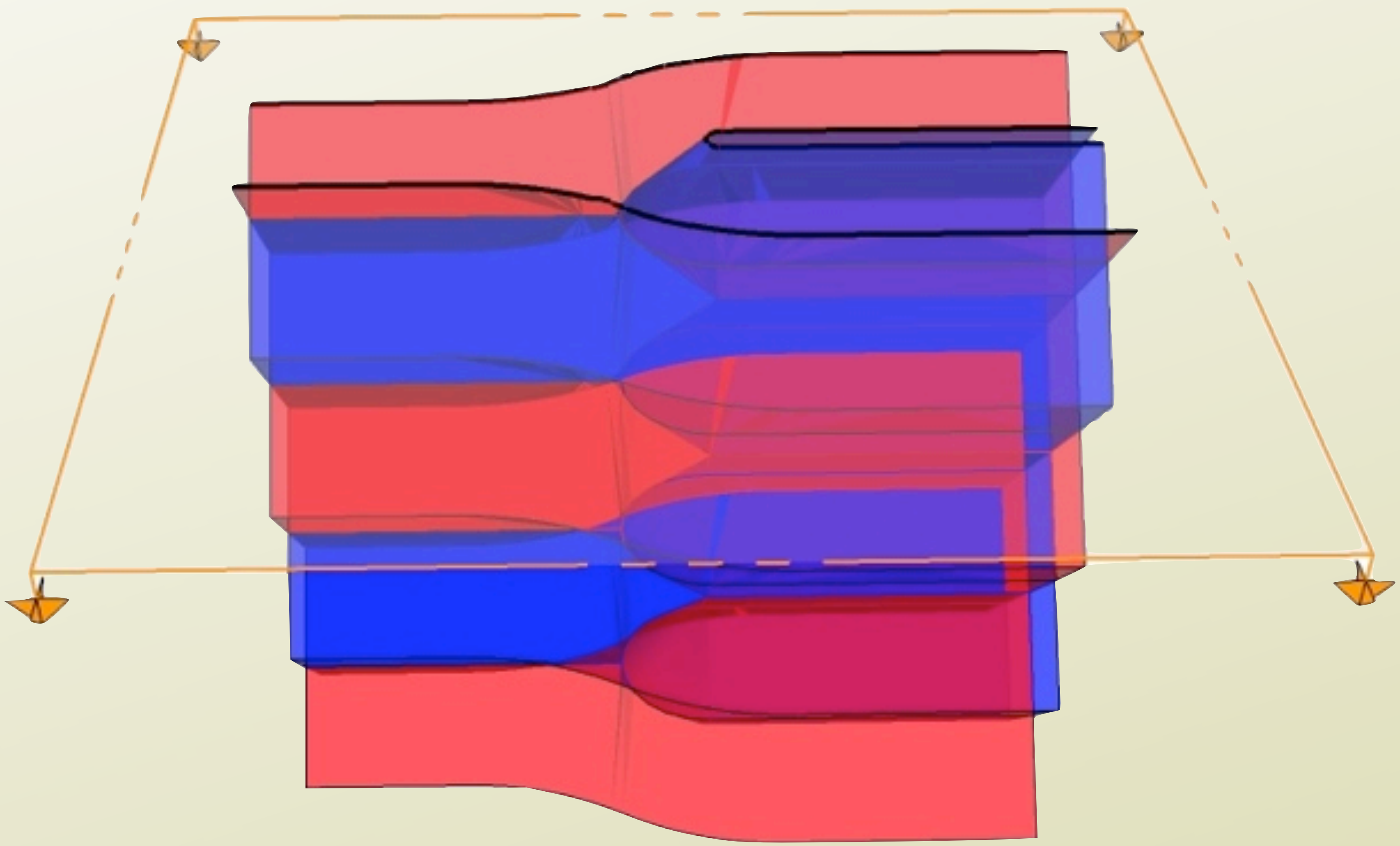

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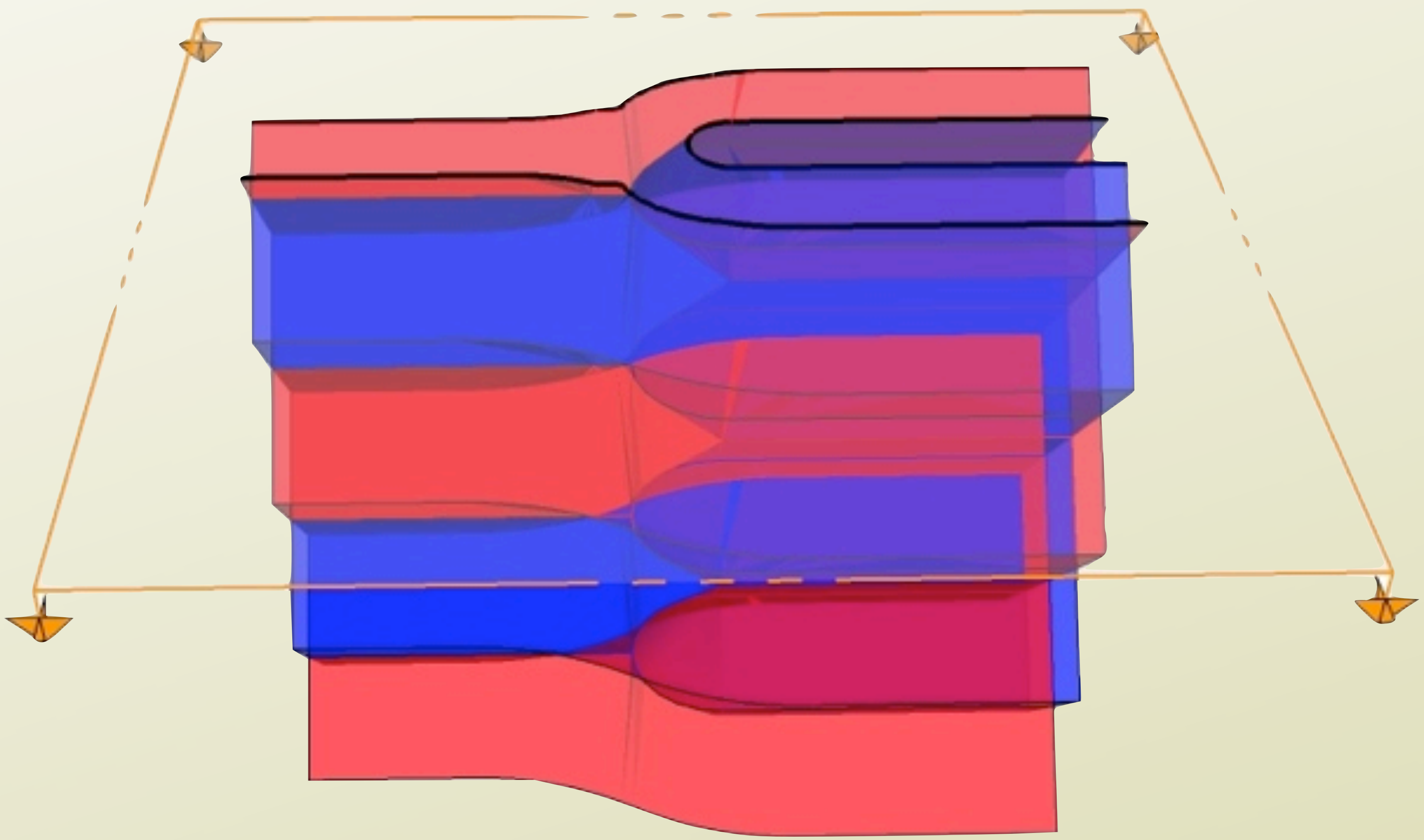

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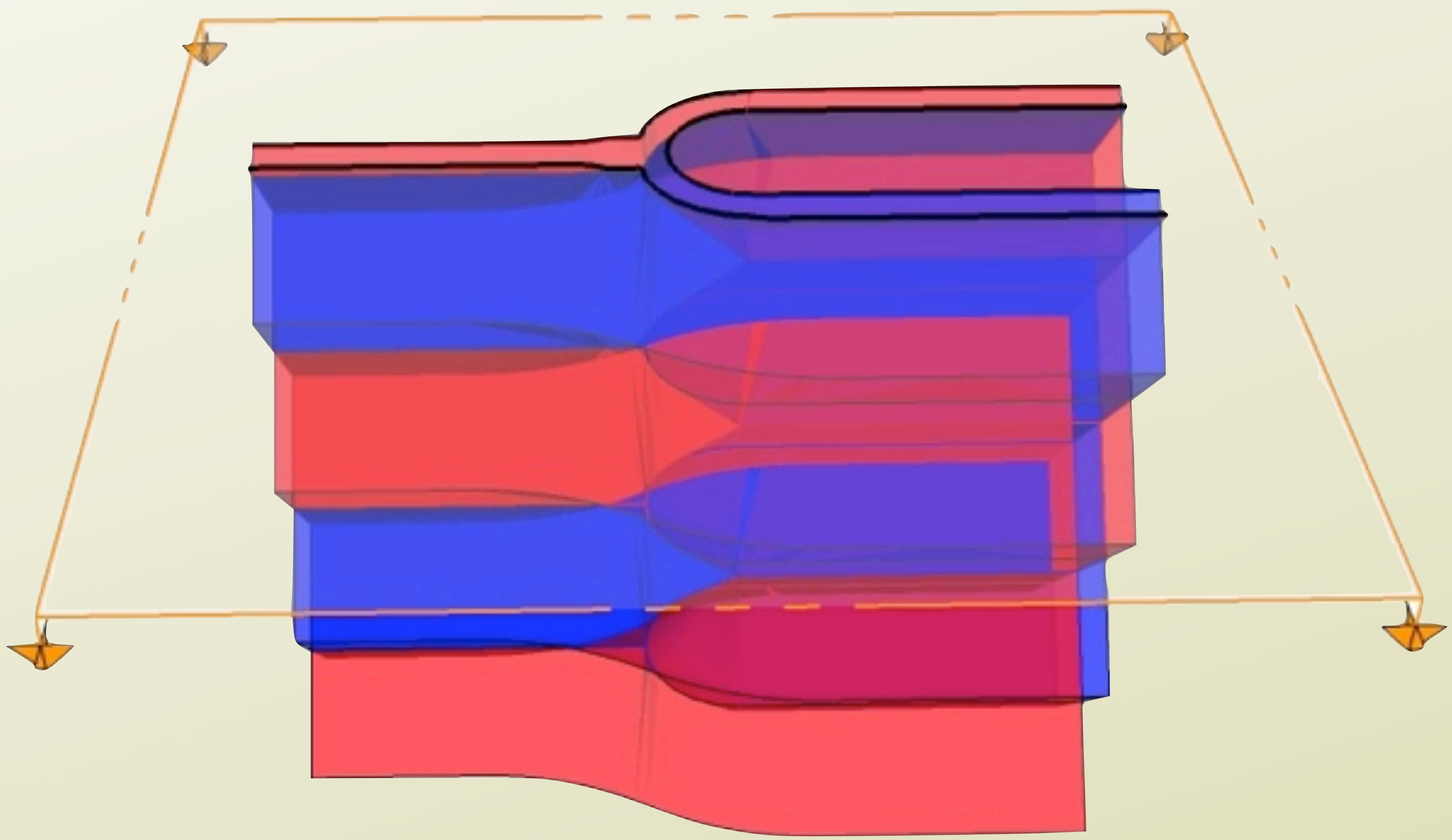

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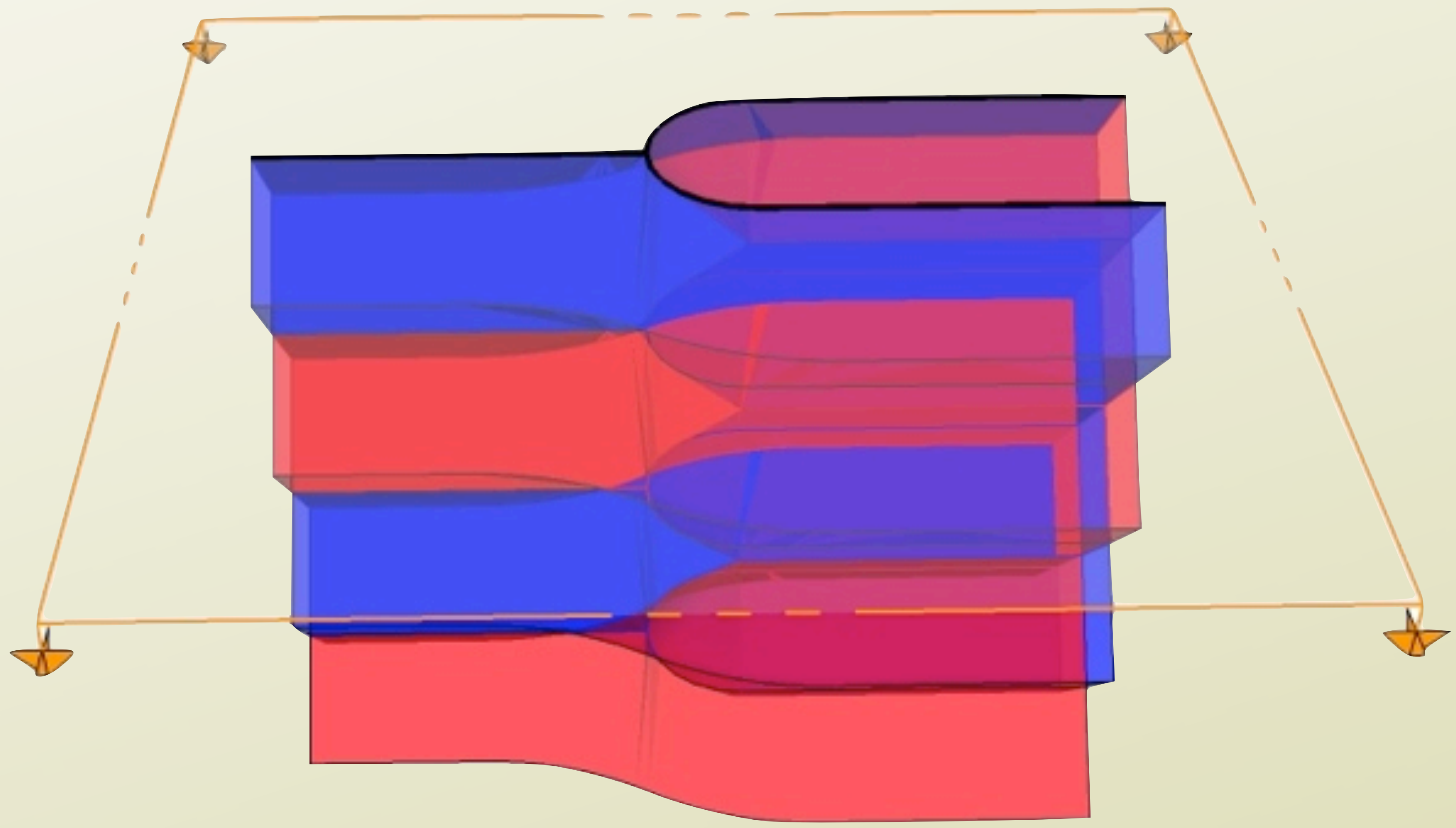

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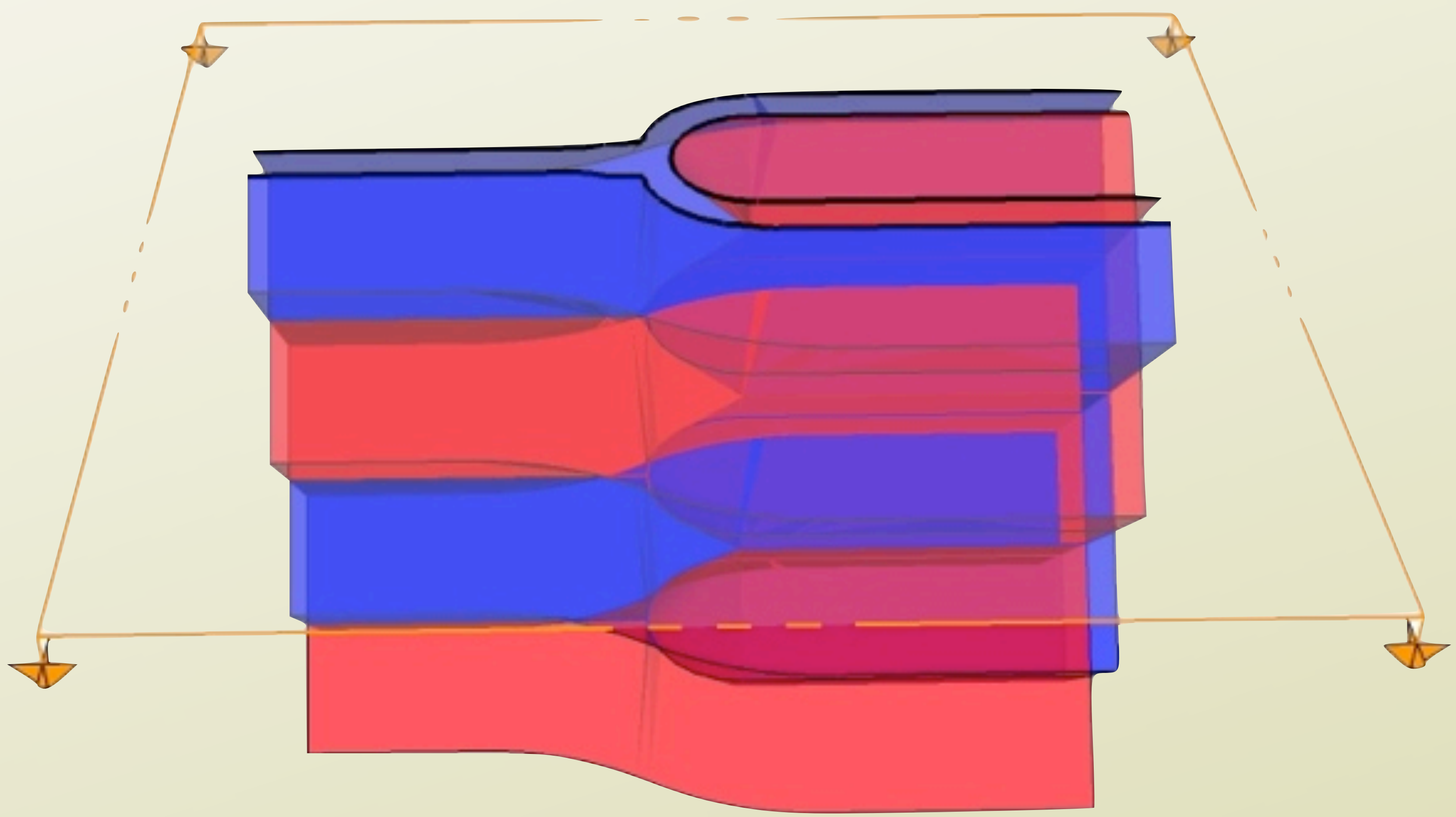

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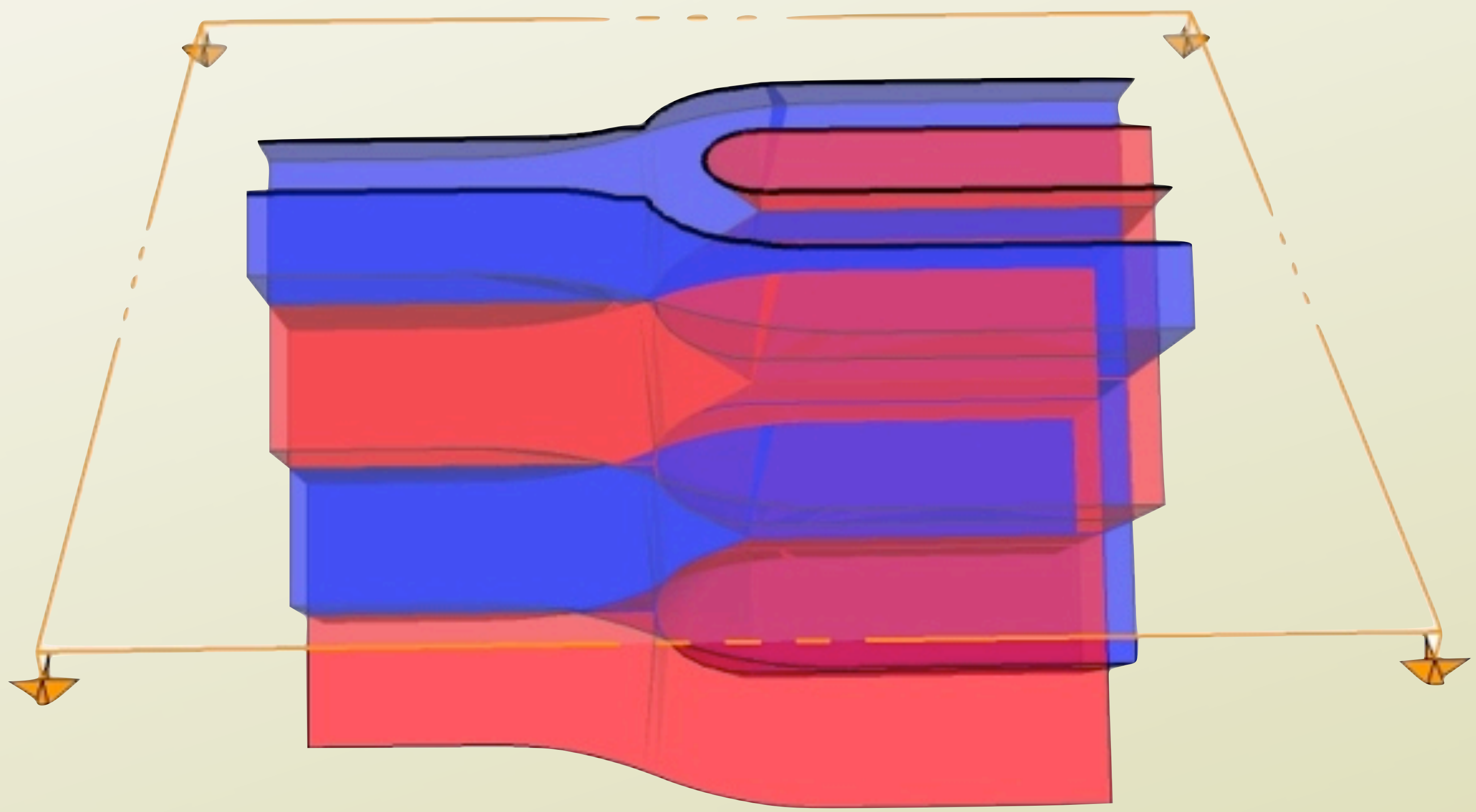

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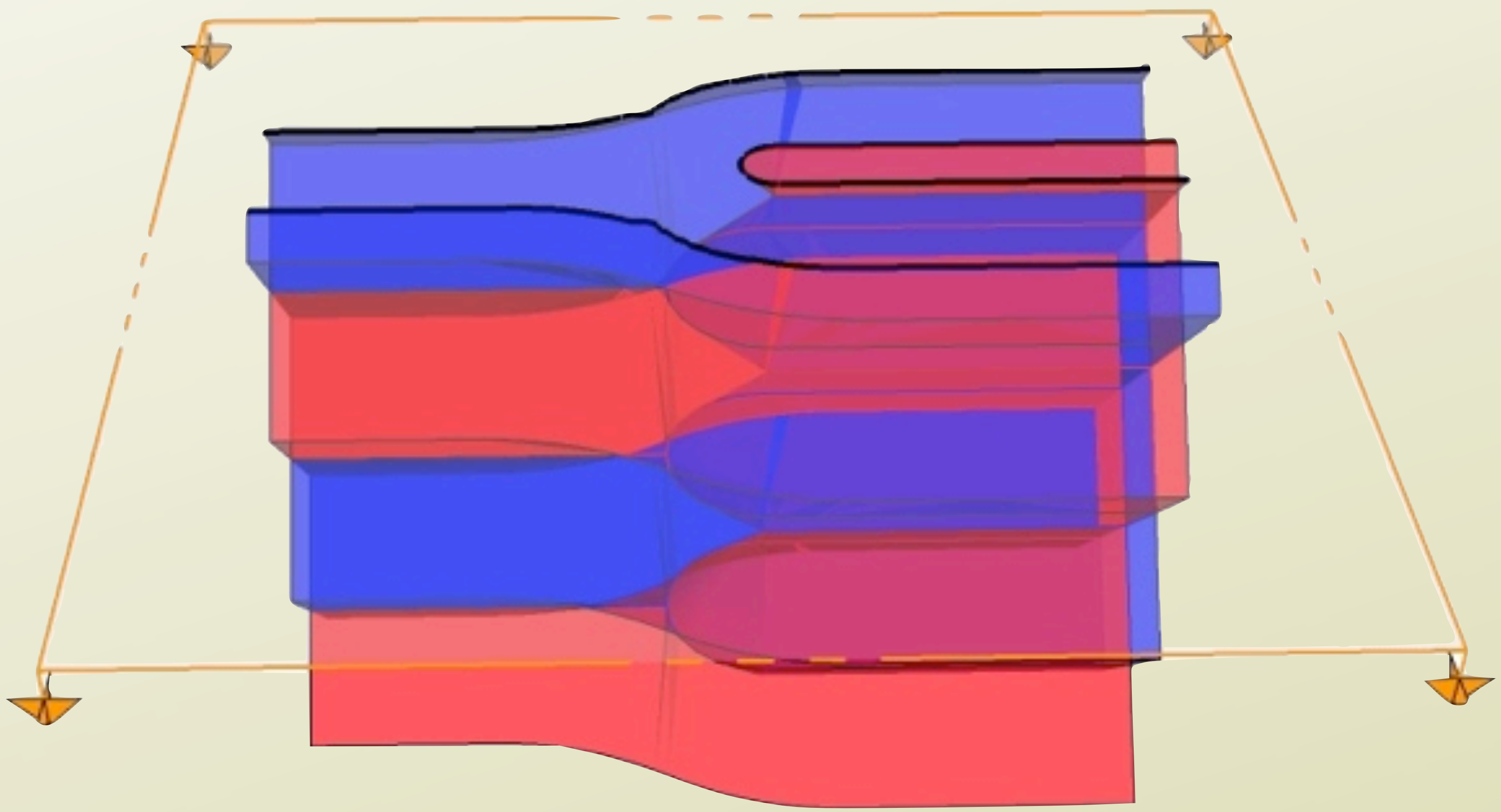

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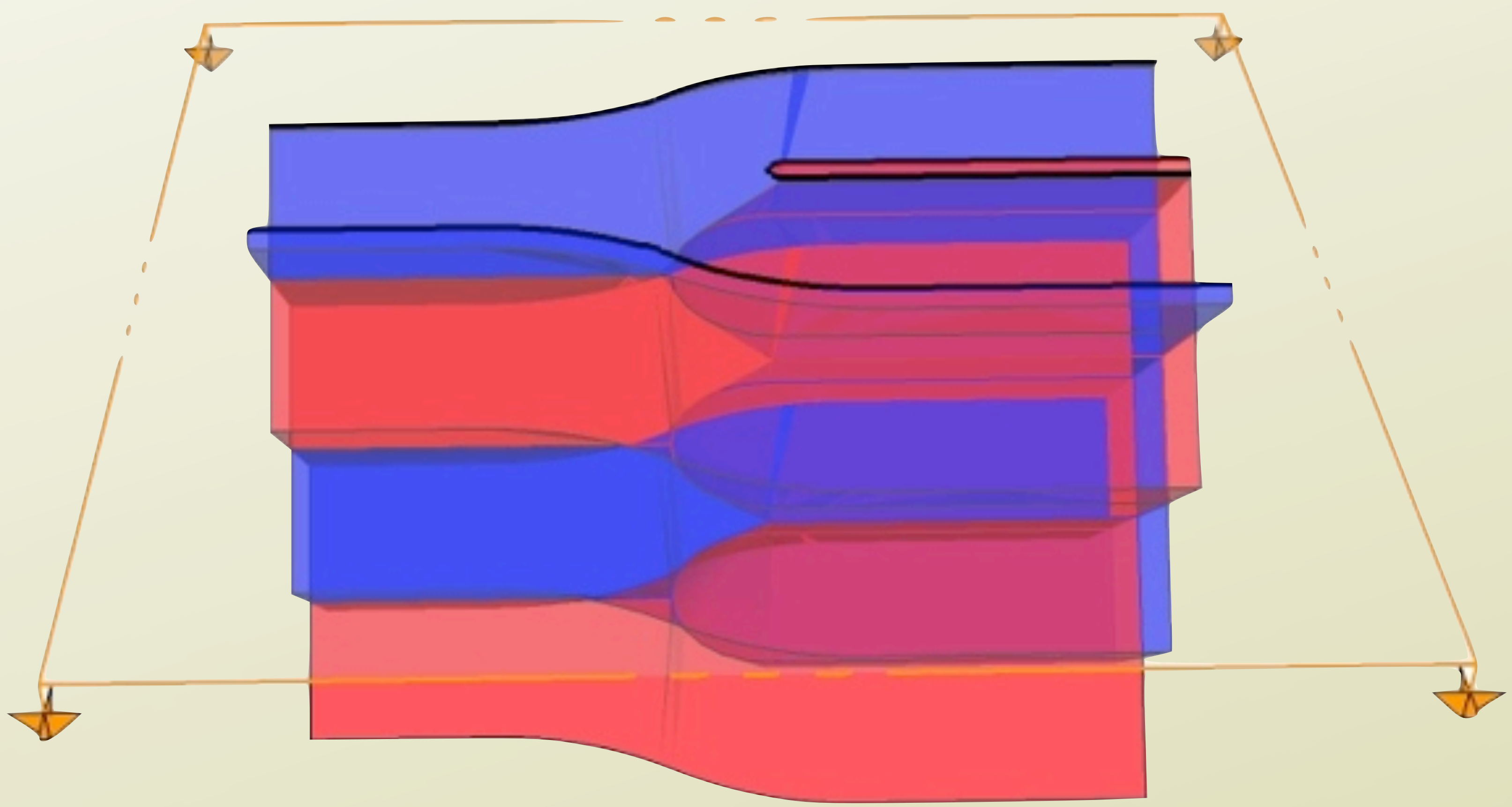

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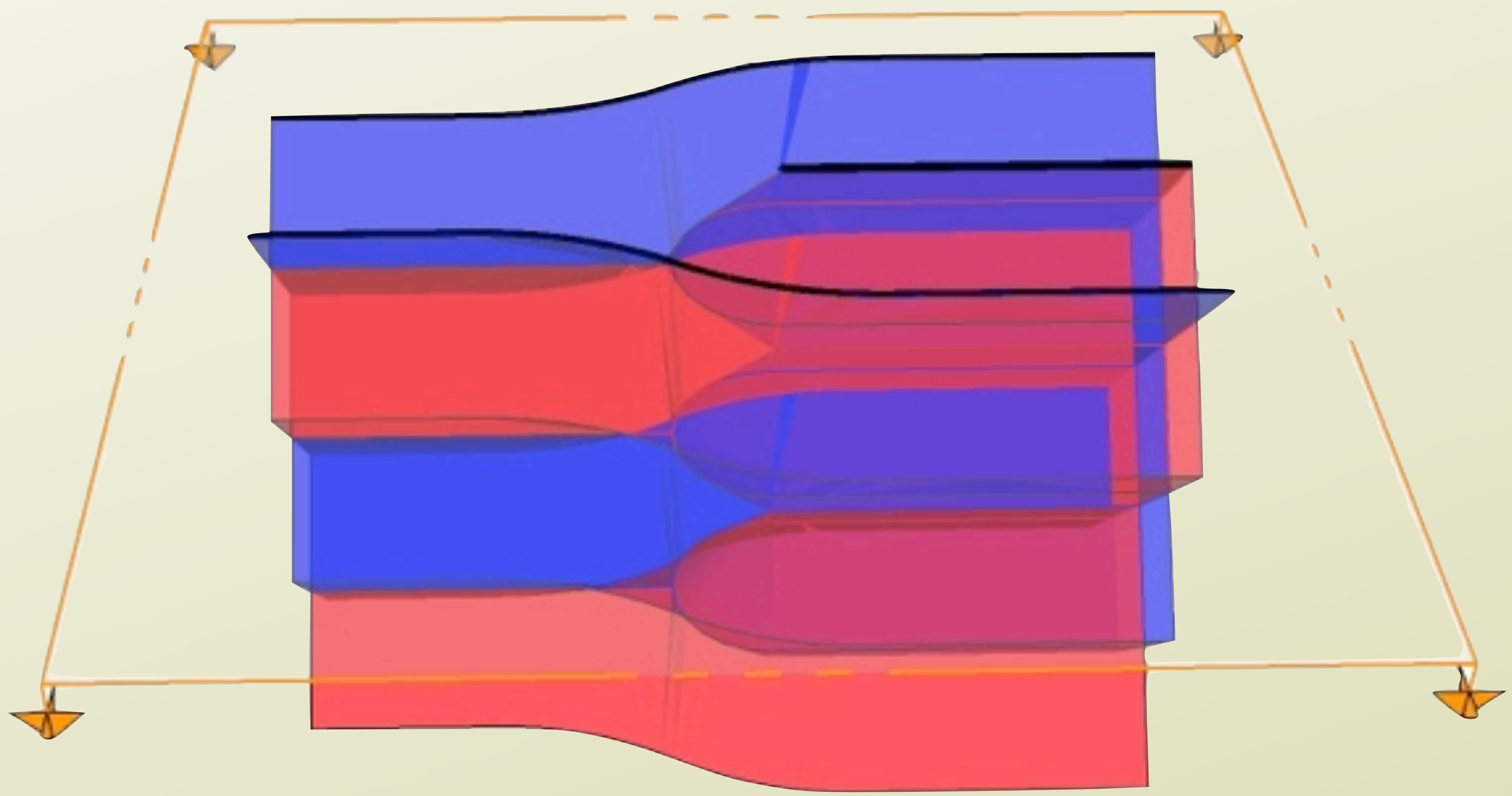

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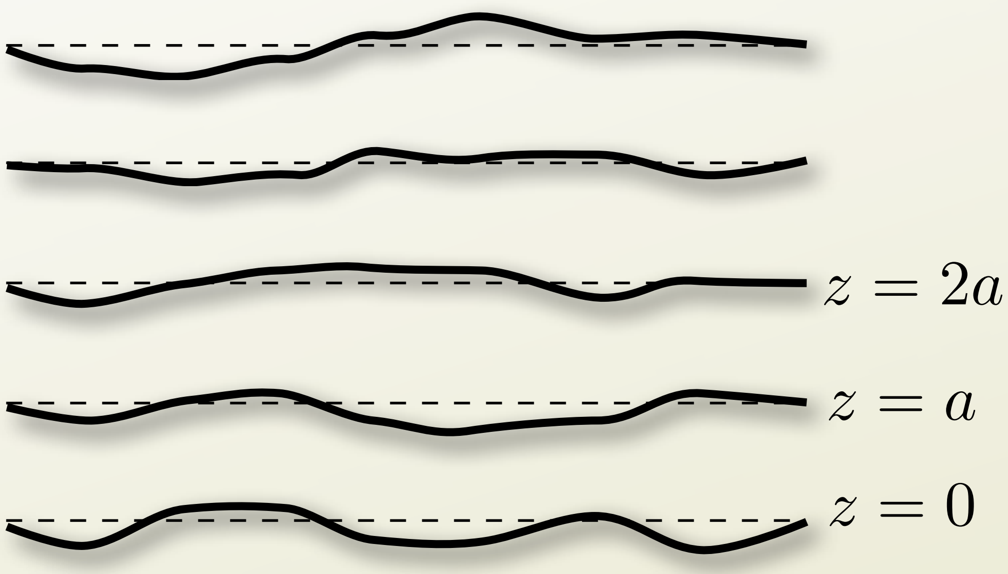
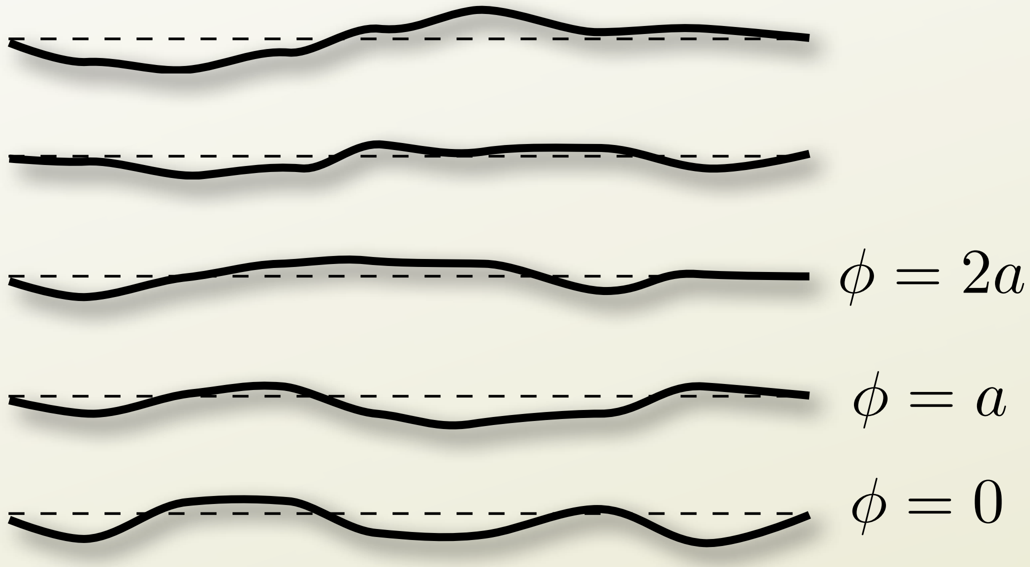
Free Energy and Rotational Invariance



density wave: $\rho \propto \cos\left(\frac{2\pi(z-u(r))}{a}\right)$

Linear elasticity: $$F = \frac{B}{2}\int \mathrm{d}^2 r \left[\left(\partial_z u\right)^2 + \lambda^2\left(\partial_\perp^2 u\right)^2\right]$$

Free Energy and Rotational Invariance



density wave: $\rho \propto \cos\left(\frac{2\pi\phi}{a}\right)$

Linear elasticity: $$F = \frac{B}{2}\int \mathrm{d}^2 r \left[\left(\partial_z u\right)^2 + \lambda^2\left(\partial_\perp^2 u\right)^2\right]$$

Nonlinear elasticity: $$F = \frac{B}{2}\int \mathrm{d}^2 r \left[\tfrac{1}{4}\left[(\nabla\phi)^2 - 1\right]^2 + \lambda^2\left(\nabla\cdot\mathbf{n}\right)^2\right]$$

$\phi = z - u(r)$

$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$

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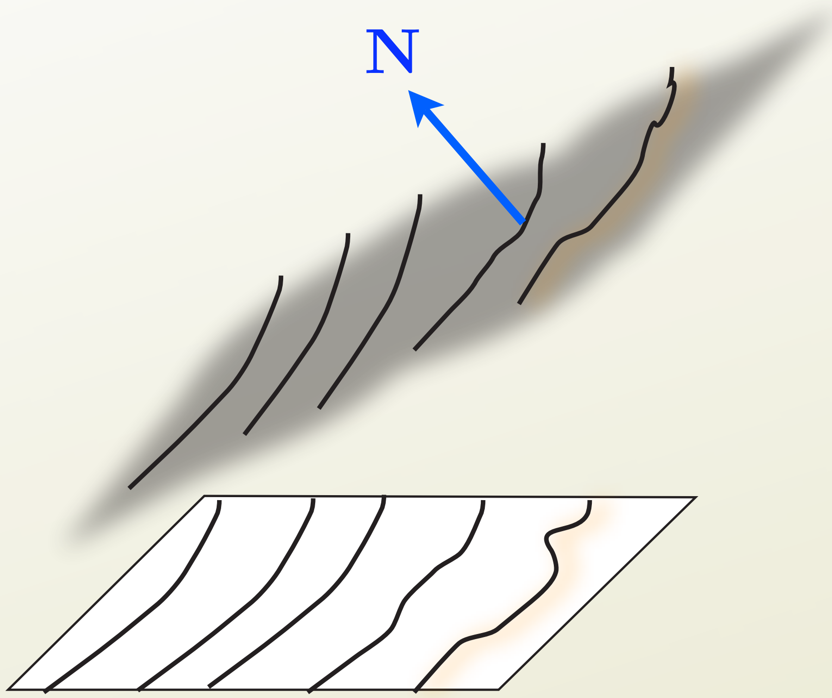
SURFACE ENERGETICS



Viewing ϕ as a graph: $$\mathbf{N} = \frac{(-\partial_x\phi, -\partial_y\phi, 1)}{\sqrt{1 + (\nabla\phi)^2}}$$

Equal spacing of curves: $$\mathbf{e}_z \cdot \mathbf{N} = \frac{1}{\sqrt{2}}$$

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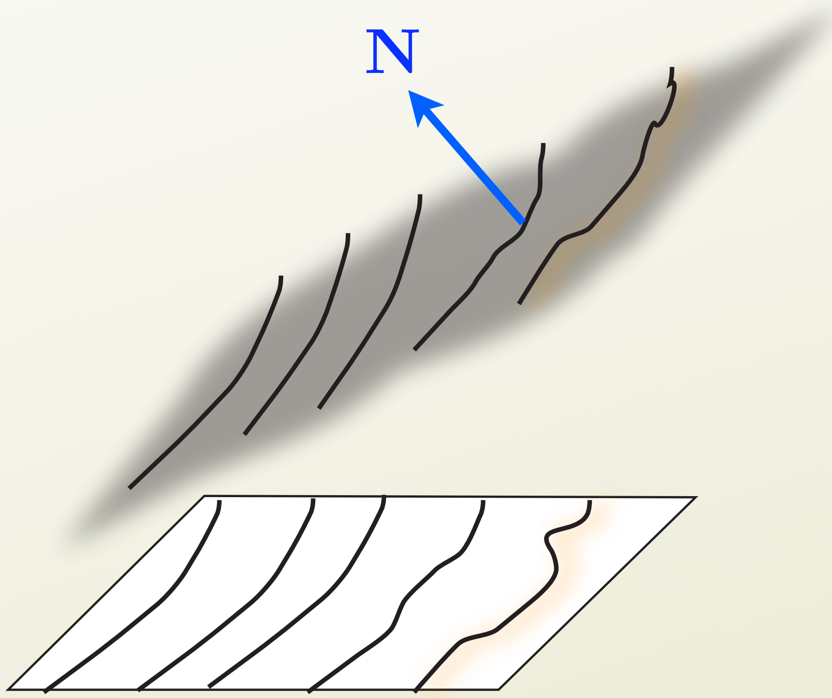
Surface Energetics



Viewing ϕ as a graph: $$\mathbf{N} = \frac{(-\partial_x\phi, -\partial_y\phi, 1)}{\sqrt{1+(\nabla\phi)^2}}$$

Equal spacing of curves: $$\mathbf{e}_z \cdot \mathbf{N} = \frac{1}{\sqrt{2}}$$

Candidate: $$F = \frac{B}{2}\int \mathrm{d}A \left[\left(\mathbf{e}_z \cdot \mathbf{N} - \tfrac{1}{\sqrt{2}}\right)^2 + \lambda^2 H^2\right]$$

$$\approx \frac{B}{2}\int \mathrm{d}^2 r \left[\left(\partial_x u\right)^2 + \left(\partial_y^2 u\right)^2\right]$$

"Willmore in a field"

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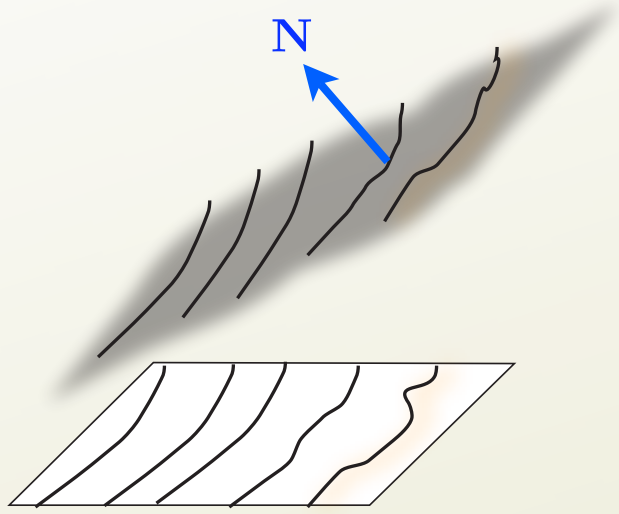
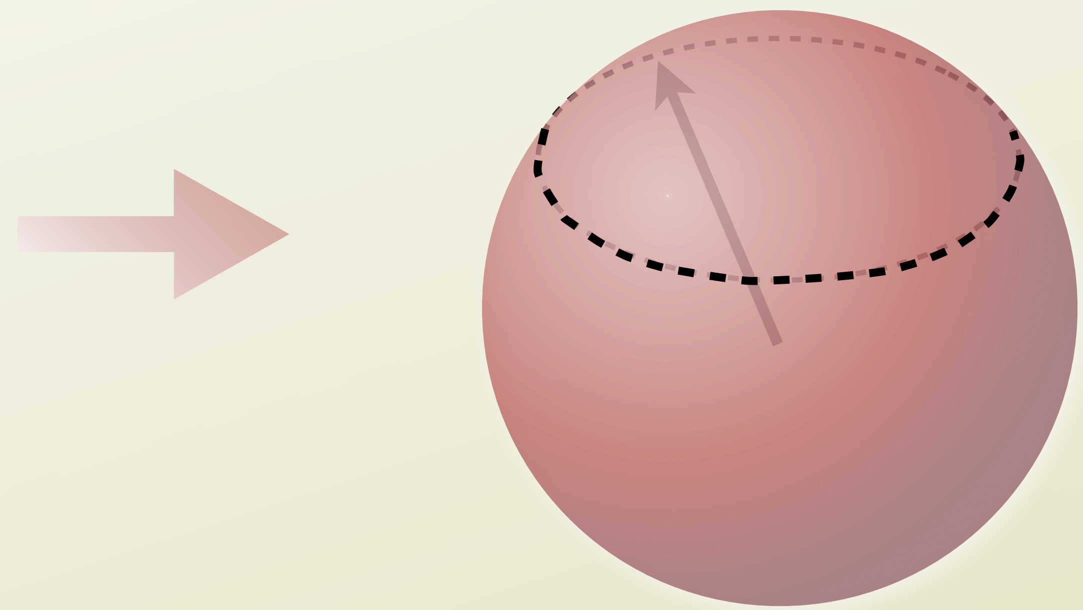
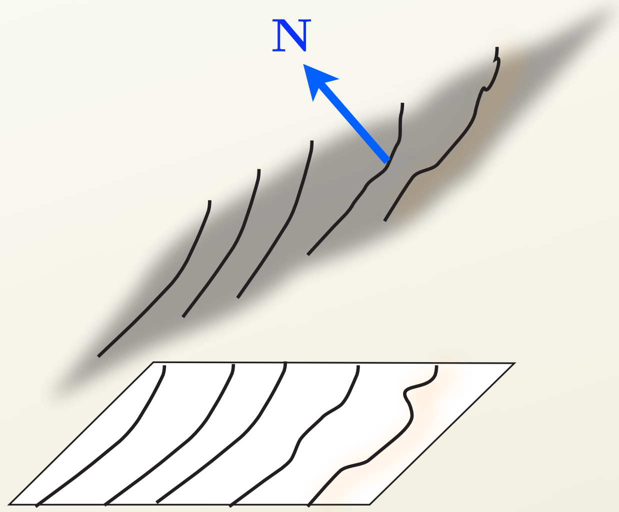
Equal Spacing



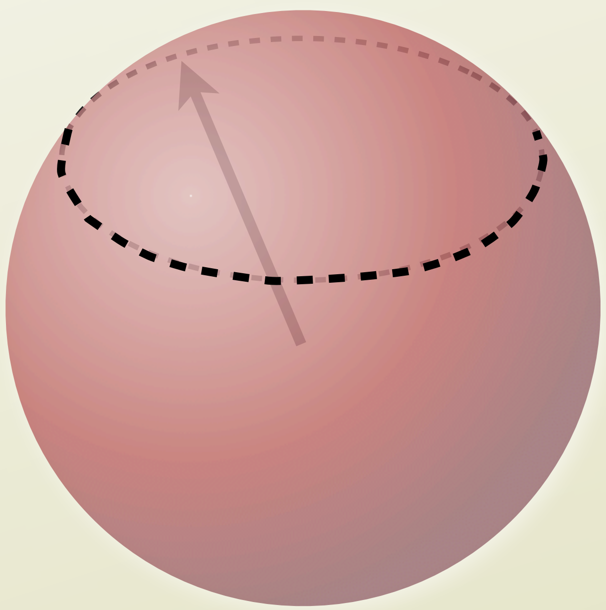
$$\mathbf{e}_z \cdot \mathbf{N} = \frac{1}{\sqrt{2}}$$



Equal Spacing



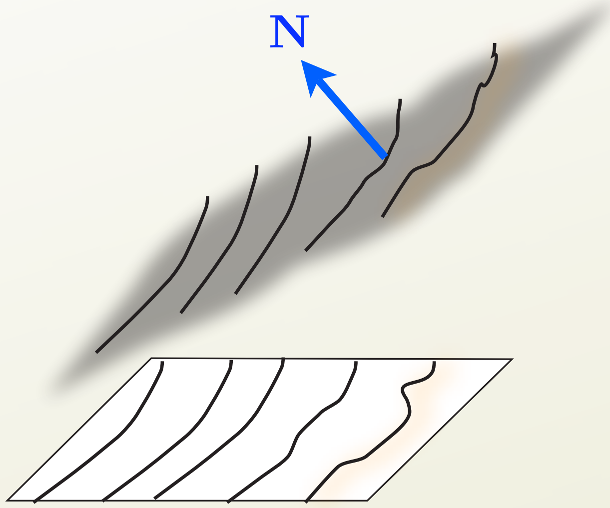
$$\mathbf{e}_z \cdot \mathbf{N} = \frac{1}{\sqrt{2}}$$



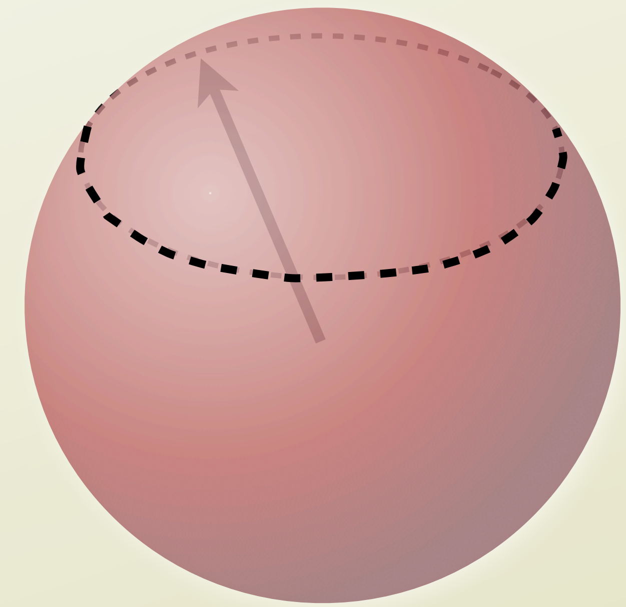
$$K = 0$$

isometric to the plane

Equal Spacing

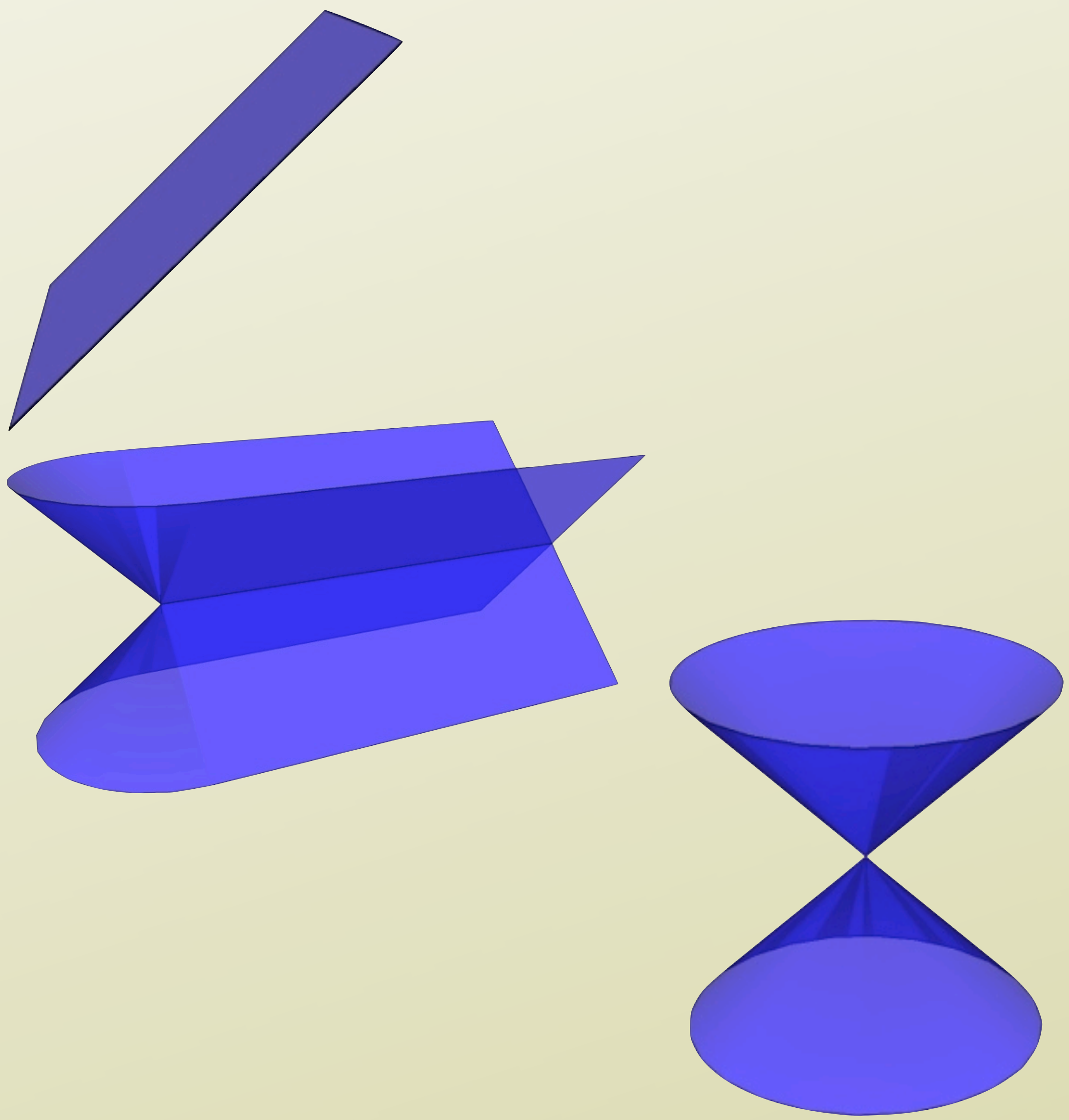


$$\mathbf{e}_z \cdot \mathbf{N} = \frac{1}{\sqrt{2}}$$

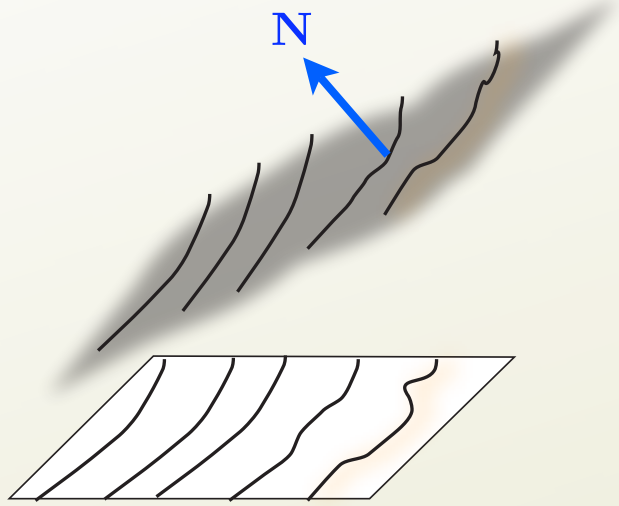


$$K = 0$$

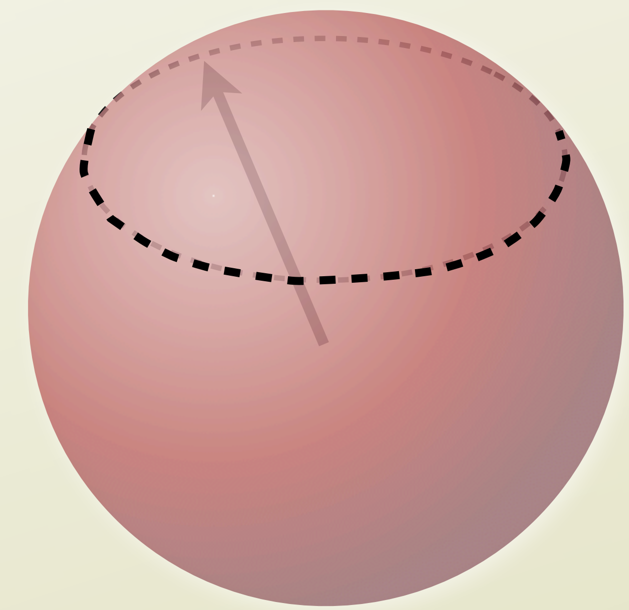
isometric to the plane



EQUAL SPACING



$$\mathbf{e}_z \cdot \mathbf{N} = \frac{1}{\sqrt{2}}$$



$$K = 0$$

isometric to the plane

FOCAL CONICS

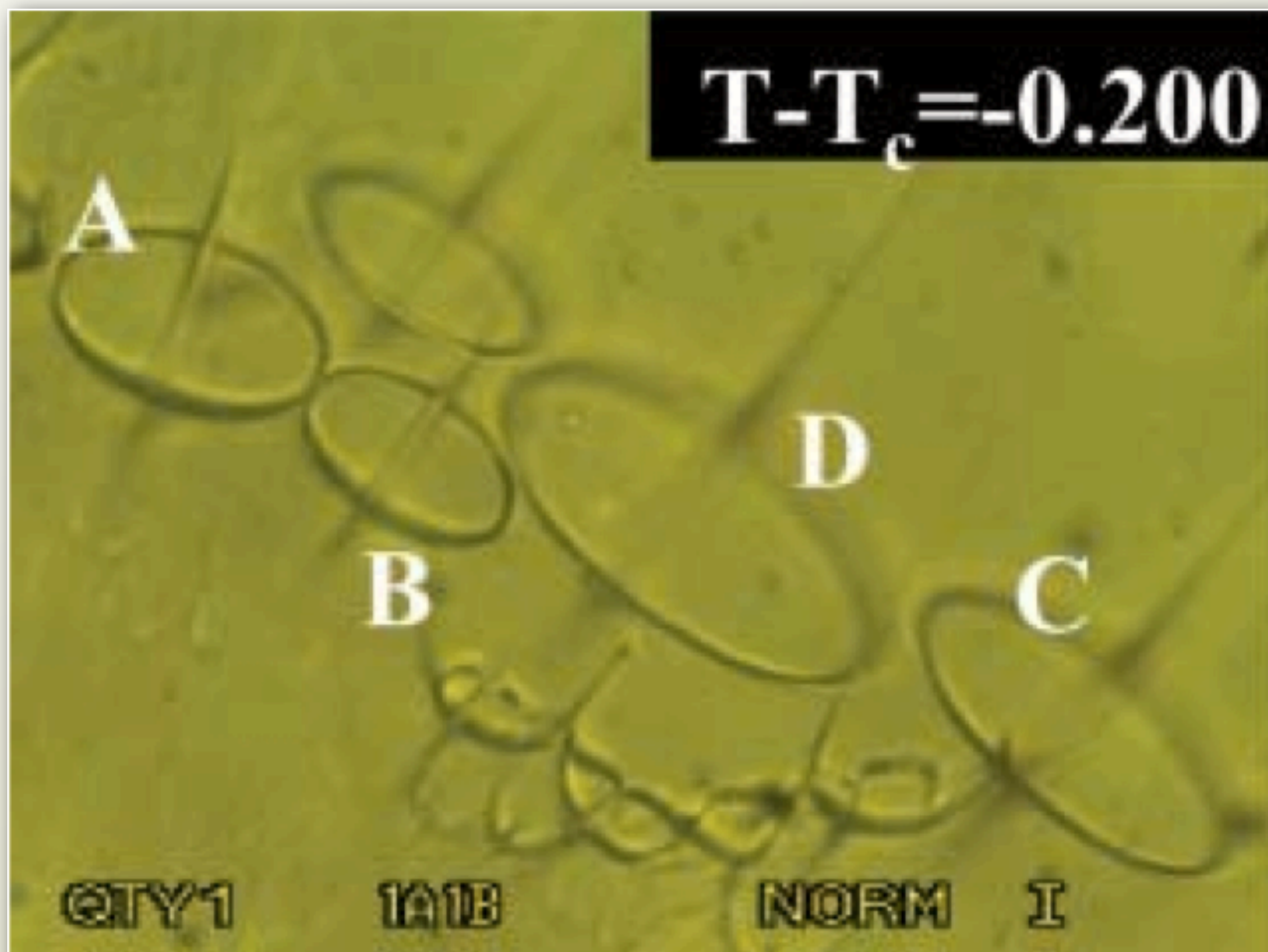
Friedel, Granjean, *Bull. Soc. Fr. Minéral.* **33**, 409-465 (1910)

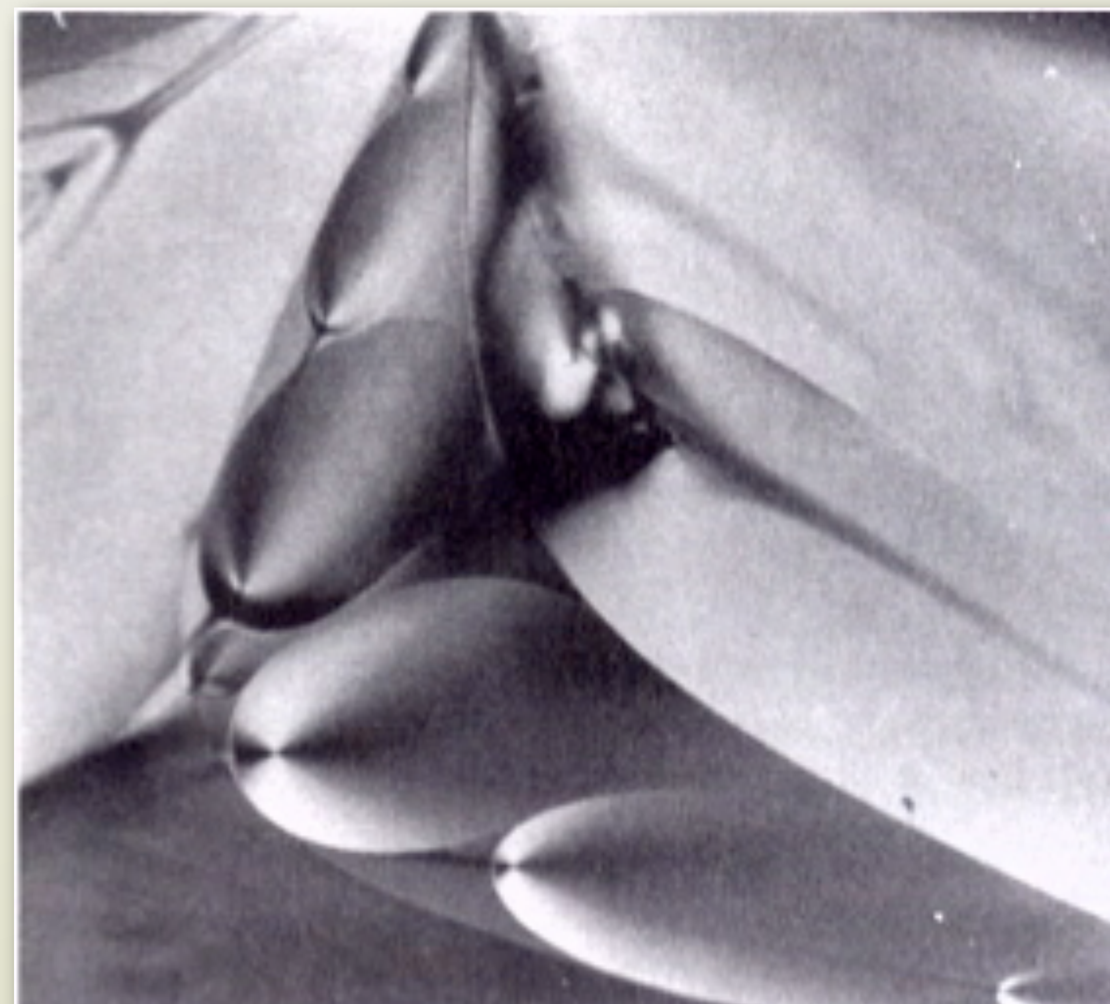
Observations géométriques sur les liquides à coniques focales;

PAR MM. G. FRIEDEL ET F. GRANDJEAN.

Nous avons signalé, dans une précédente Note (1), les étranges figures géométriques que renferment certains liquides anisotropes. Ces figures, qui sont des groupes de *coniques focales* associées suivant des lois simples, s'observent dans le par-

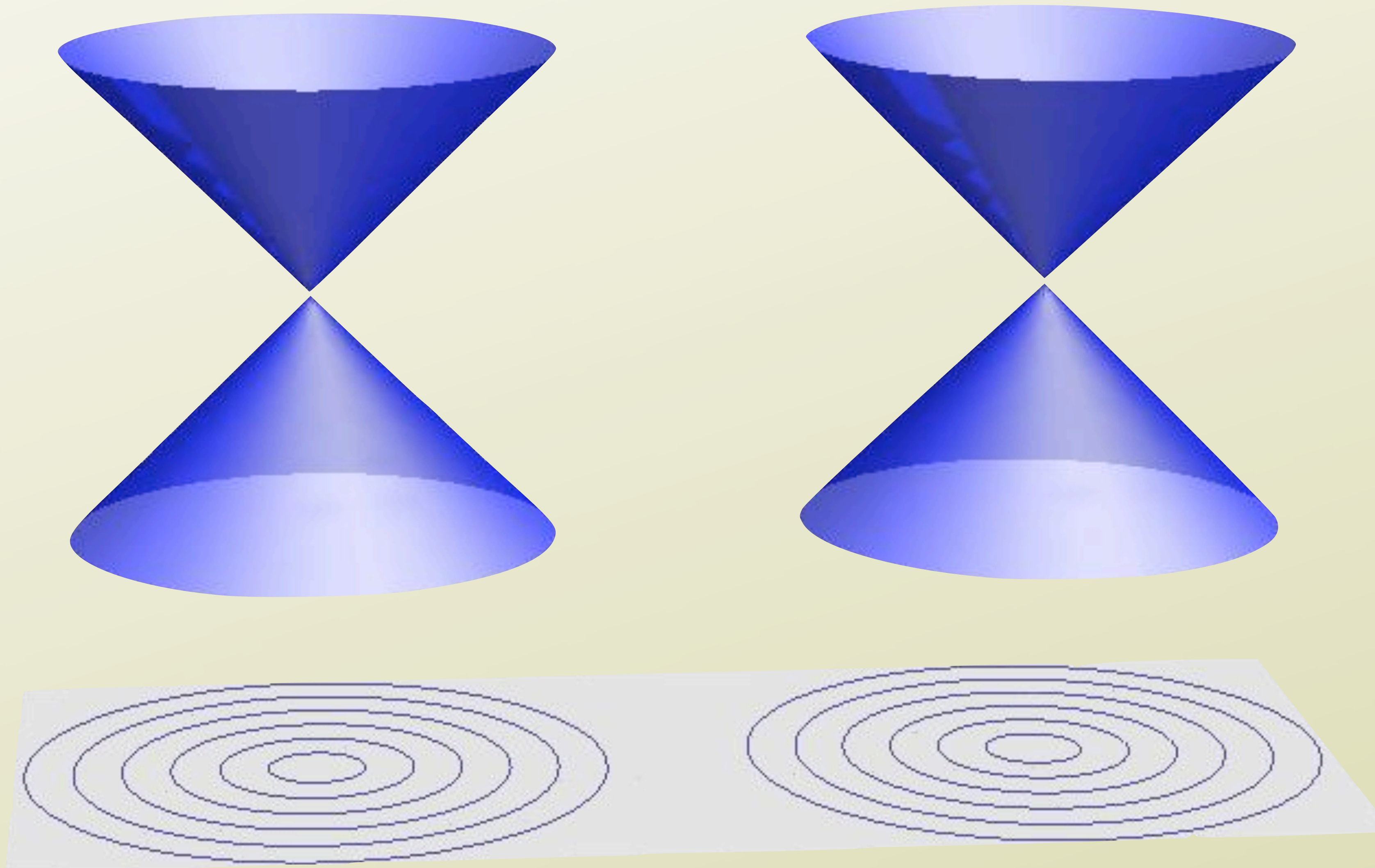
(1) Les liquides à coniques focales (*Comptes rendus de l'Académie des Sciences*, t. 151, 31 octobre 1910, p. 762).





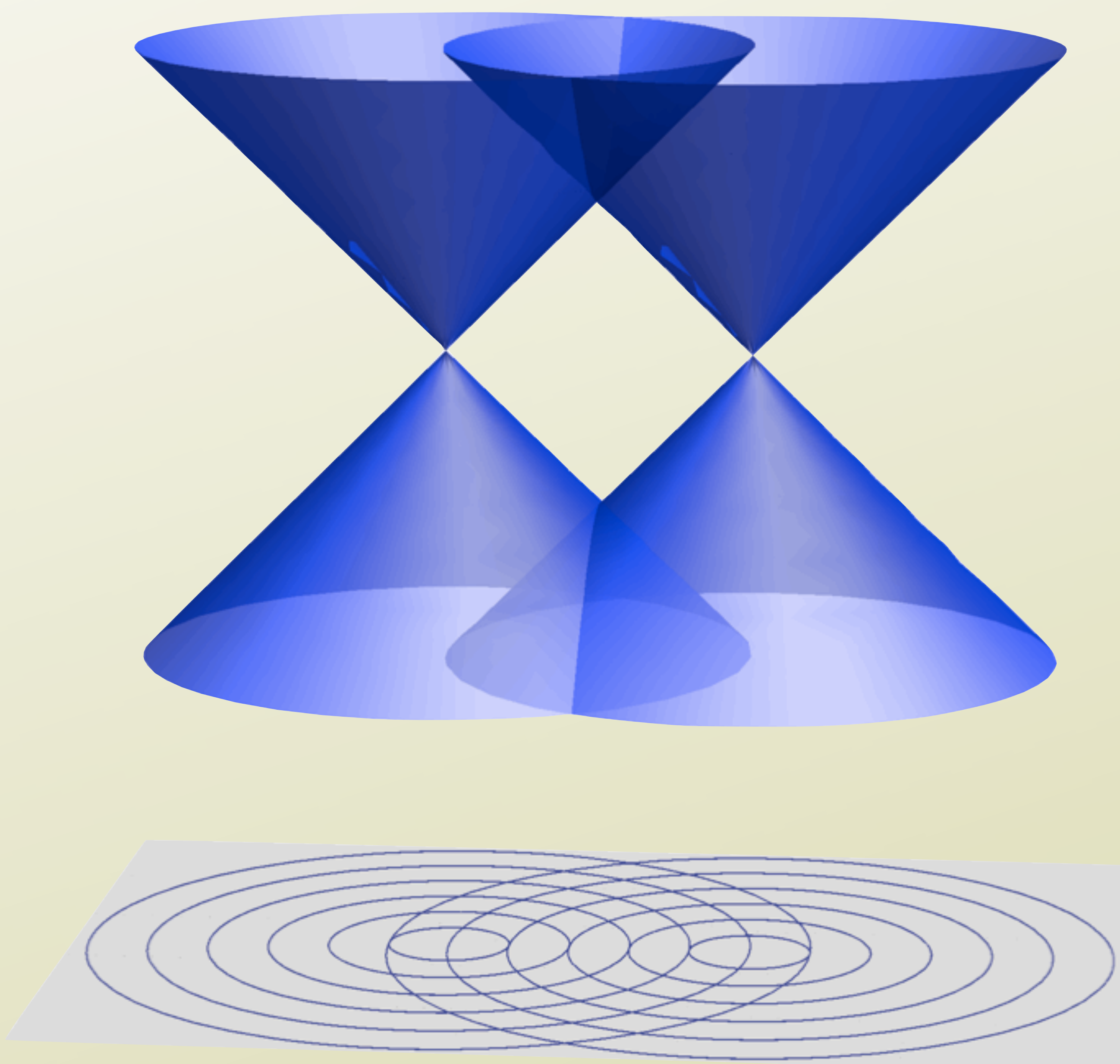
Nastishin, Meyer, and Kléman (2008), C. Williams, from de Gennes & Prost

Two Cones



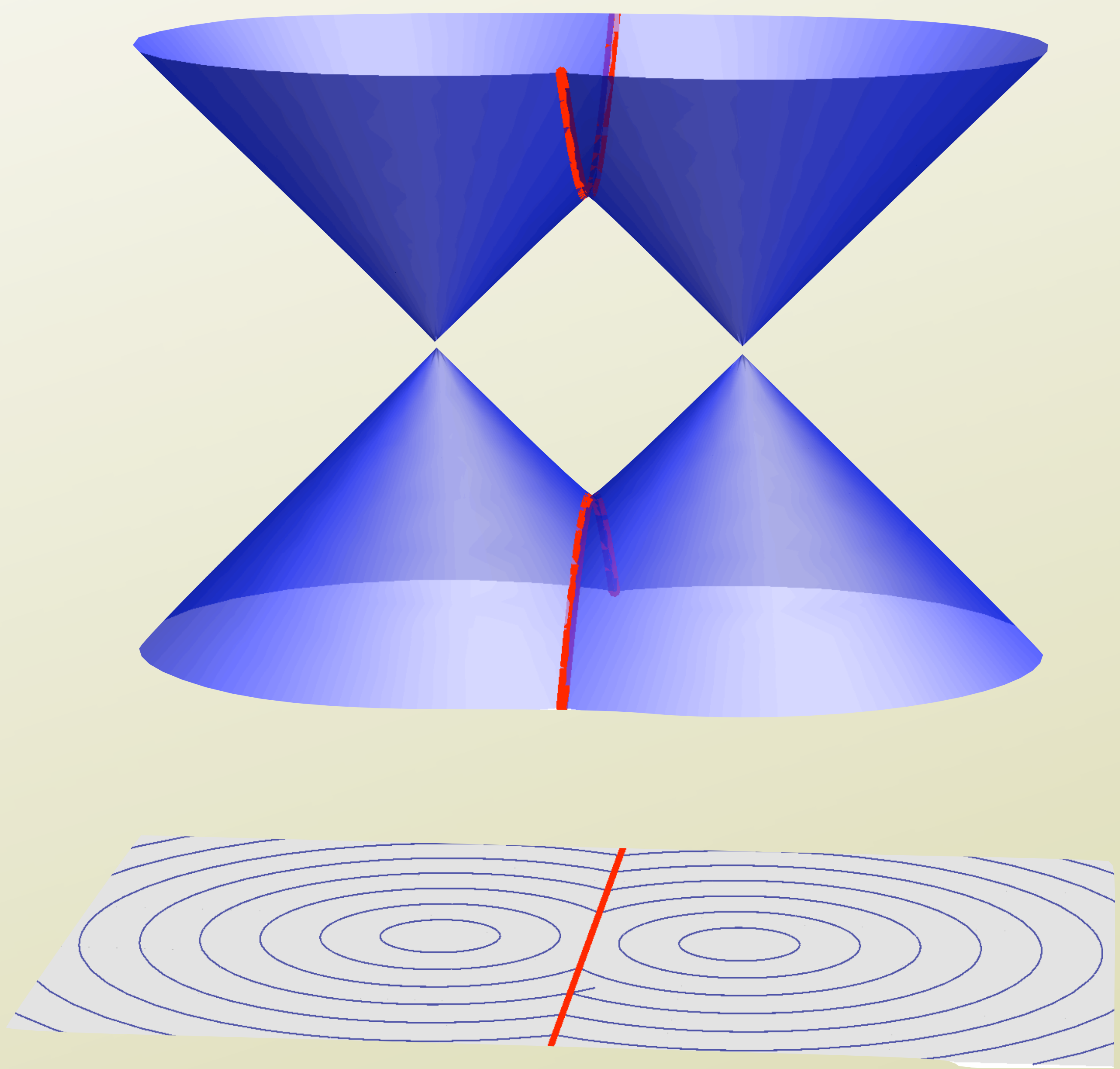
Alexander, Chen, Matsumoto, Kamien, (2010)

TWO CONES



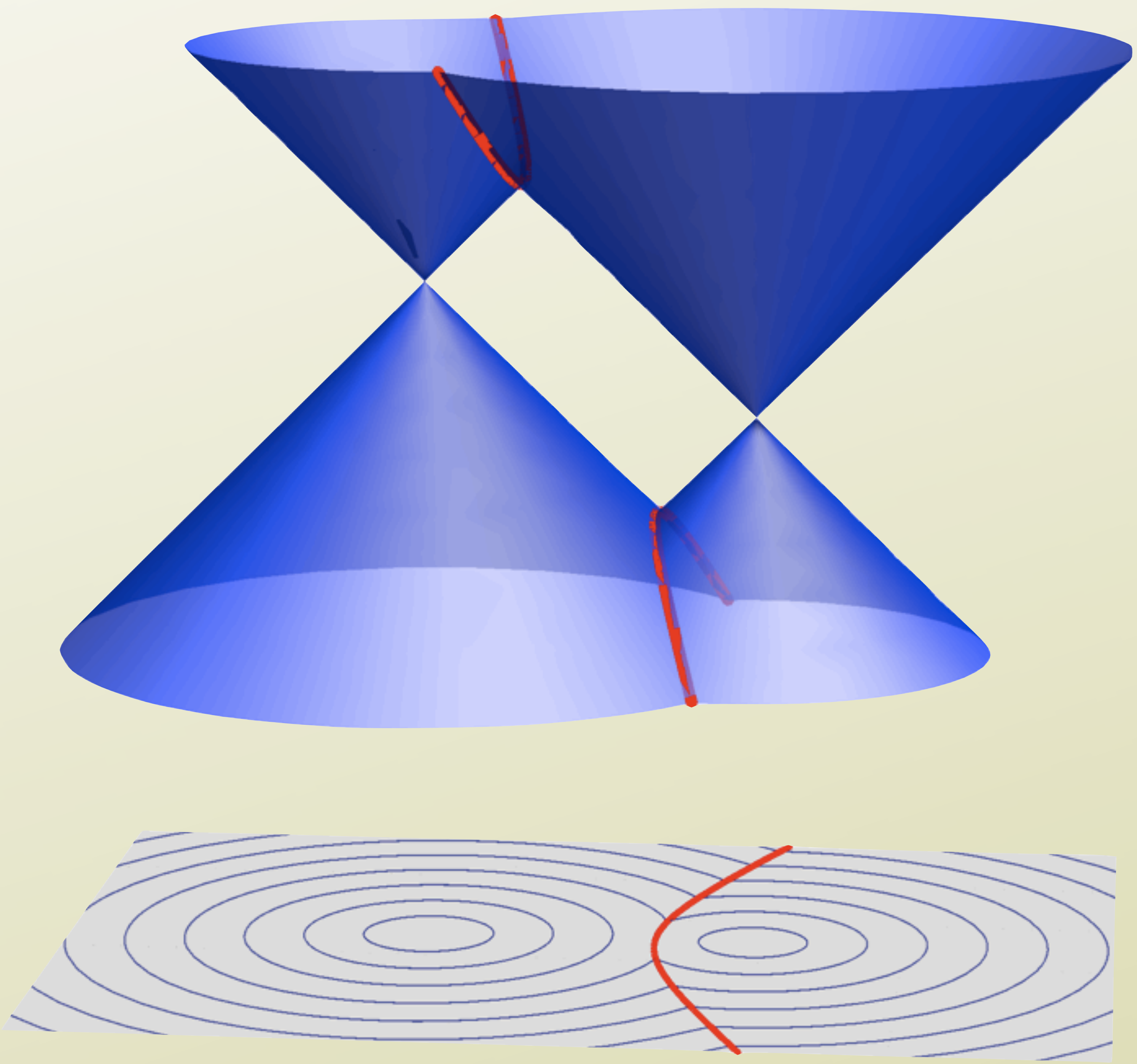
Alexander, Chen, Matsumoto, Kamien, (2010)

Two Cones



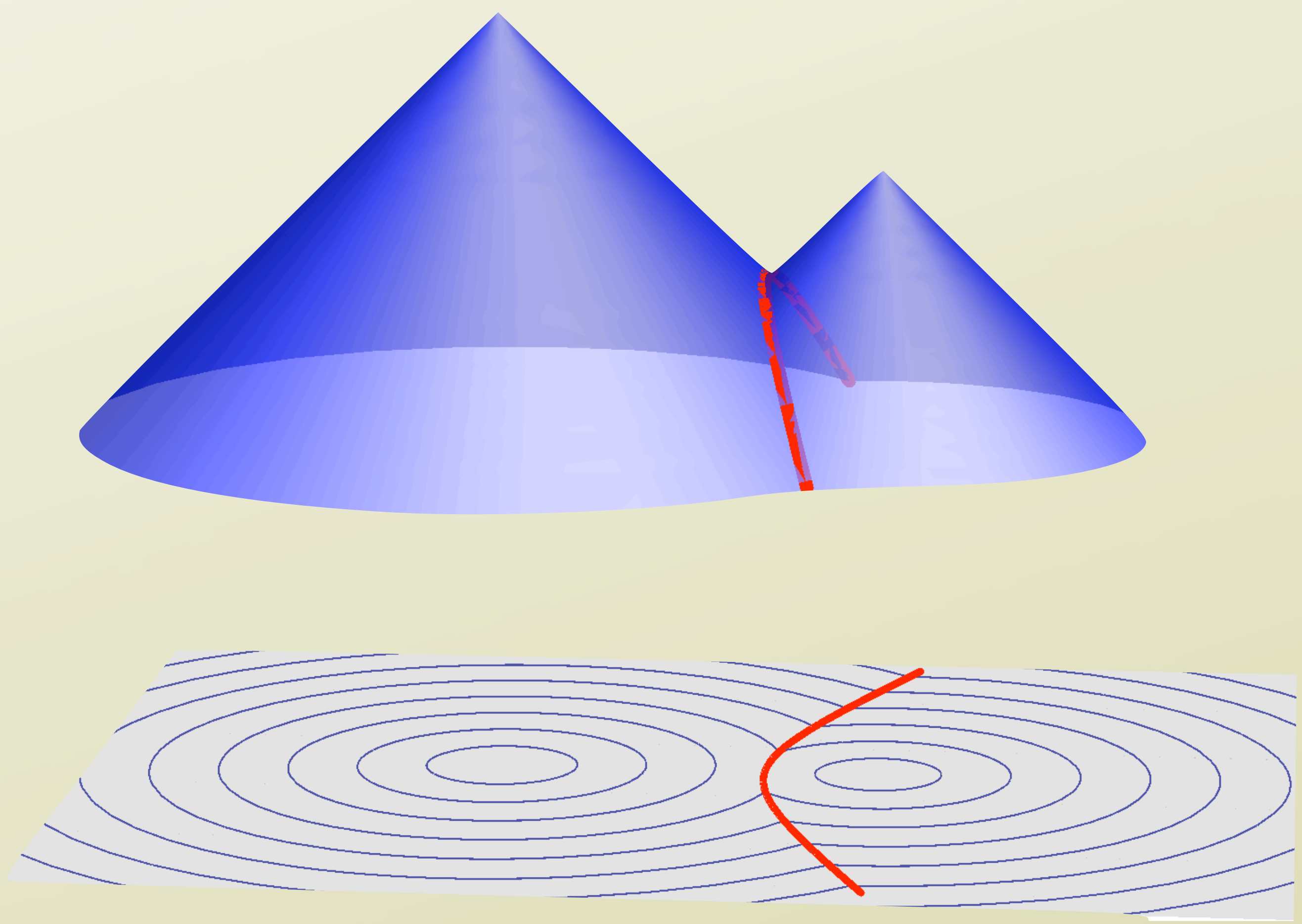
Alexander, Chen, Matsumoto, Kamien, (2010)

Two Cones



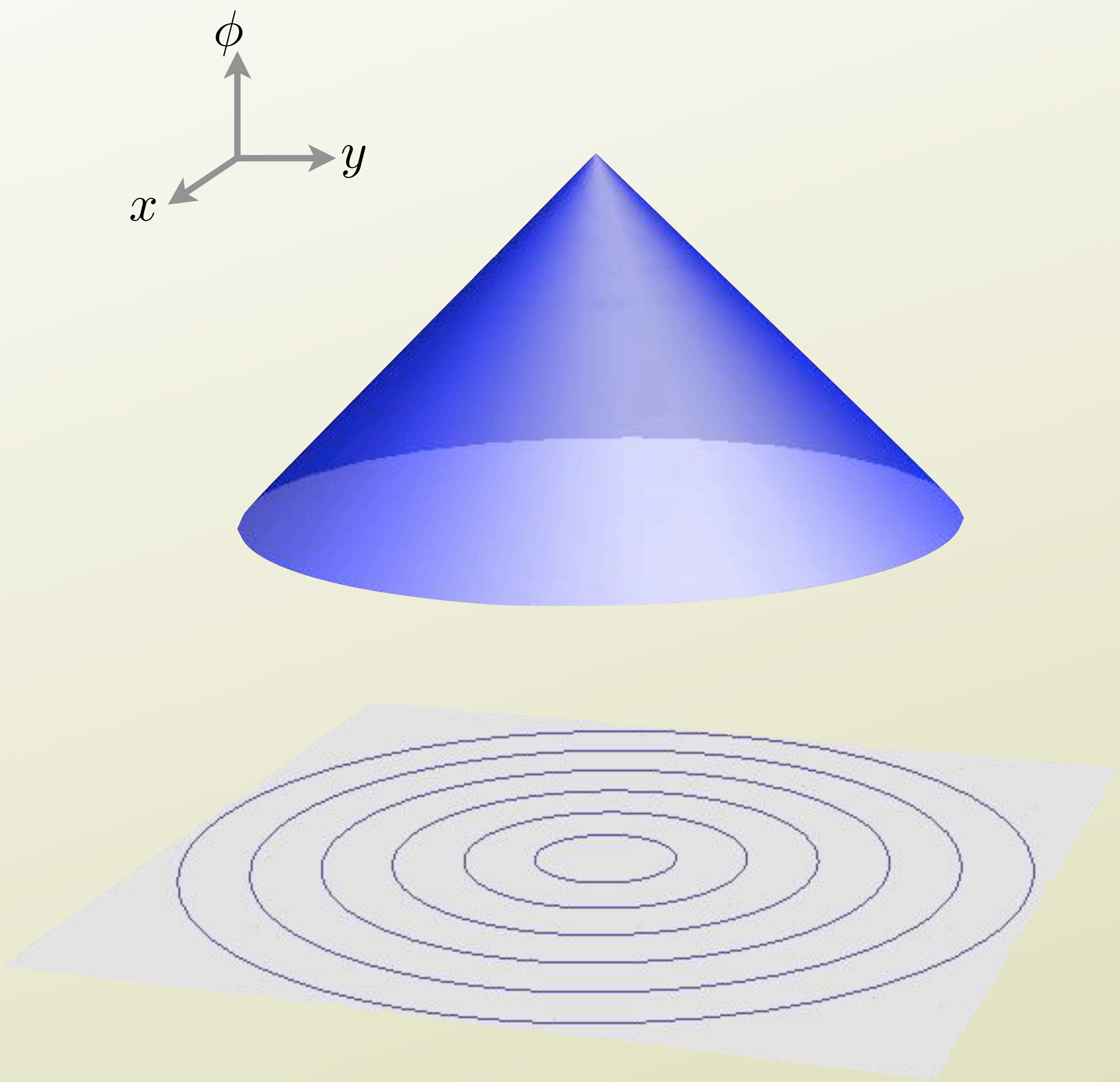
Alexander, Chen, Matsumoto, Kamien, (2010)

Two Cones



Alexander, Chen, Matsumoto, Kamien, (2010)

SHEDDING LIGHT ON FOCAL CONICS



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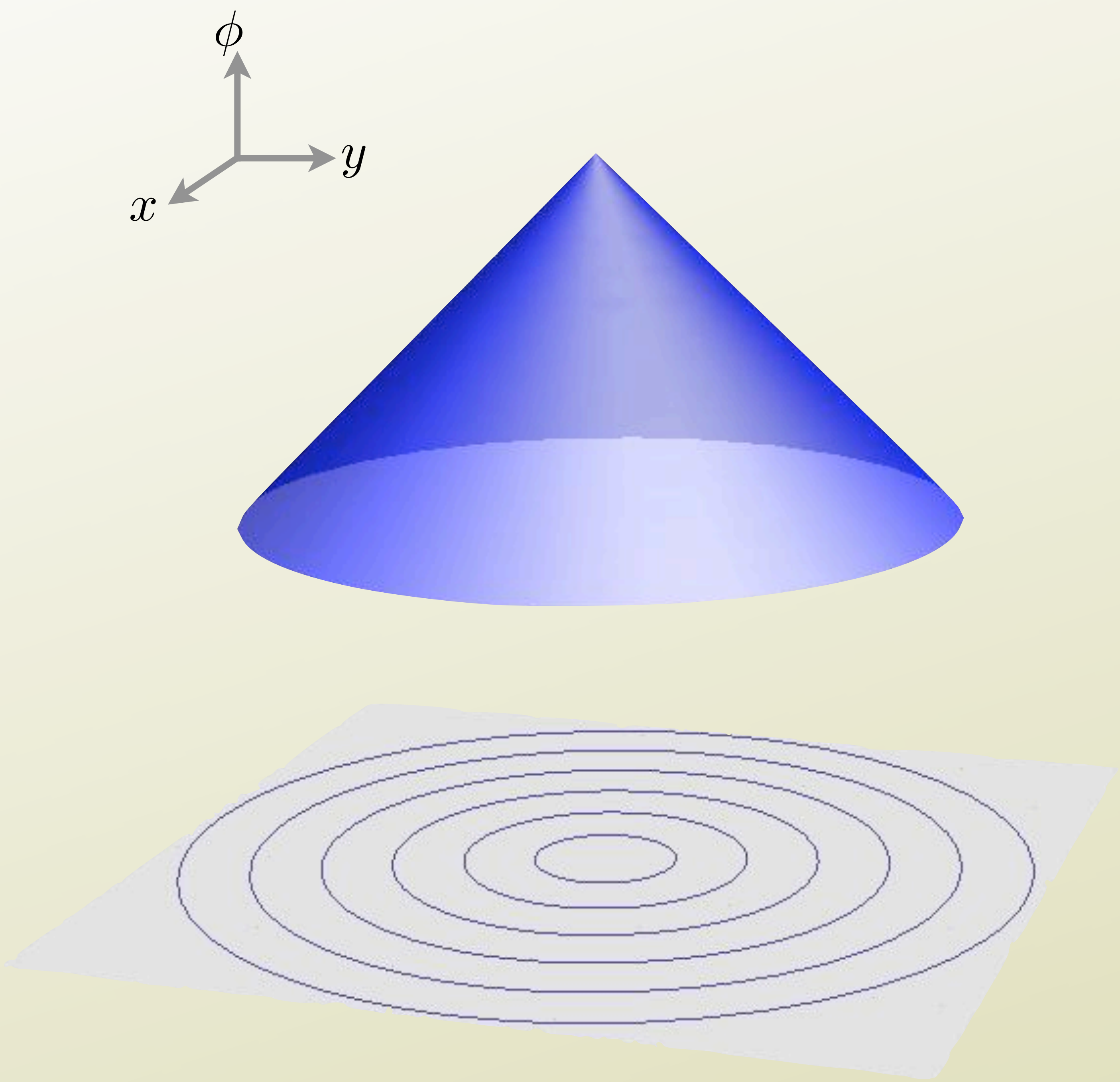
Alexander, Chen, Matsumoto, Kamien, (2010)

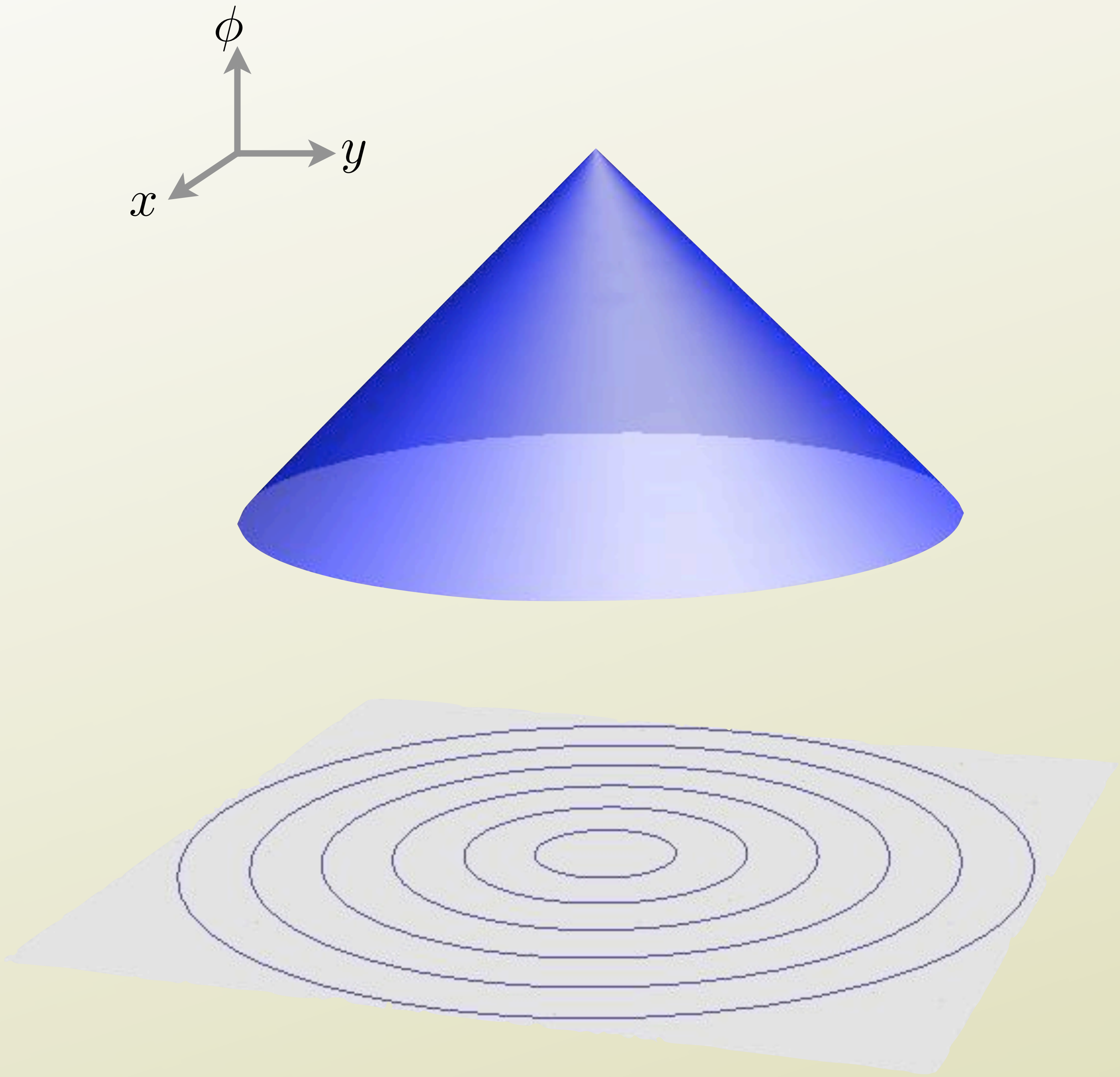
SHEDDING LIGHT ON FOCAL CONICS



$$\phi = -\sqrt{x^2 + y^2}$$

Alexander, Chen, Matsumoto, Kamien, (2010)

Shedding Light on Focal Conics



$$\phi^2 = x^2 + y^2$$

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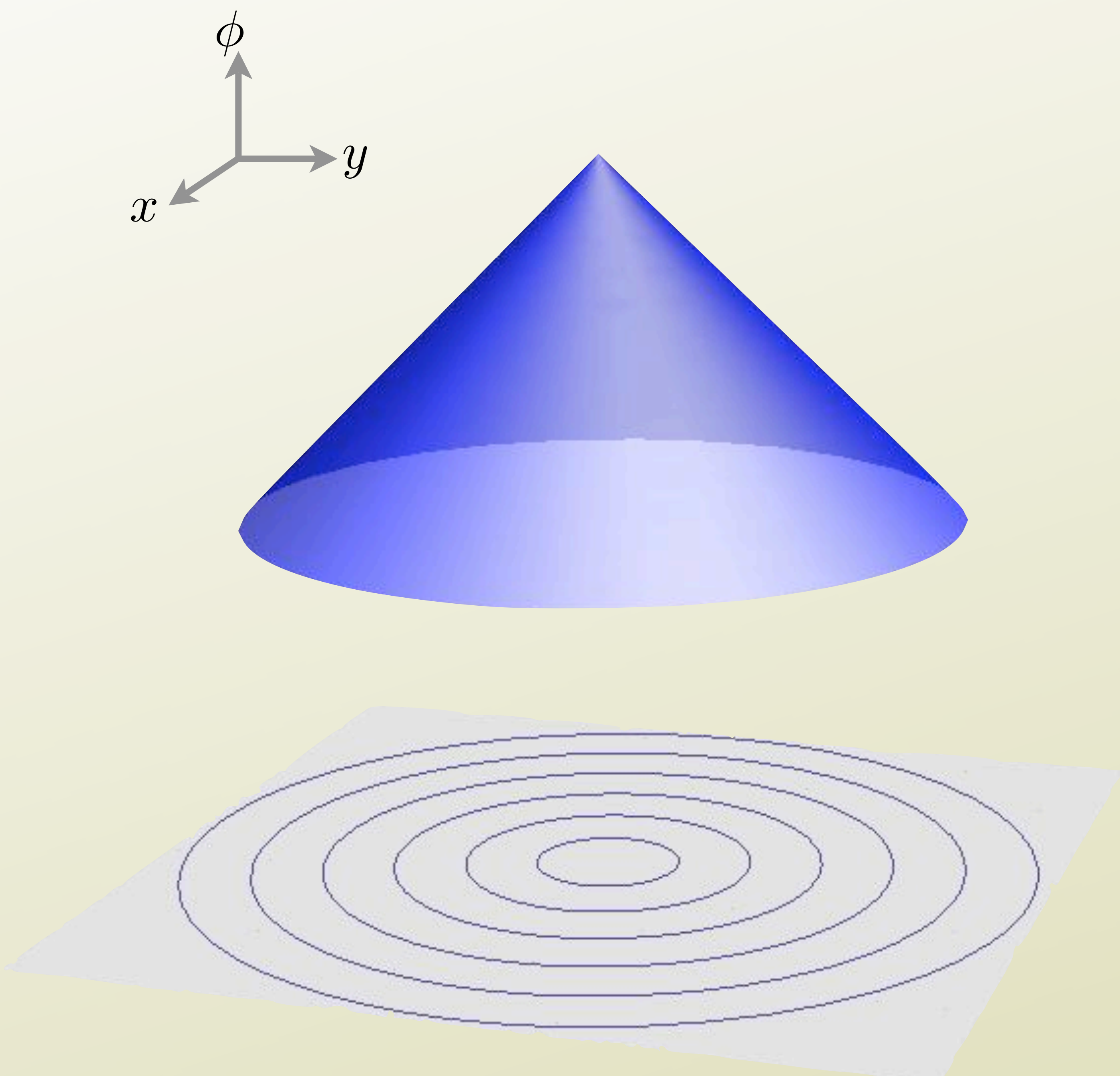
Alexander, Chen, Matsumoto, Kamien, (2010)

Shedding Light on Focal Conics

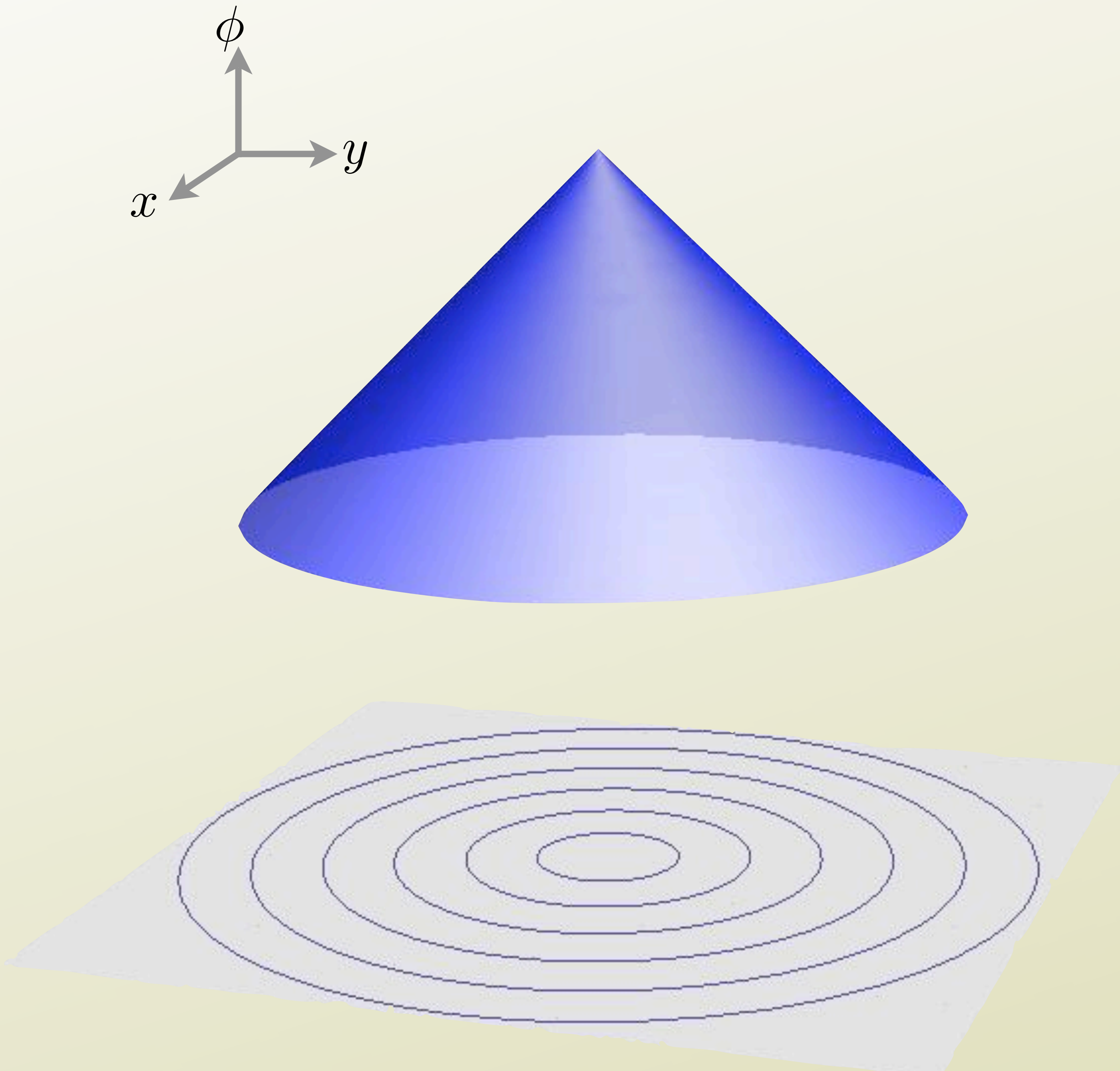


$$-\phi^2 + x^2 + y^2 = 0$$

$\| \cdot \|^2_{\mathbb{M}^3}$ light cone

Alexander, Chen, Matsumoto, Kamien, (2010)

Shedding Light on Focal Conics



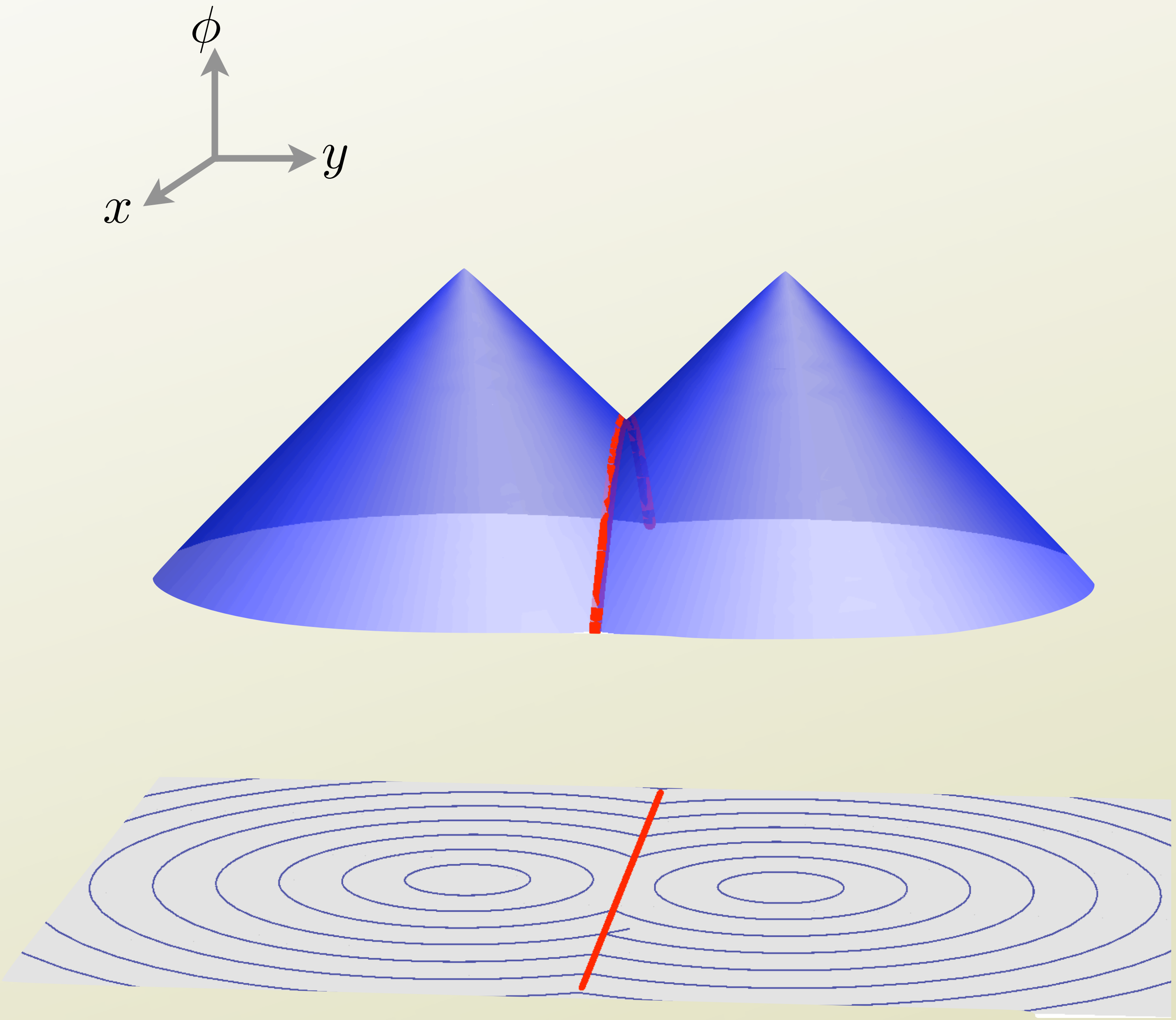
$$-\phi^2 + x^2 + y^2 = 0$$

$\| \cdot \|^2_{\mathbb{M}^3}$ light cone

Equal spacing ⇔ Null hypersurface

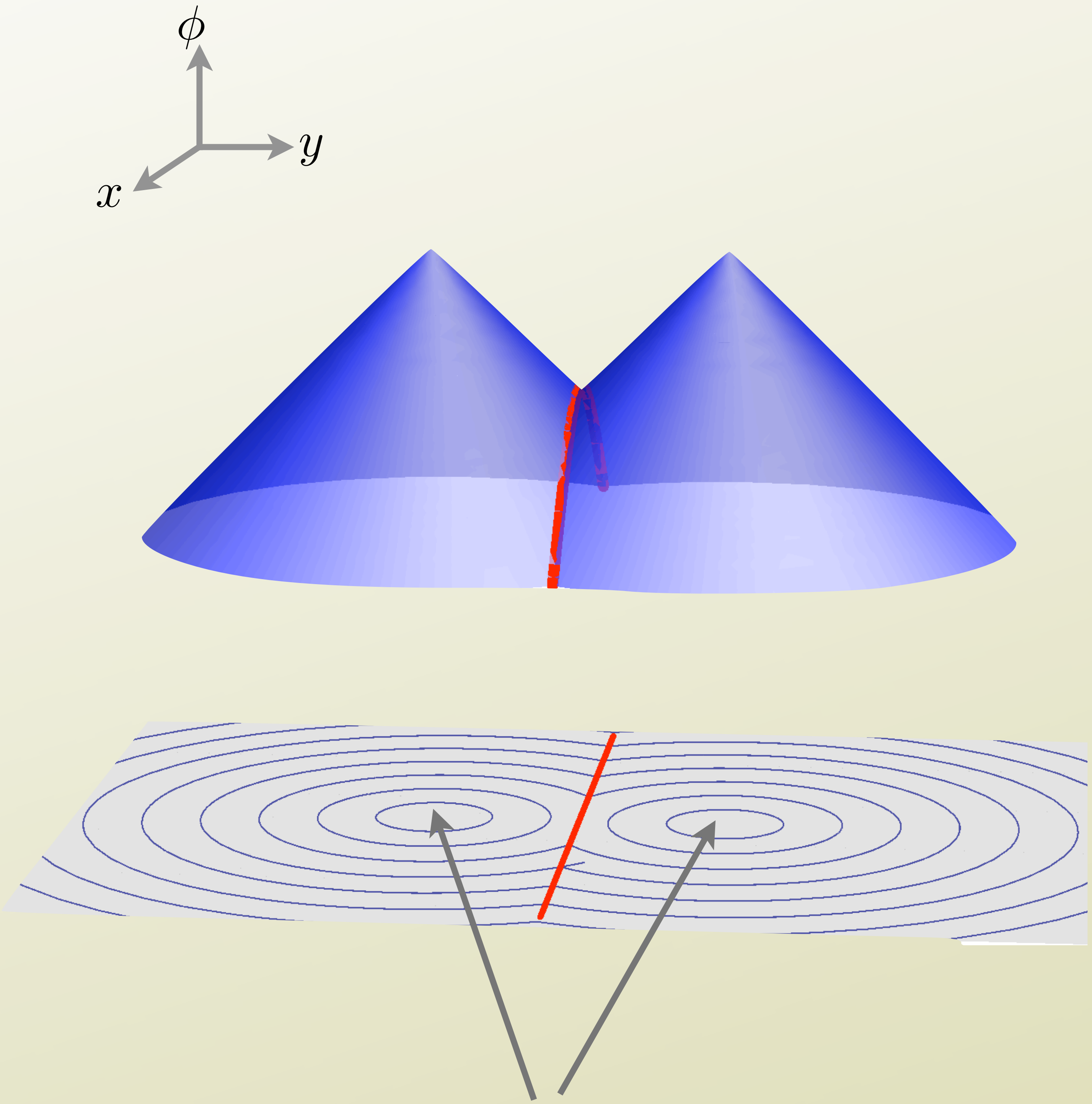
Alexander, Chen, Matsumoto, Kamien, (2010)

Space–Like Separated Events



Alexander, Chen, Matsumoto, Kamien, (2010)

SPACE–LIKE SEPARATED EVENTS



events $\quad e_1, e_2 = (0, 0, \pm r)$

foci $\quad (0, \pm r)$

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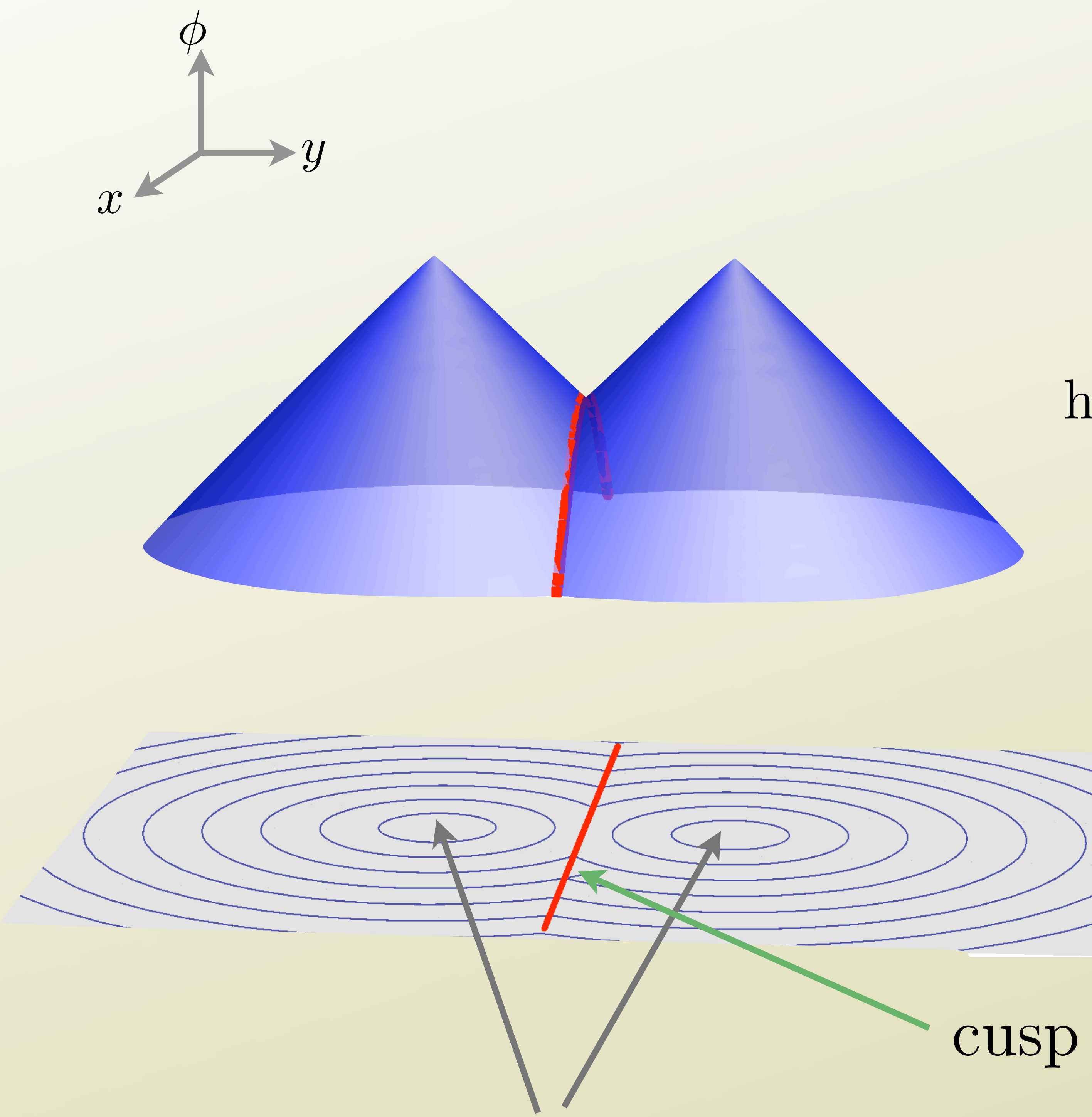
Alexander, Chen, Matsumoto, Kamien, (2010)

Space–Like Separated Events



events $\quad e_1, e_2 = (0, 0, \pm r)$

hyperbola $\quad -\phi^2 + x^2 = -r^2, \; y = 0$

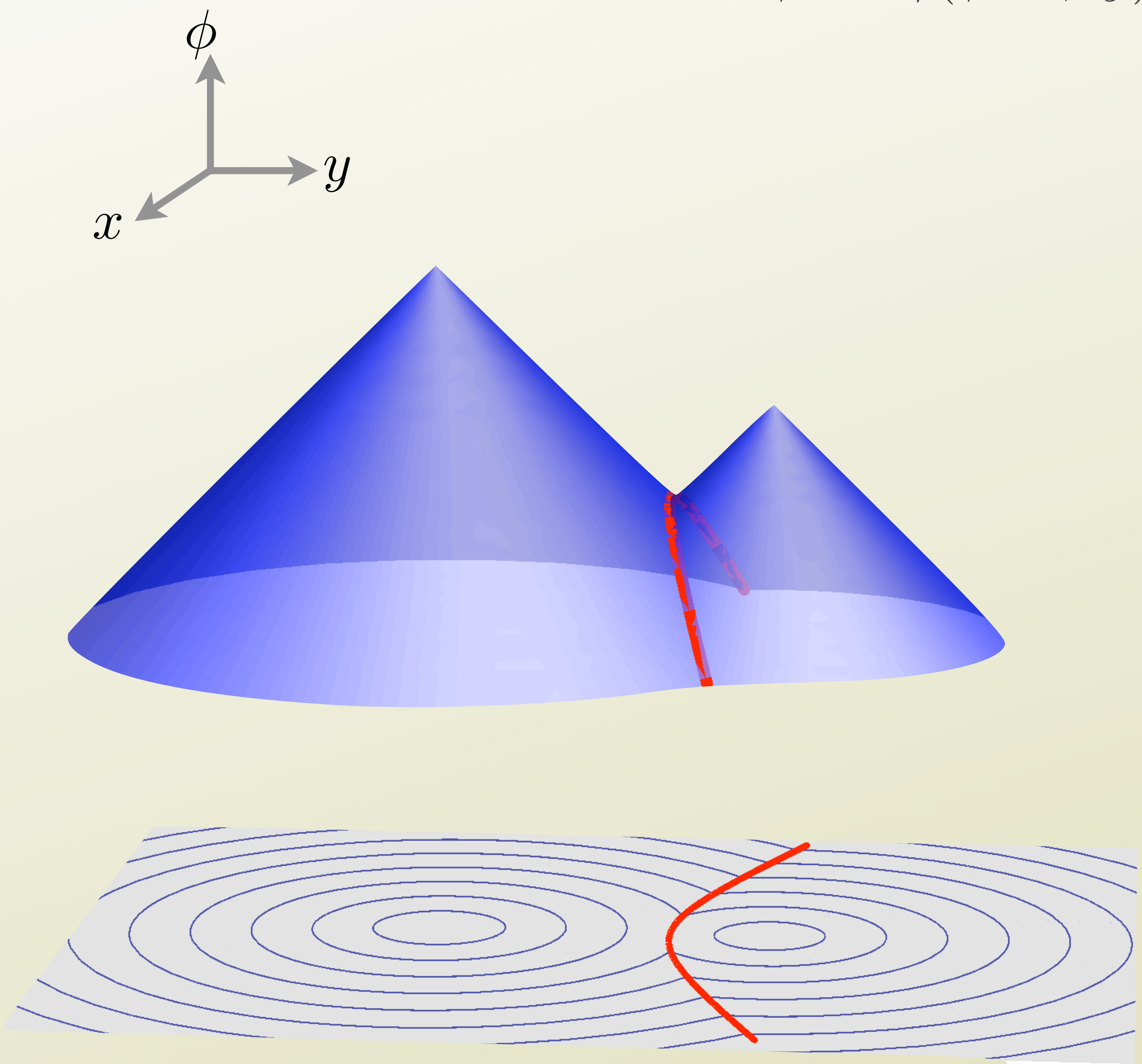
cusp $\quad y = 0$

foci $\quad (0, \pm r)$

Alexander, Chen, Matsumoto, Kamien, (2010)

SPACE–LIKE SEPARATED EVENTS

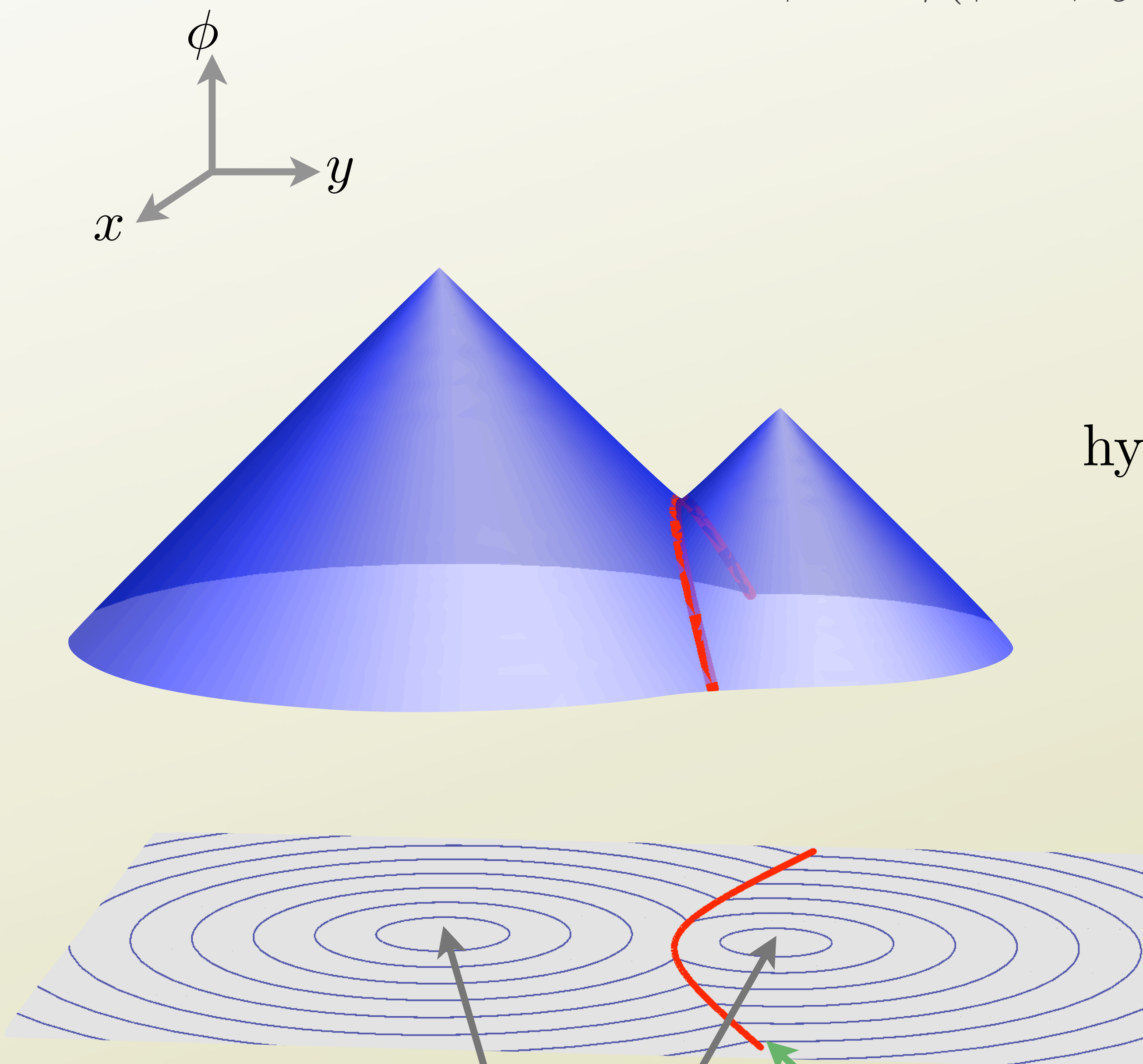
Lorentz $\phi' = \gamma(\phi - \beta y),\ x' = x,\ y' = \gamma(y - \beta\phi)$



Alexander, Chen, Matsumoto, Kamien, (2010)

Space–Like Separated Events

Lorentz $\phi' = \gamma(\phi - \beta y),\ x' = x,\ y' = \gamma(y - \beta\phi)$



events $e_1, e_2 = (\mp\gamma\beta r, 0, \pm\gamma r)$

hyperbola $-(\phi'/\gamma)^2 + x'^2 = -r^2,\ y' = -\beta\phi'$

cusp $(y/\gamma\beta)^2 - x^2 = r^2$

foci $(0, \pm\gamma r)$

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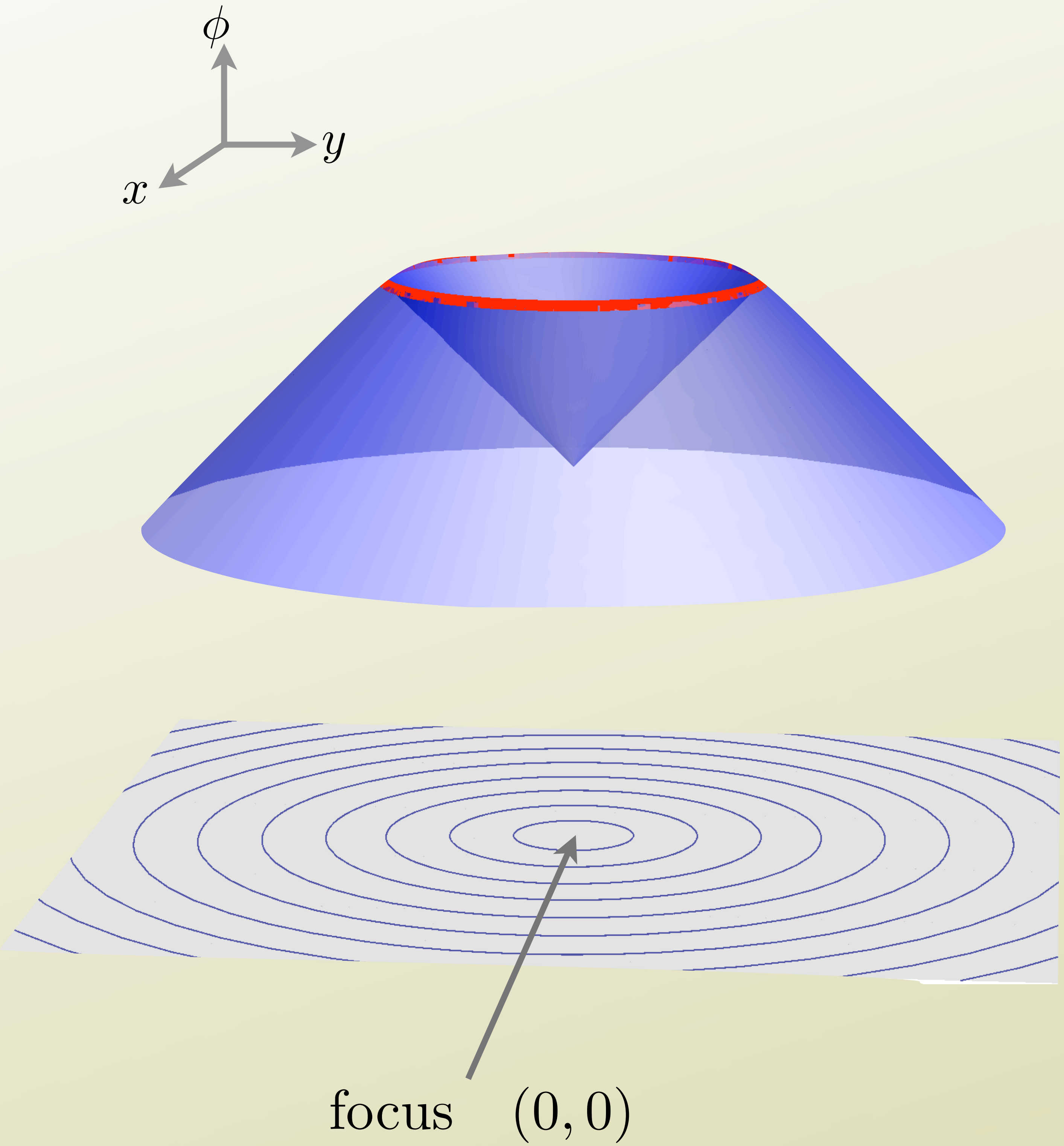
Alexander, Chen, Matsumoto, Kamien, (2010)

Time–Like Separated Events



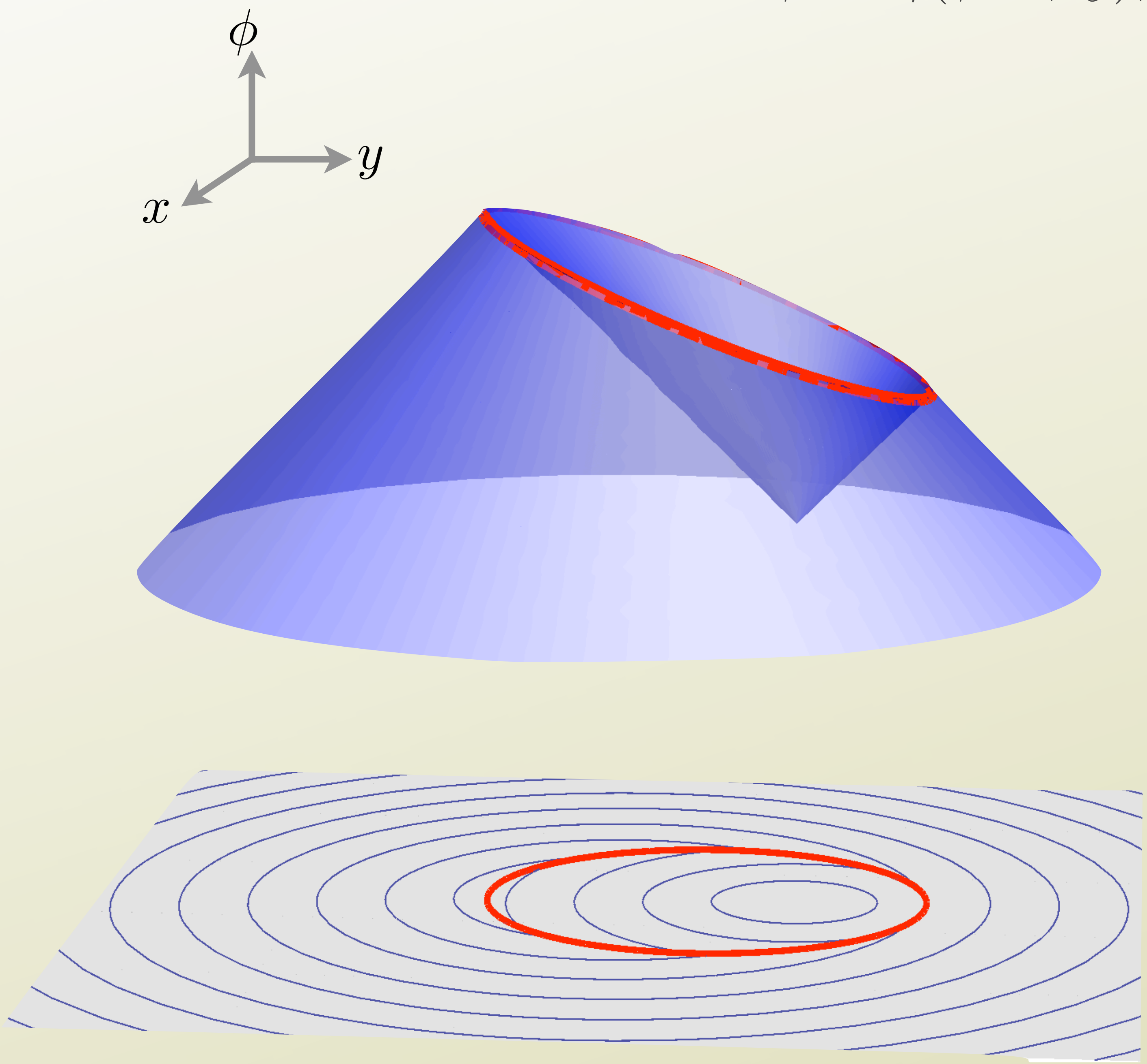
events $(\pm r, 0, 0)$

circle $x^2 + y^2 = r^2,\ \phi = 0$

Alexander, Chen, Matsumoto, Kamien, (2010)

Time–Like Separated Events

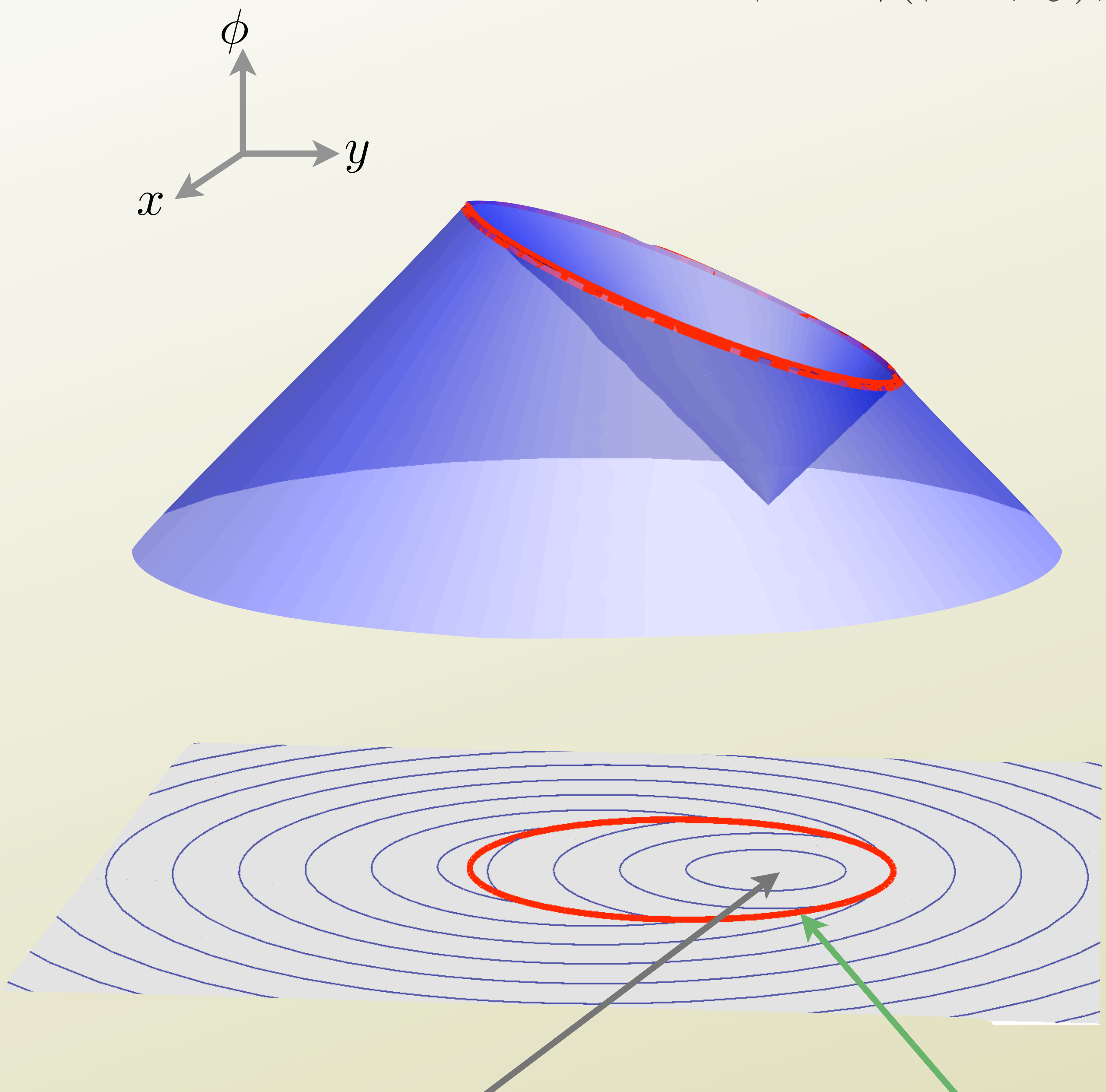
Lorentz $\phi' = \gamma(\phi - \beta y),\ x' = x,\ y' = \gamma(y - \beta\phi)$



Alexander, Chen, Matsumoto, Kamien, (2010)

Time–Like Separated Events

Lorentz $\quad \phi' = \gamma(\phi - \beta y),\ x' = x,\ y' = \gamma(y - \beta\phi)$



events $\quad (\pm\gamma r, 0, \mp\gamma\beta r)$

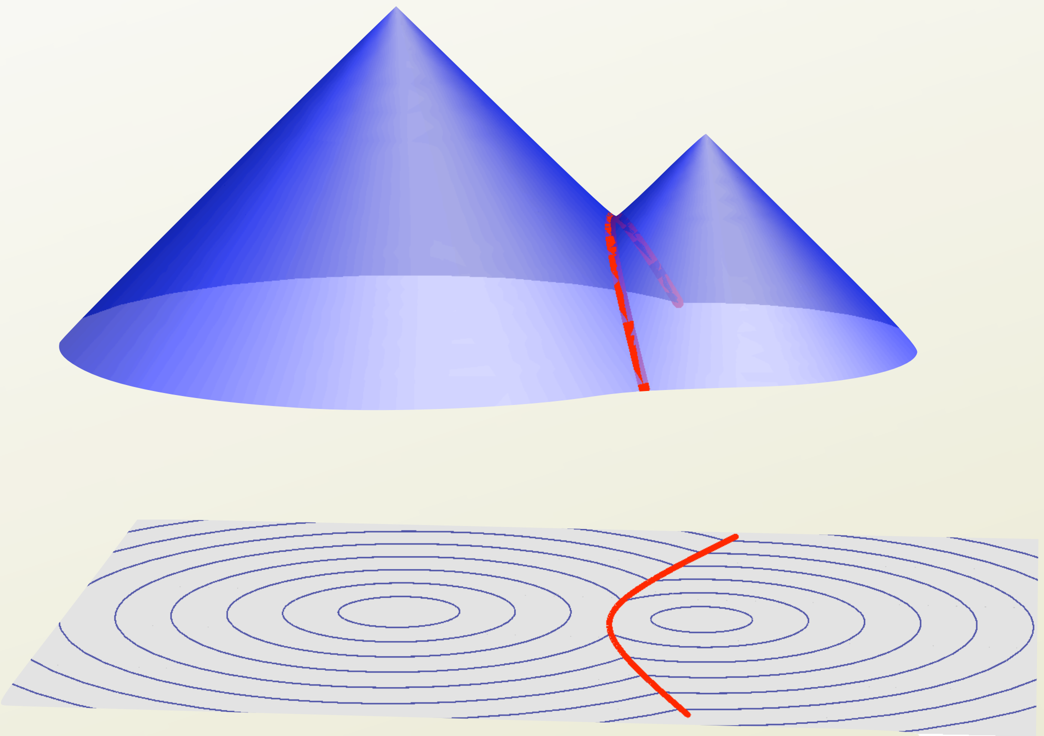
circle $\quad x'^2 + (y'/\gamma)^2 = r^2,\ \phi' = -\beta y'$

focus $\quad (0, \gamma\beta r)$

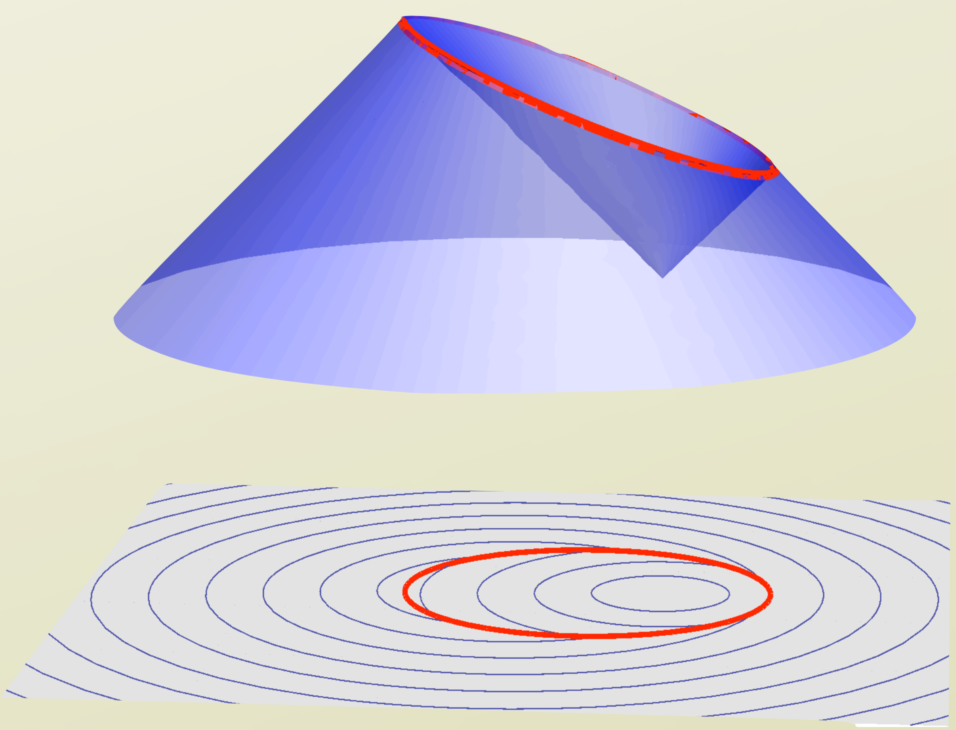
cusp $\quad x^2 + (y/\gamma)^2 = r^2$

Alexander, Chen, Matsumoto, Kamien, (2010)

FOCAL SETS



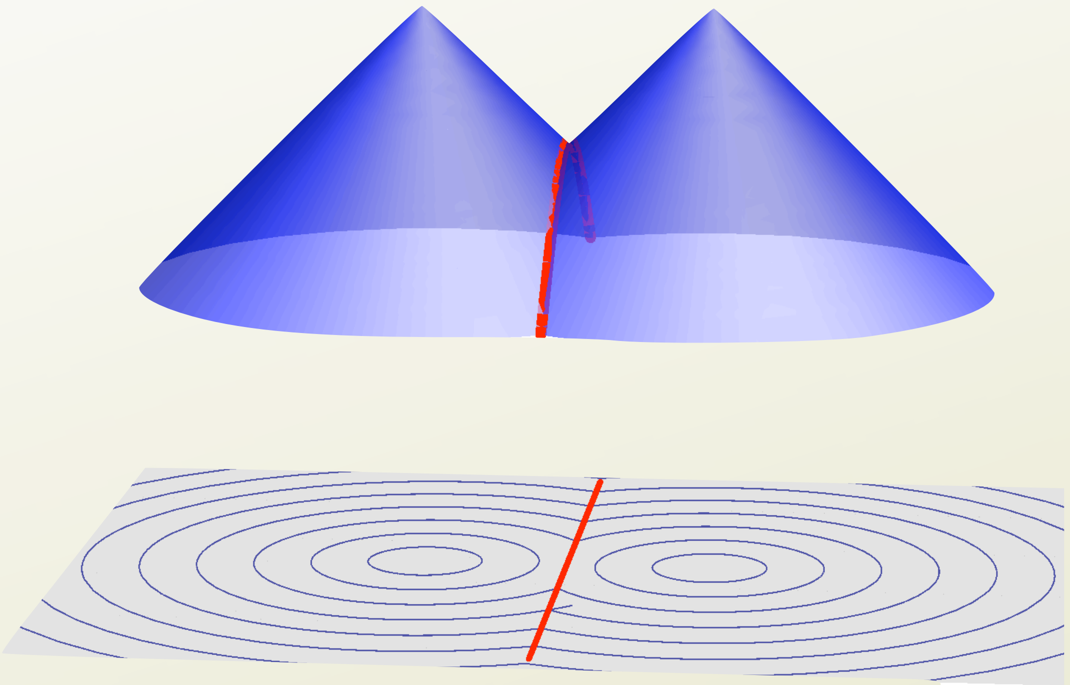
space-like separated events



time-like separated events

Alexander, Chen, Matsumoto, Kamien, (2010)

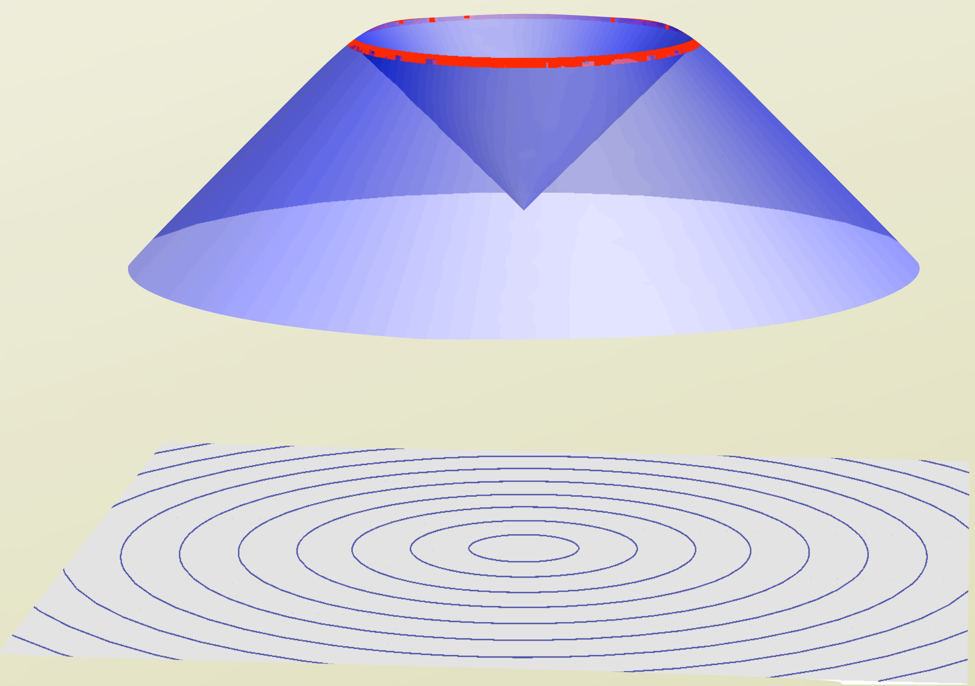
Focal Sets



space-like separated events

$$\Sigma = \{(0,0,y) \text{ s.t. } y^2 = r^2\}$$

$$\overline{\Sigma} = \{(\phi,x,0) \text{ s.t. } -\phi^2 + x^2 = -r^2\}$$



time-like separated events

$$\Sigma = \{(0,x,y) \text{ s.t. } x^2 + y^2 = r^2\}$$

$$\overline{\Sigma} = \{(\phi,0,0) \text{ s.t. } -\phi^2 = -r^2\}$$

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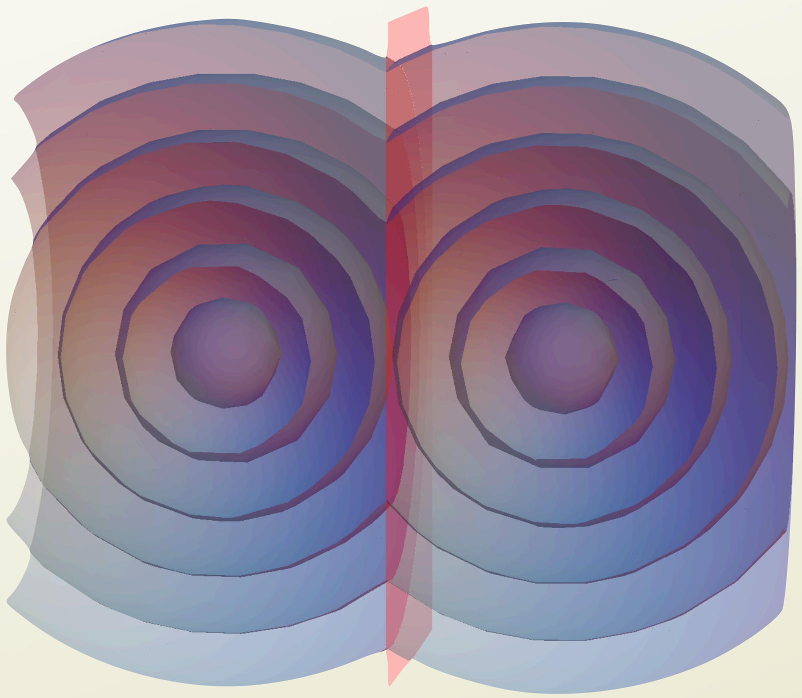
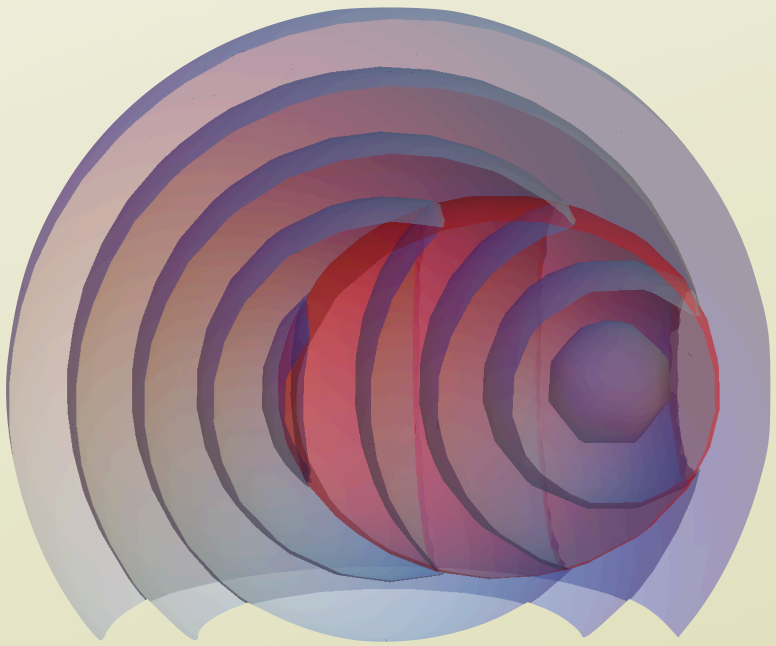
Alexander, Chen, Matsumoto, Kamien, (2010)
F. G. Friedlander, *Math. Proc. Camb. Phil. Soc.* **43**, 360-373 (1947)

Three Dimensions



space-like separated events

$$\Sigma = \{(0,0,0,z)\ \mathrm{s.t.}\ z^2 = r^2\}$$
$$\overline{\Sigma} = \{(\phi,x,y,0)\ \mathrm{s.t.}\ -\phi^2 + x^2 + y^2 = -r^2\}$$



time-like separated events

$$\Sigma = \{(0,x,y,z)\ \mathrm{s.t.}\ x^2 + y^2 + z^2 = r^2\}$$
$$\overline{\Sigma} = \{(\phi,0,0,0)\ \mathrm{s.t.}\ -\phi^2 = -r^2\}$$

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Alexander, Chen, Matsumoto, Kamien, (2010)

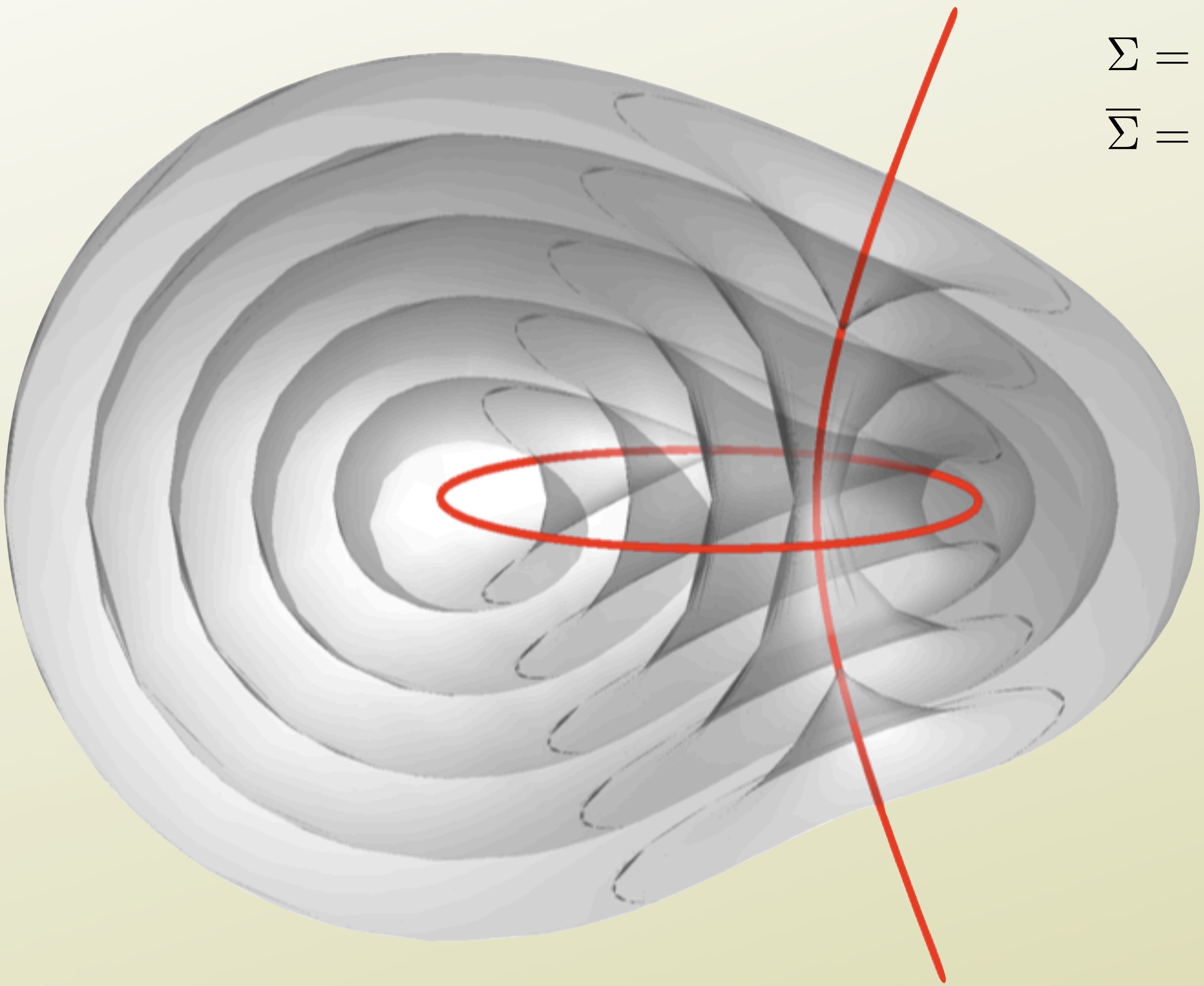
F. G. Friedlander, *Math. Proc. Camb. Phil. Soc.* **43**, 360-373 (1947)

Dupin Cyclides

two one-dimensional focal sets - "confocal conics"

$$\Sigma = \{(0,0,y,z) \text{ s.t. } y^2 + z^2 = r^2\}$$

$$\overline{\Sigma} = \{(\phi,x,0,0) \text{ s.t } -\phi^2 + x^2 = -r^2\}$$



Alexander, Chen, Matsumoto, Kamien, (2010)

F. G. Friedlander, *Math. Proc. Camb. Phil. Soc.* **43**, 360-373 (1947)

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Dupin Cyclides

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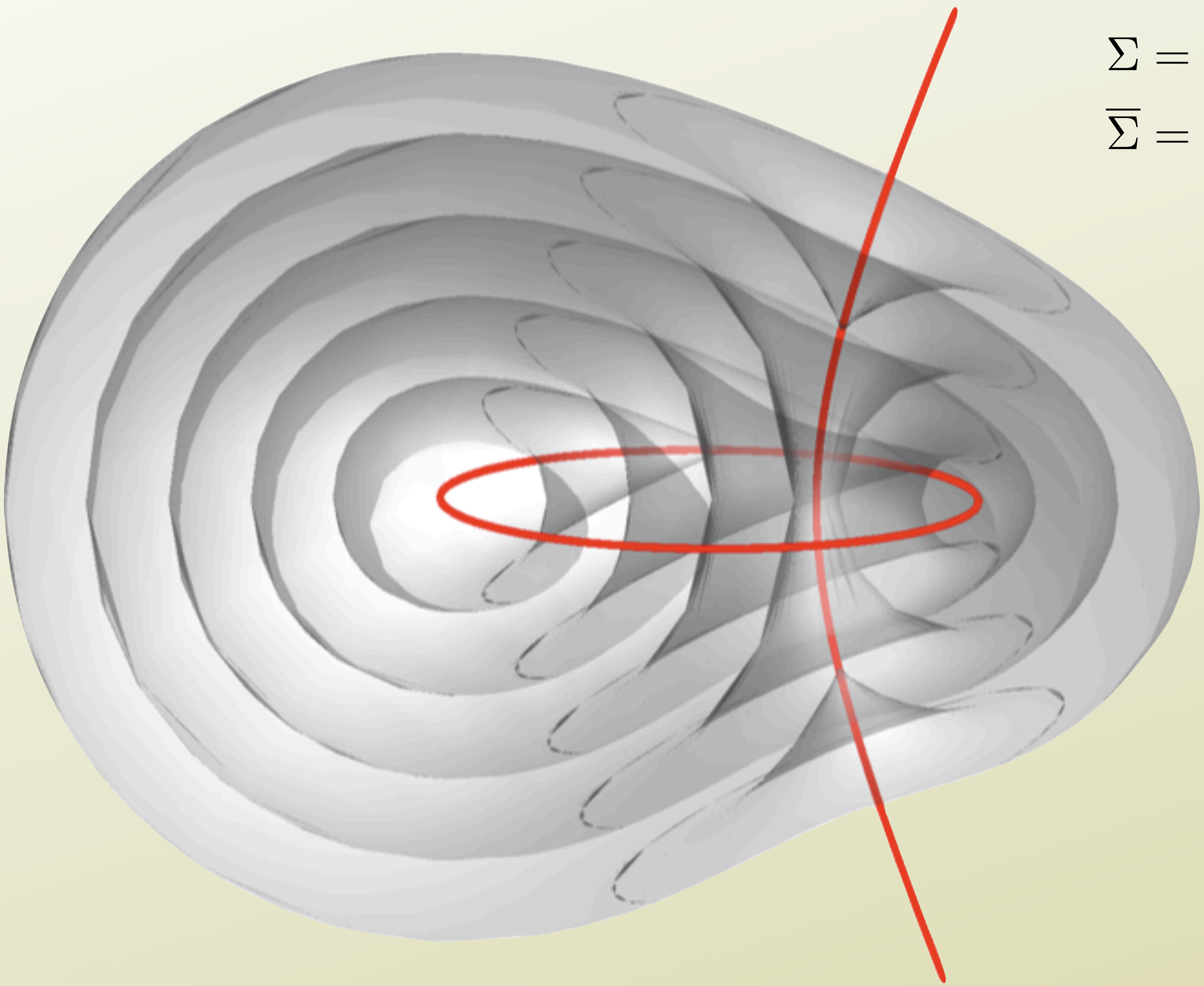
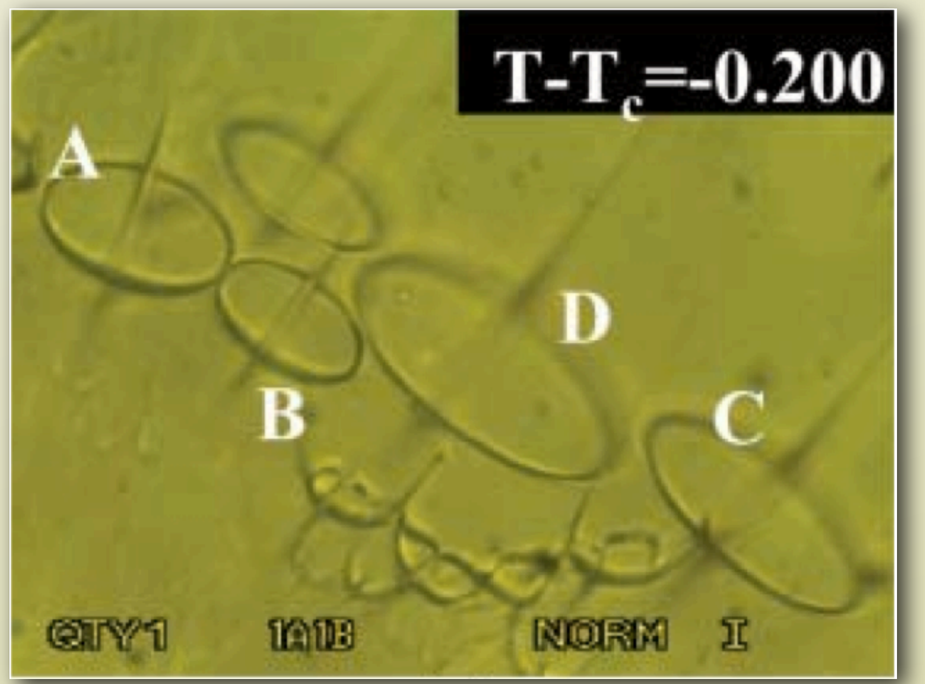
$$\Sigma = \{(0,0,y,z)\text{ s.t. } y^2+z^2=r^2\}$$
$$\overline{\Sigma} = \{(\phi,x,0,0)\text{ s.t } -\phi^2+x^2=-r^2\}$$





Nastishin, Meyer, and Kléman (2008)

Alexander, Chen, Matsumoto, Kamien, (2010)

F. G. Friedlander, *Math. Proc. Camb. Phil. Soc.* **43**, 360-373 (1947)

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FOCAL CONICS

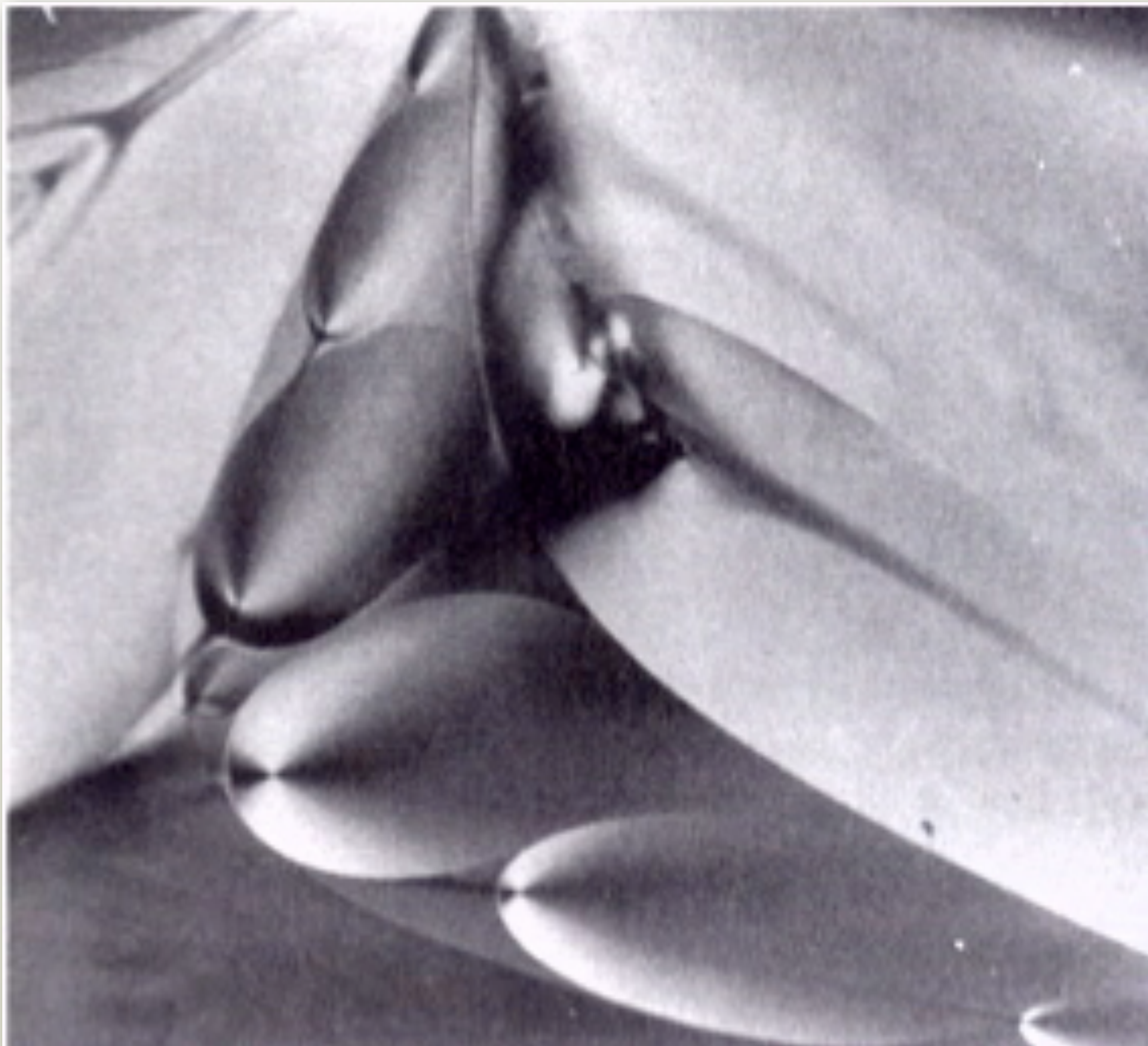


Photo: C. Williams, from de Gennes & Prost

Focal Conics

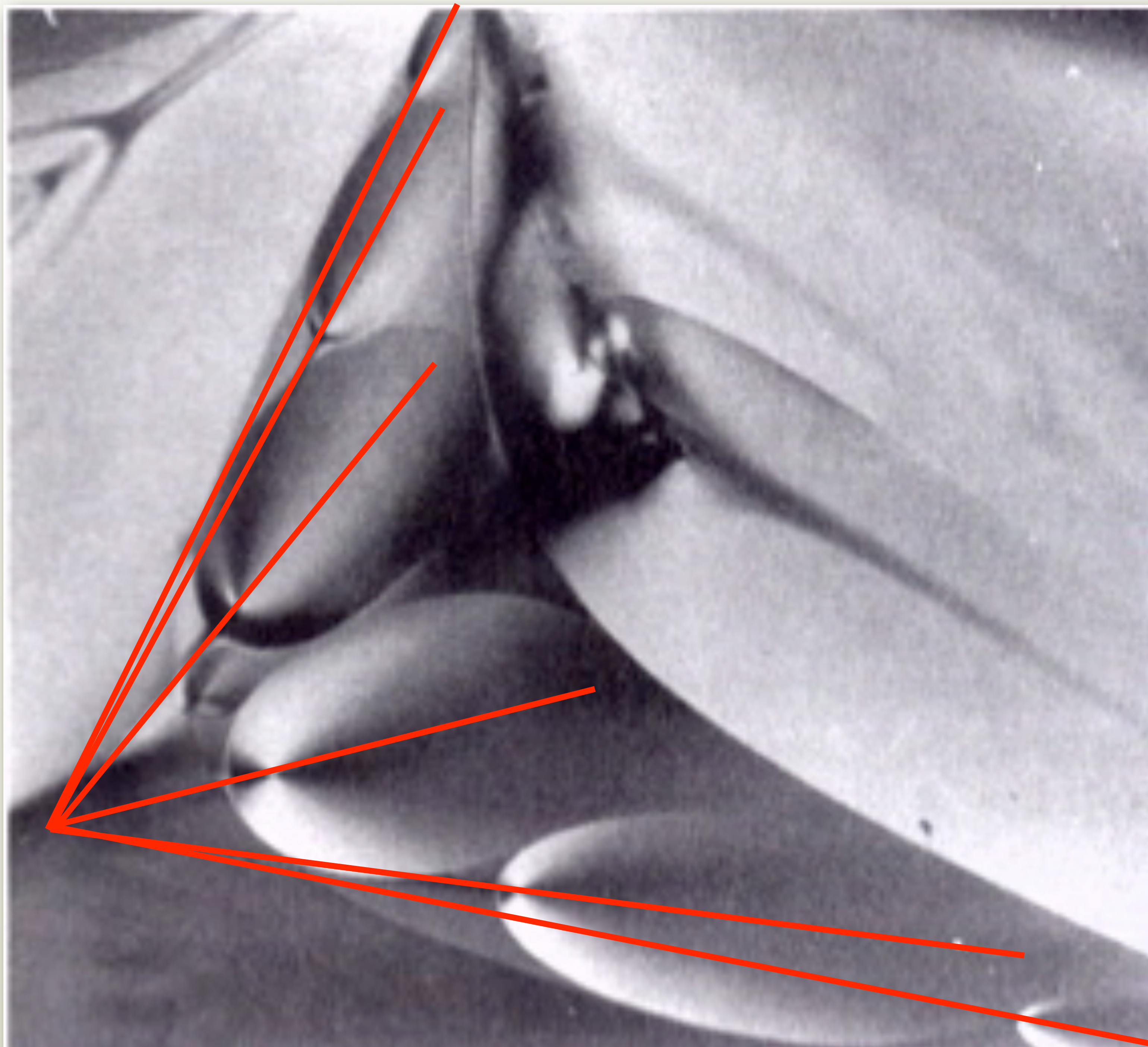
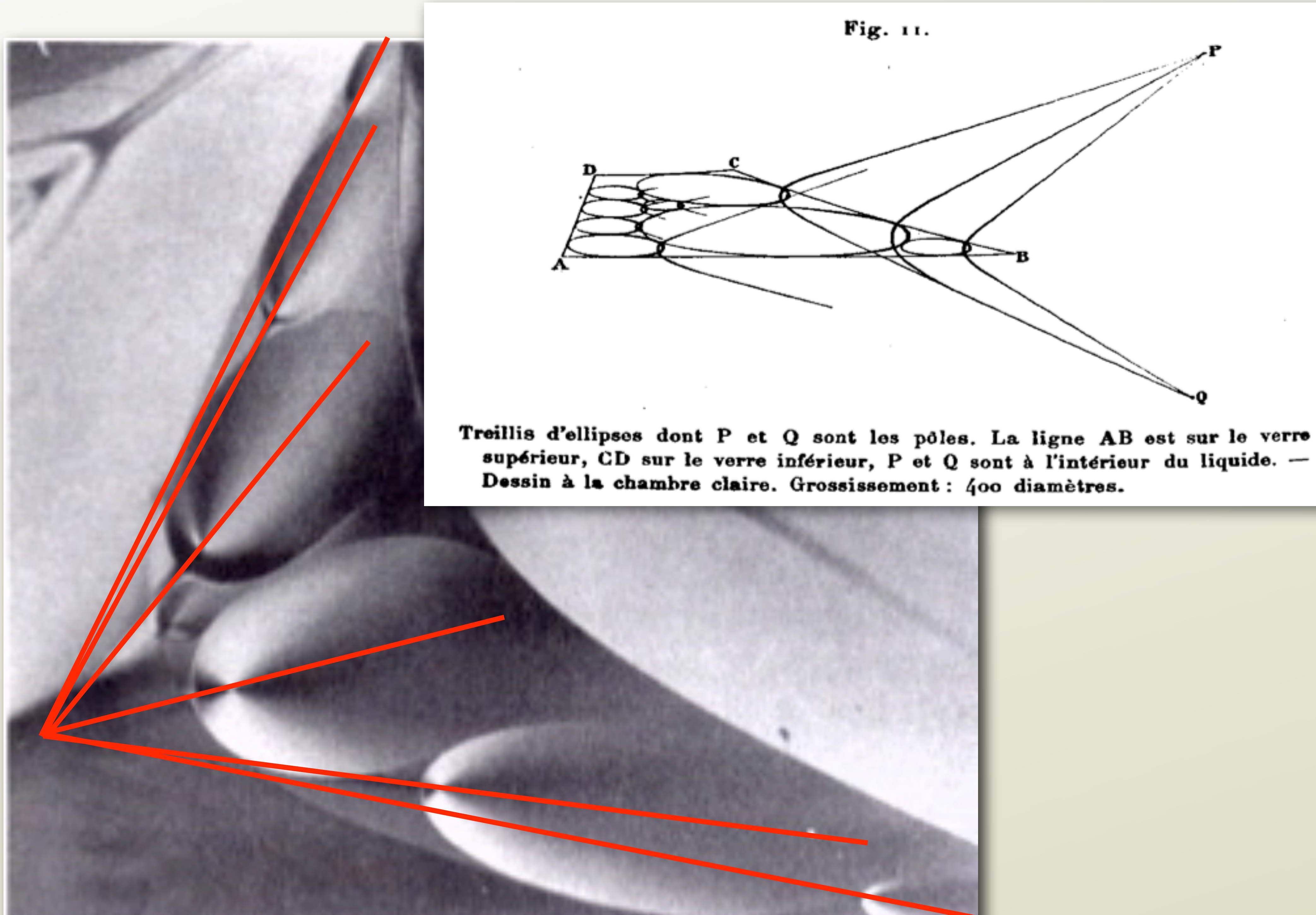


Photo: C. Williams, from de Gennes & Prost

FOCAL CONICS

Friedel, Granjean, *Bull. Soc. Fr. Minéral.* **33**, 409-465 (1910)

Fig. 11.



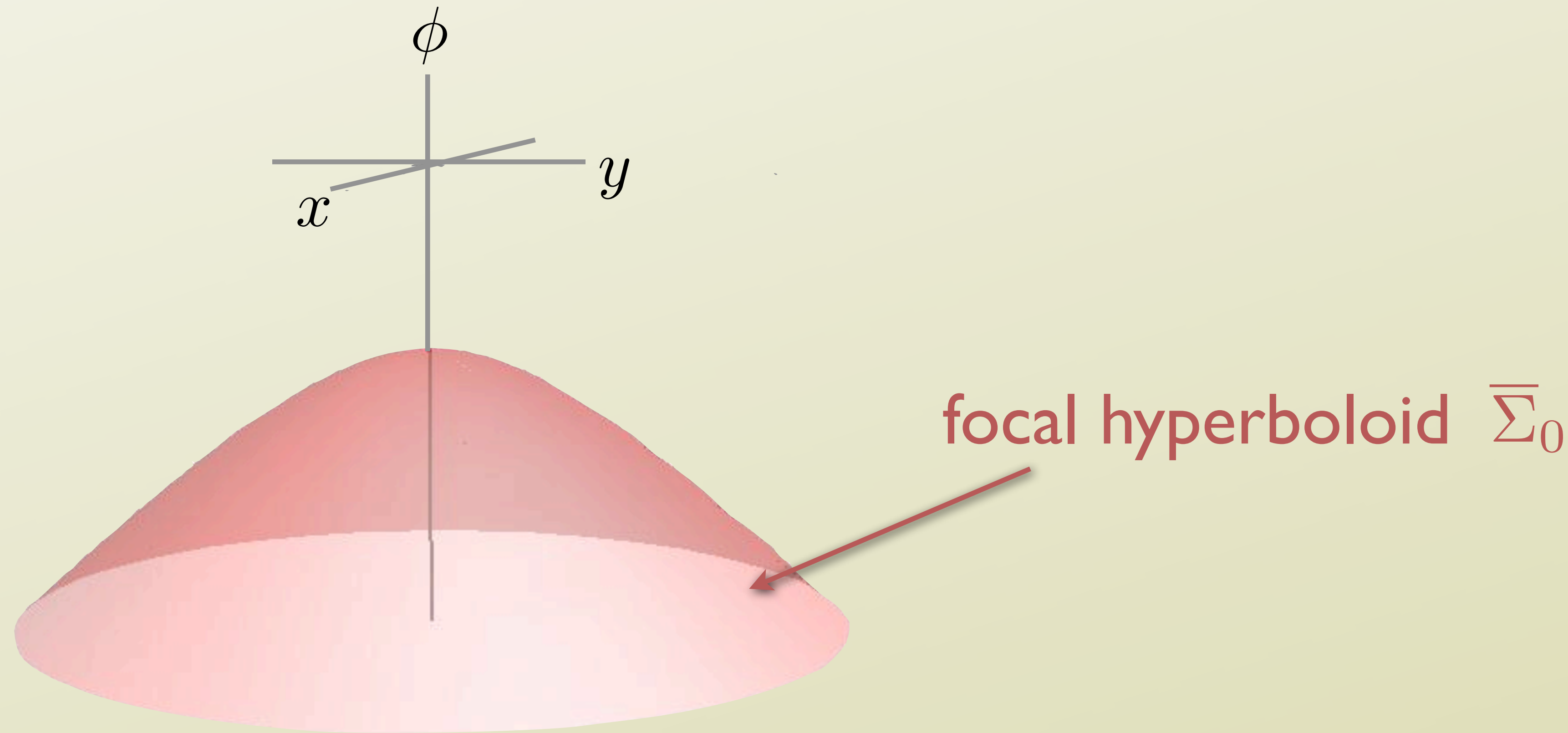
Treillis d'ellipses dont P et Q sont les pôles. La ligne AB est sur le verre supérieur, CD sur le verre inférieur, P et Q sont à l'intérieur du liquide. — Dessin à la chambre claire. Grossissement : 400 diamètres.

Photo: C. Williams, from de Gennes & Prost

NESTED FOCAL SETS

Many ellipses are organised through common points - *view this as a pair of events*

$$\Sigma_0 = \{(0,0,0,z) \text{ s.t. } z^2 = R^2\}$$
$$\overline{\Sigma}_0 = \{(\phi,x,y,0) \text{ s.t. } -\phi^2 + x^2 + y^2 = -R^2\}$$



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Alexander, Chen, Matsumoto, Kamien, (2010)

NESTED FOCAL SETS

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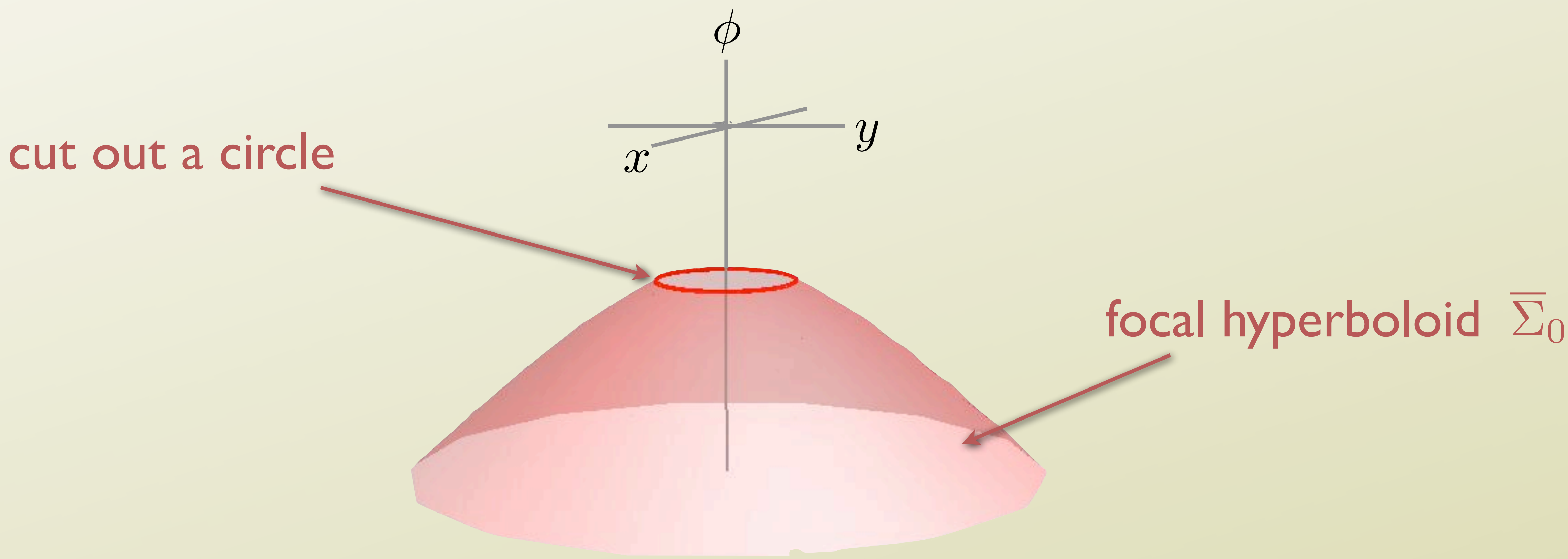

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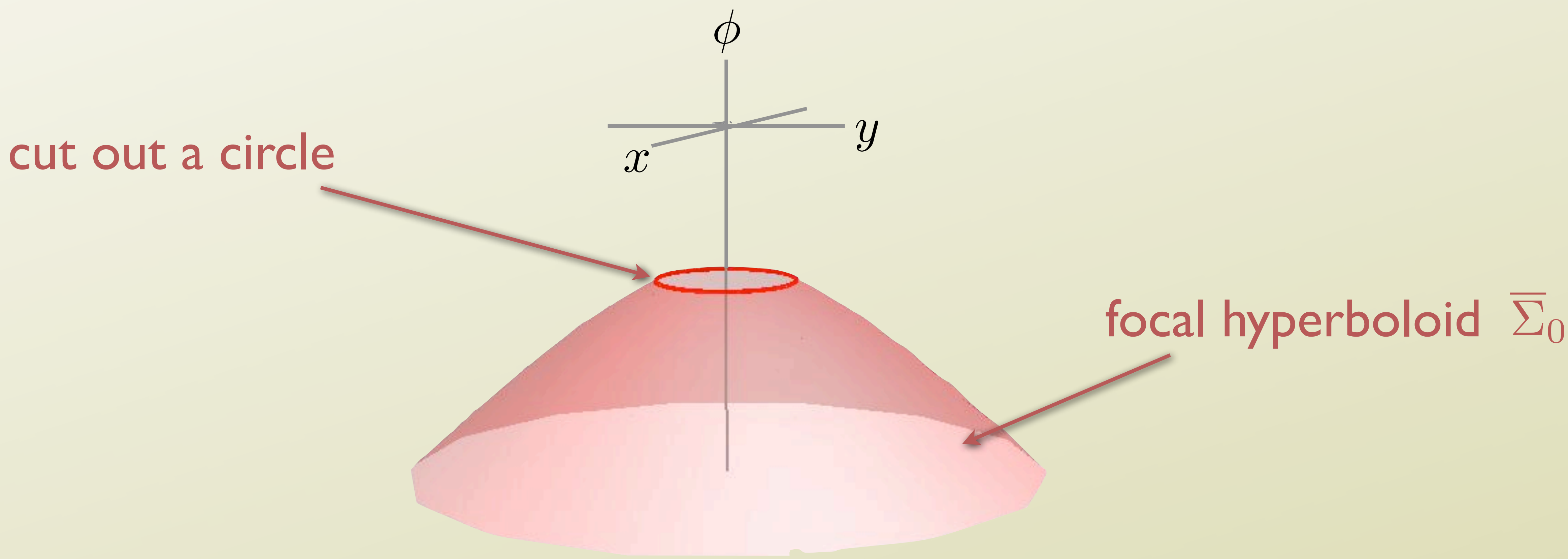
Alexander, Chen, Matsumoto, Kamien, (2010)

NESTED FOCAL SETS

Many ellipses are organised through common points - *view this as a pair of events*

$$\Sigma_0 = \{(0,0,0,z) \text{ s.t. } z^2 = R^2\}$$
$$\overline{\Sigma}_0 = \{(\phi,x,y,0) \text{ s.t. } -\phi^2 + x^2 + y^2 = -R^2\}$$



$$\Sigma_1 = \{(-\sqrt{r^2+R^2},x,y,0) \text{ s.t. } x^2 + y^2 = r^2\}$$
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Alexander, Chen, Matsumoto, Kamien, (2010)

Nested Focal Sets

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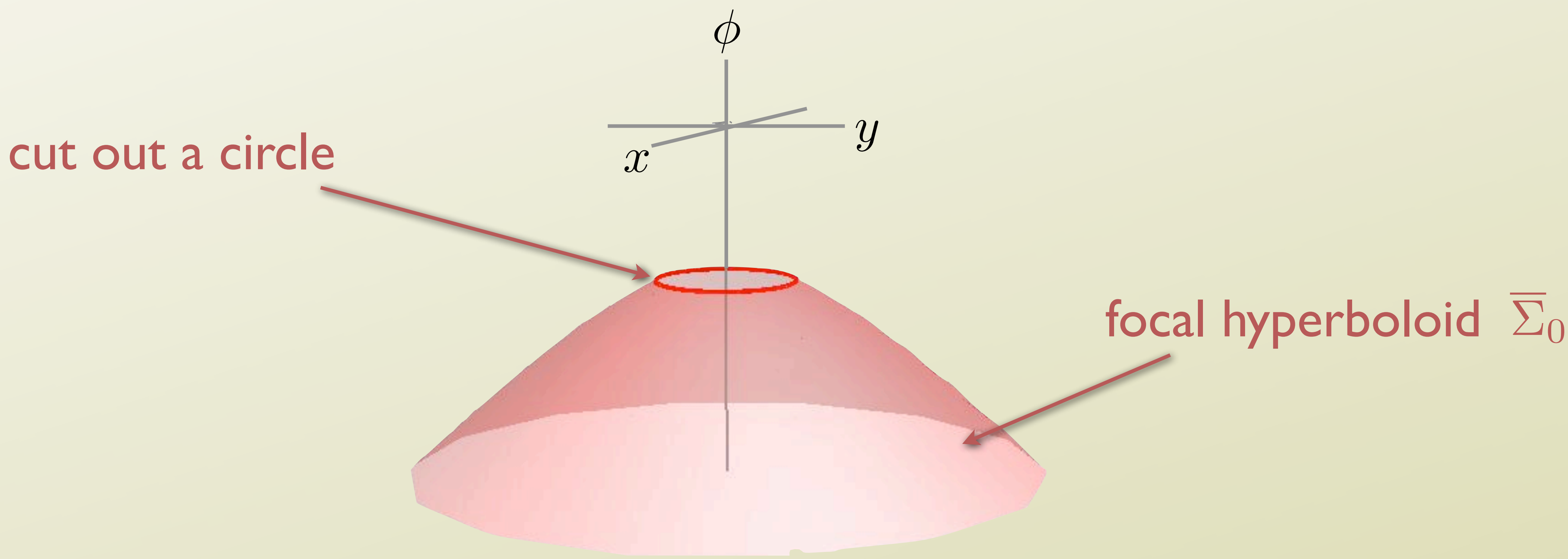

$$\overline{\Sigma}_0 \supset \ \Sigma_1 = \{(-\sqrt{r^2+R^2},x,y,0) \text{ s.t. } x^2+y^2 = r^2\}$$
$$\Sigma_0 \subset \ \overline{\Sigma}_1 = \{(\phi,0,0,z) \text{ s.t. } -(\phi+\sqrt{r^2+R^2})^2 + z^2 = -r^2\}$$

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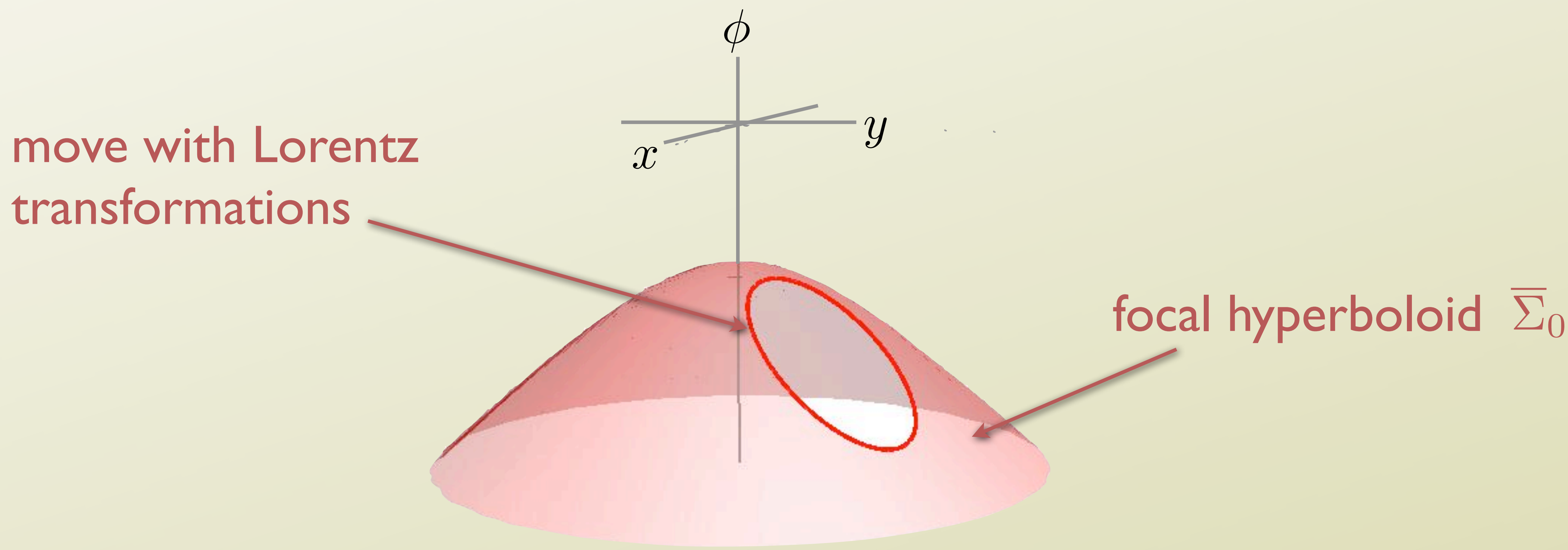

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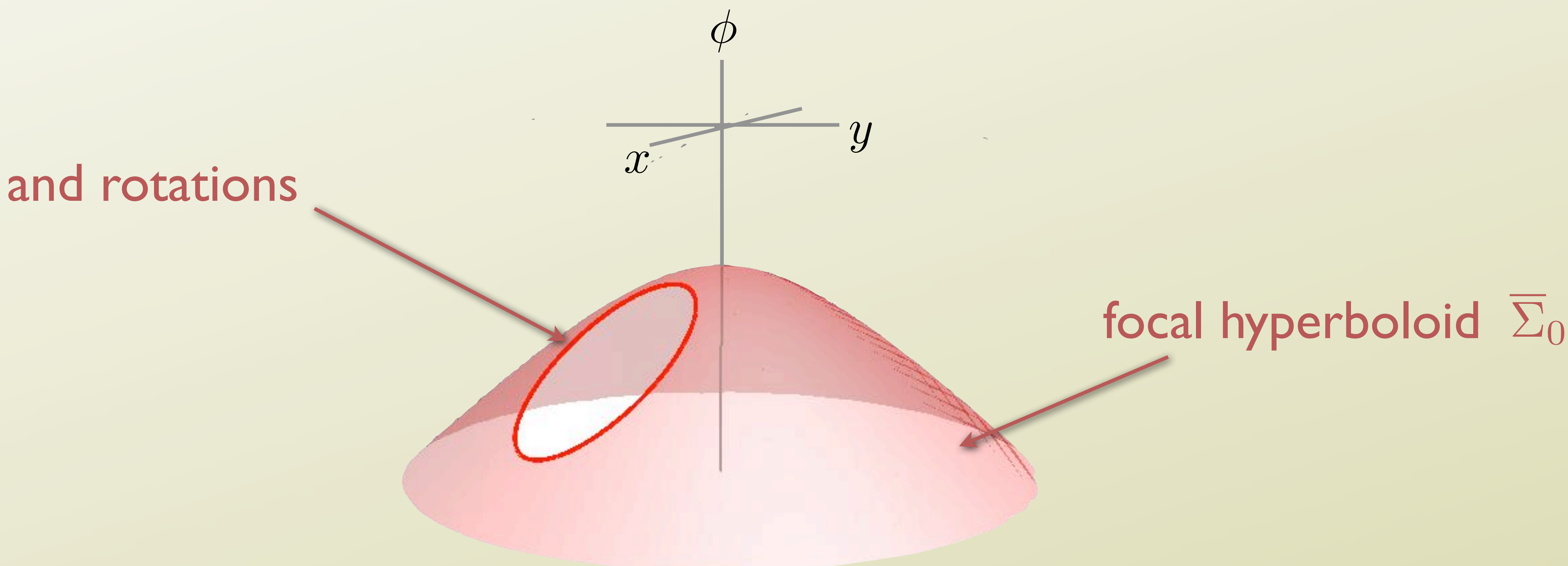

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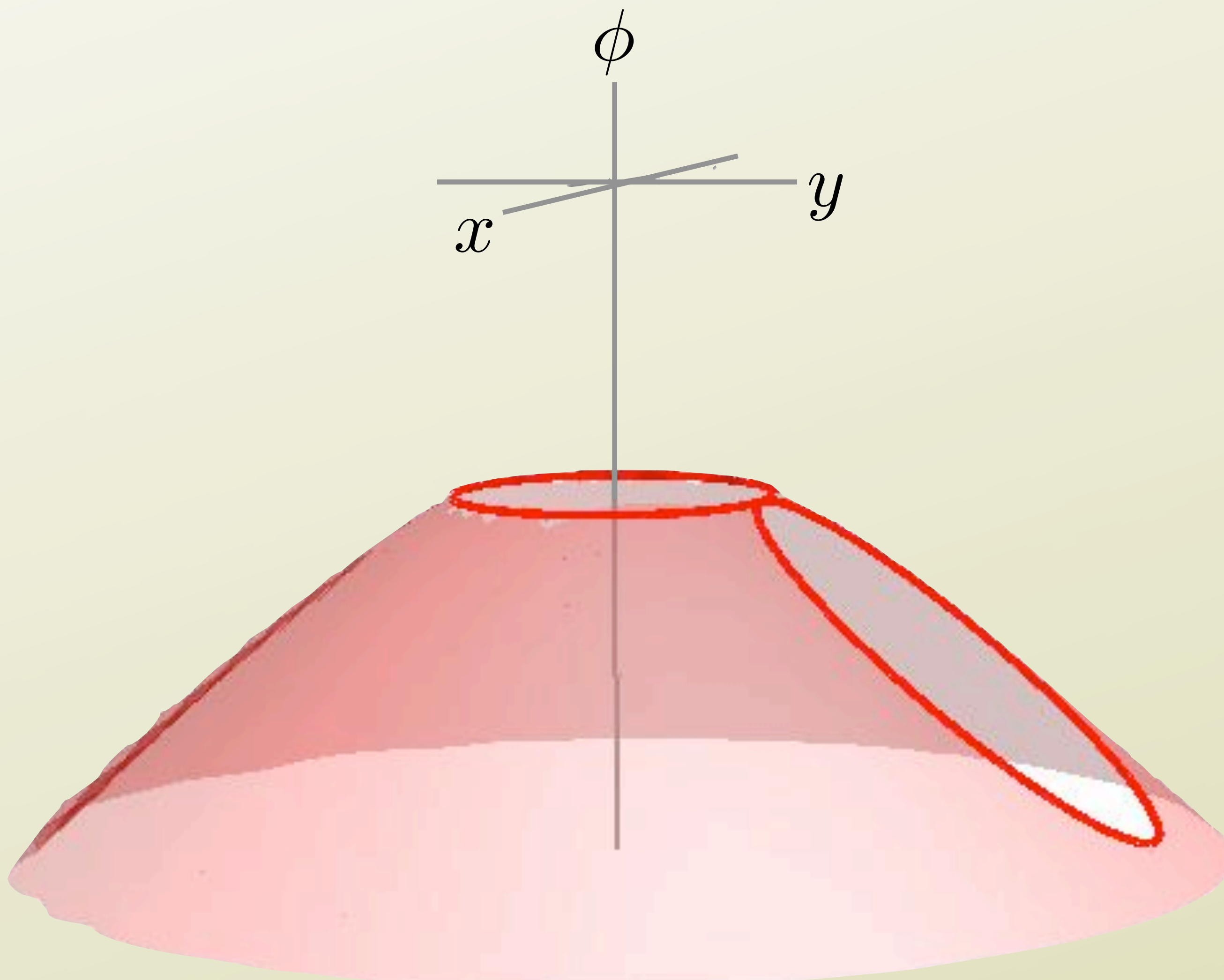
NULL SEPARATION – CORRESPONDING CONES

Two circular subsets with a point in common



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NULL SEPARATION – CORRESPONDING CONES

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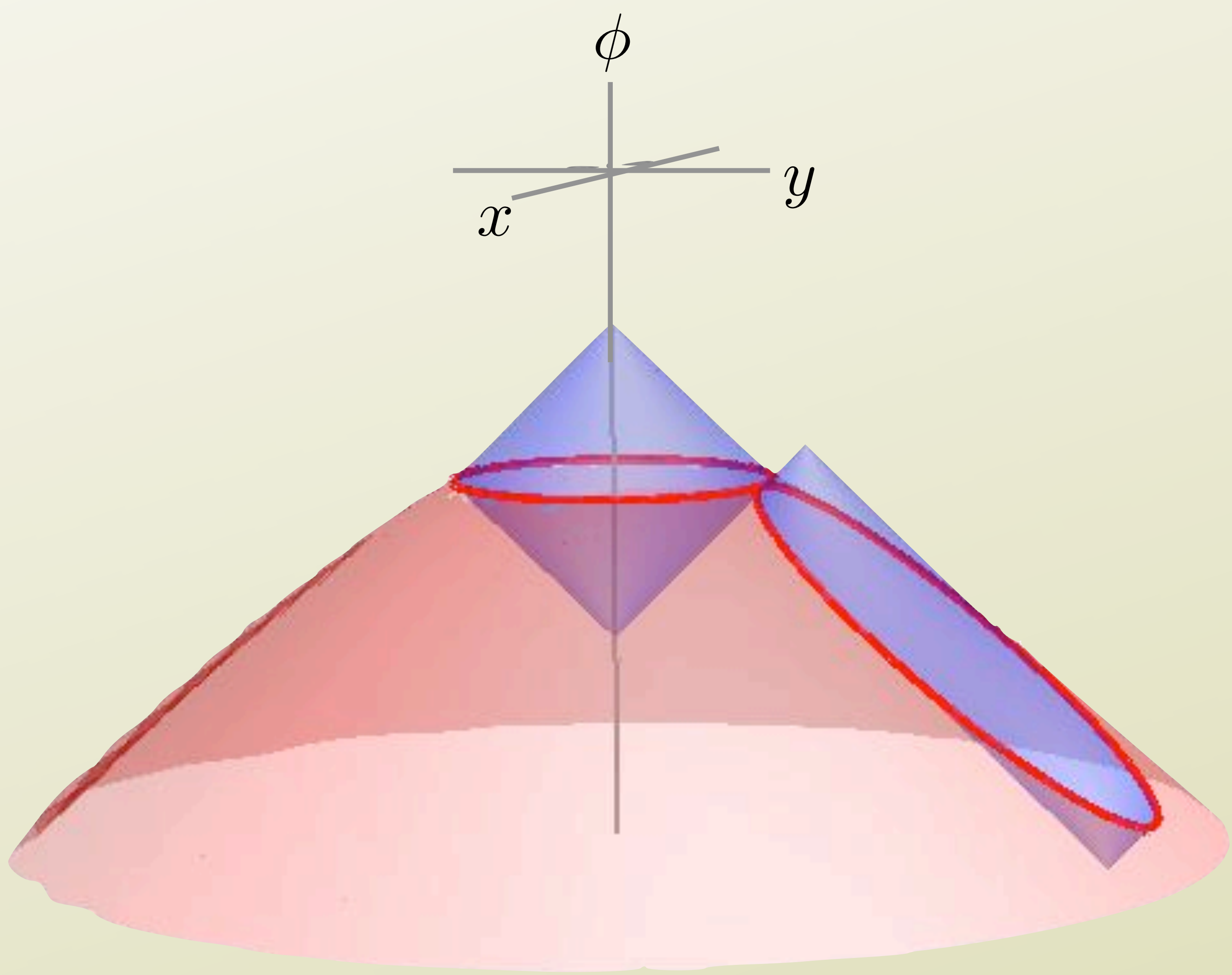
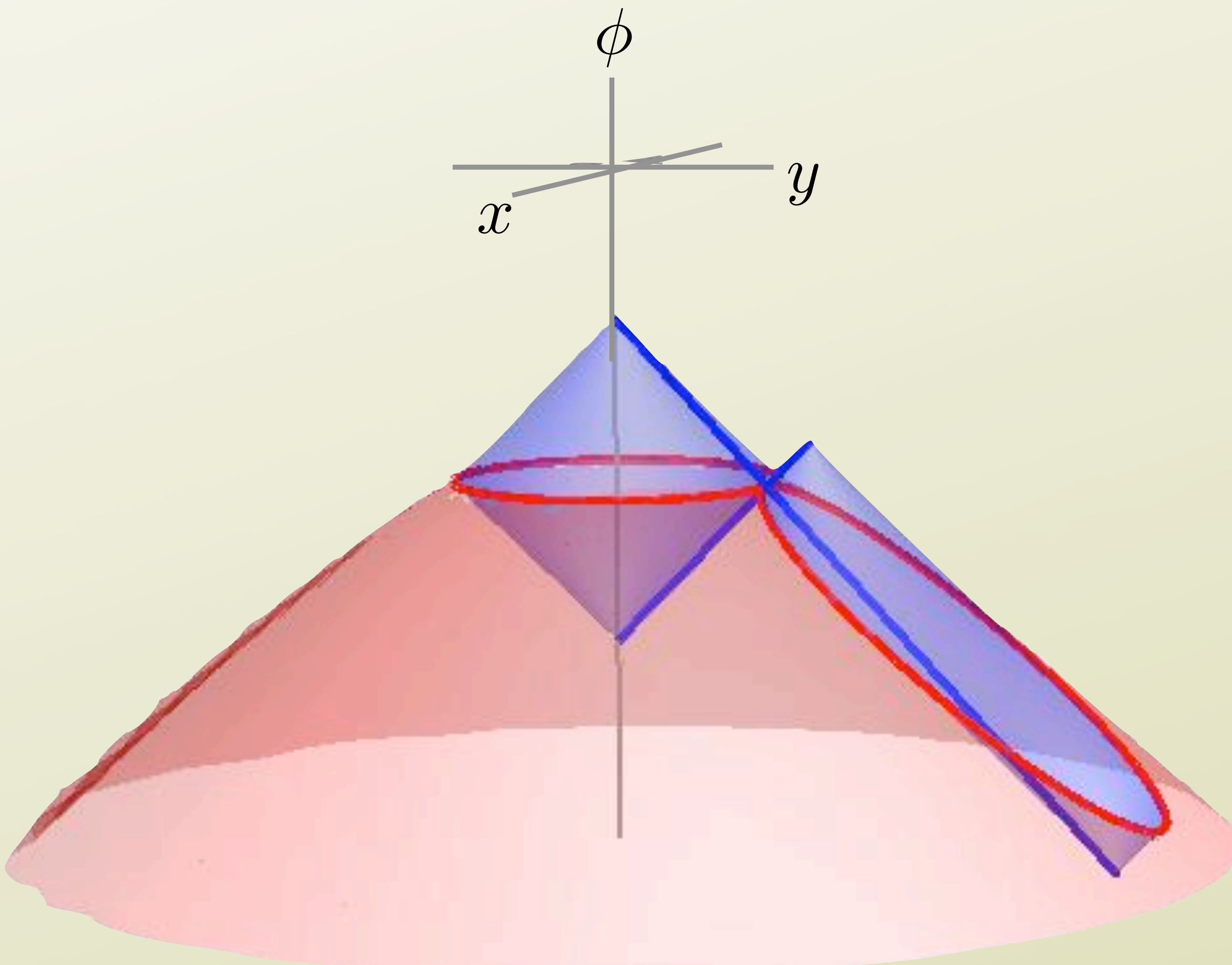


Alexander, Chen, Matsumoto, Kamien, (2010)

Null Separation – Corresponding Cones

Two circular subsets with a point in common



Mutually tangent iff foci are null separated

Alexander, Chen, Matsumoto, Kamien, (2010)

POLYGONAL TEXTURES: TRÉILLIS ET RÉSEAUX

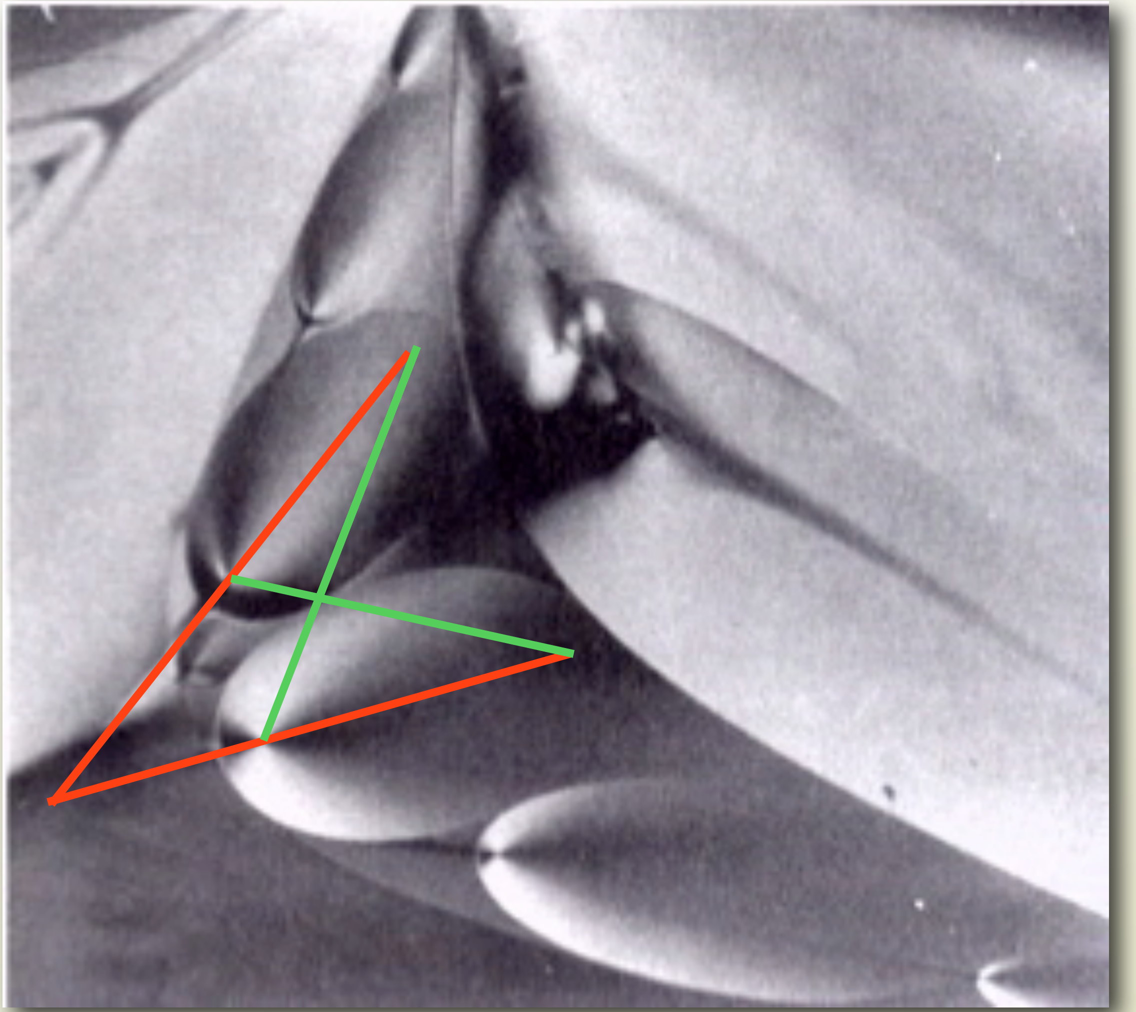
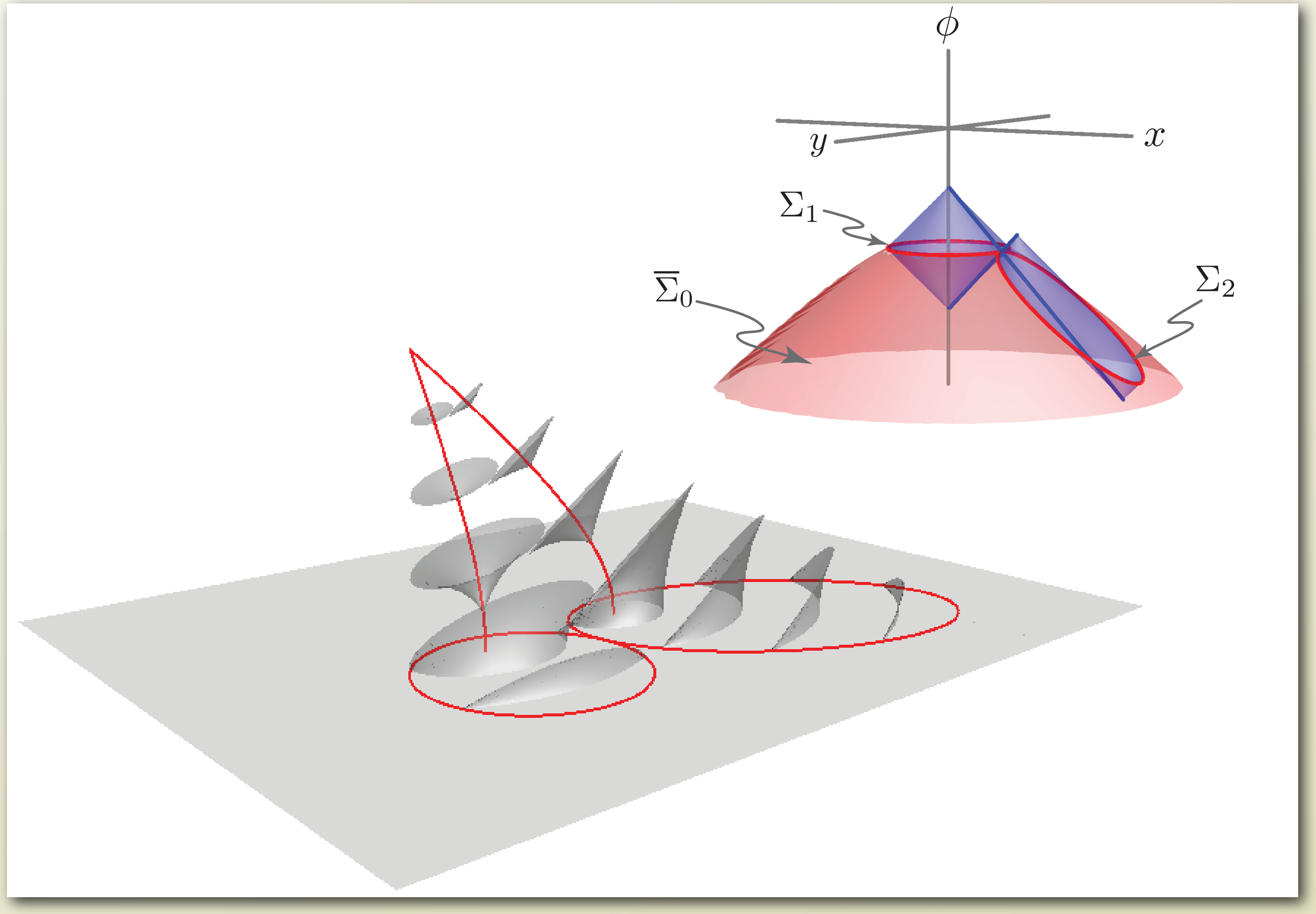


Photo: C. Williams, from de Gennes & Prost



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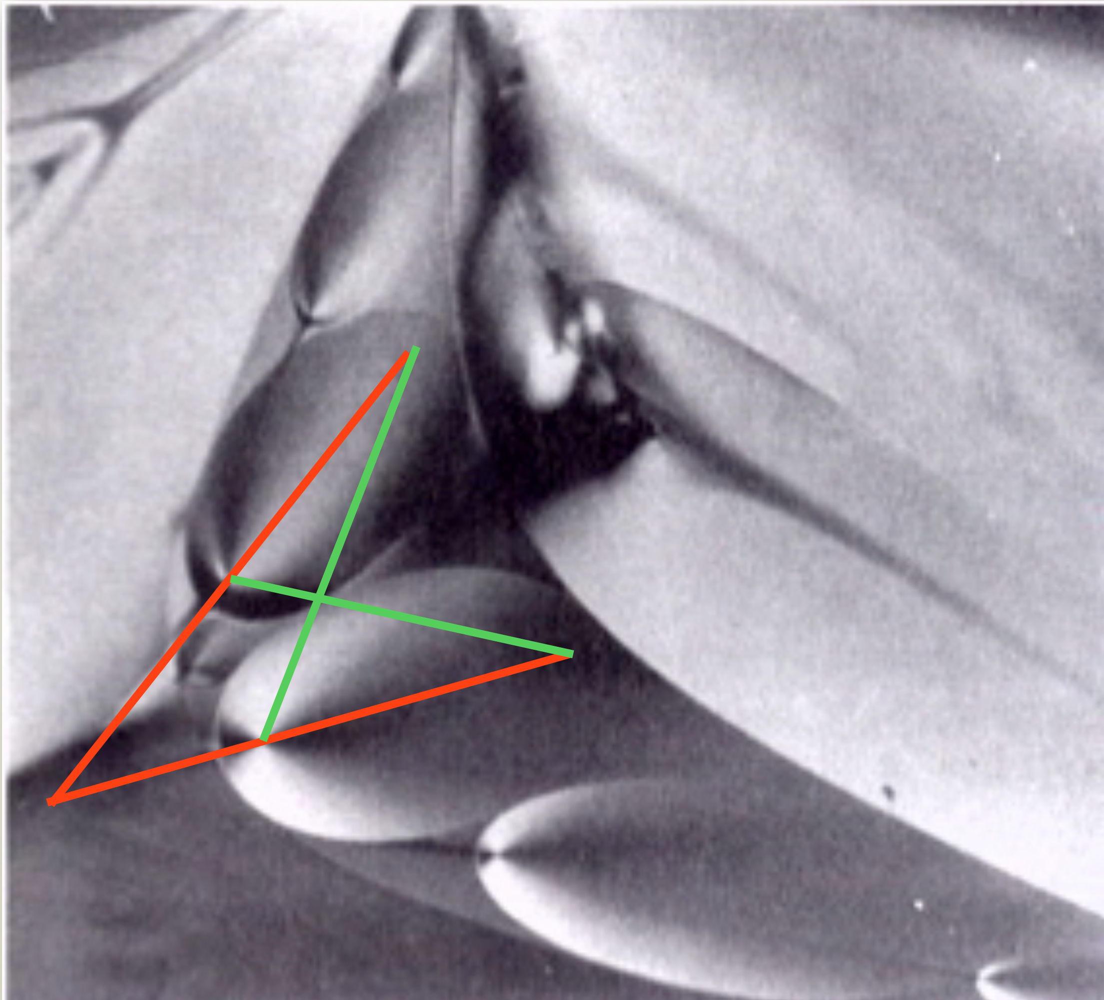
POLYGONAL TEXTURES: TRÉILLIS ET RÉSEAUX



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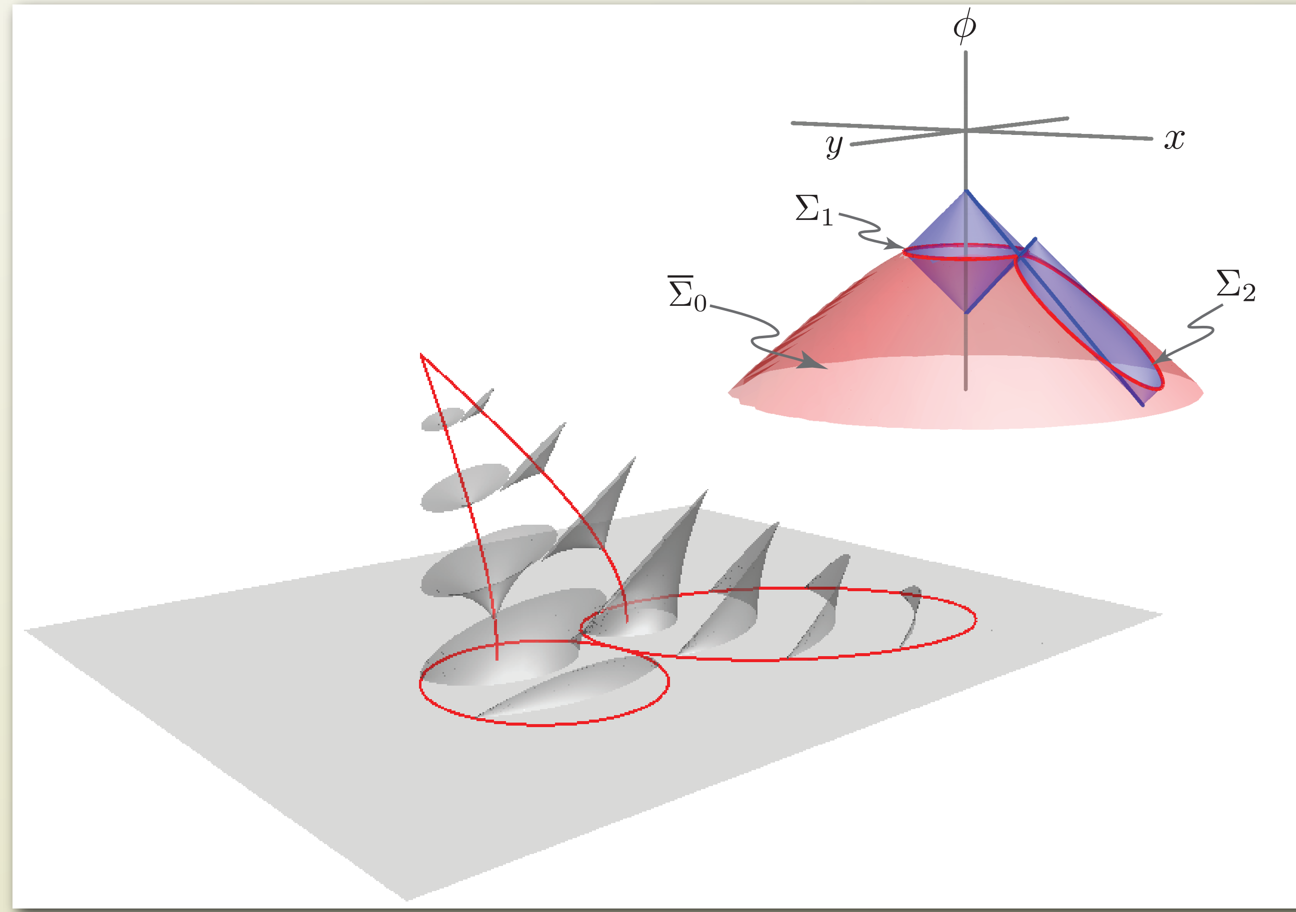

- Multiple tangency of ellipses $\Rightarrow$ Apollonian packing
- "Curvatures" satisfy the *hyperbolic* Déscartes-Soddy-Gossett theorem
- Polygonal boundaries correspond to intersections of hyperboloids

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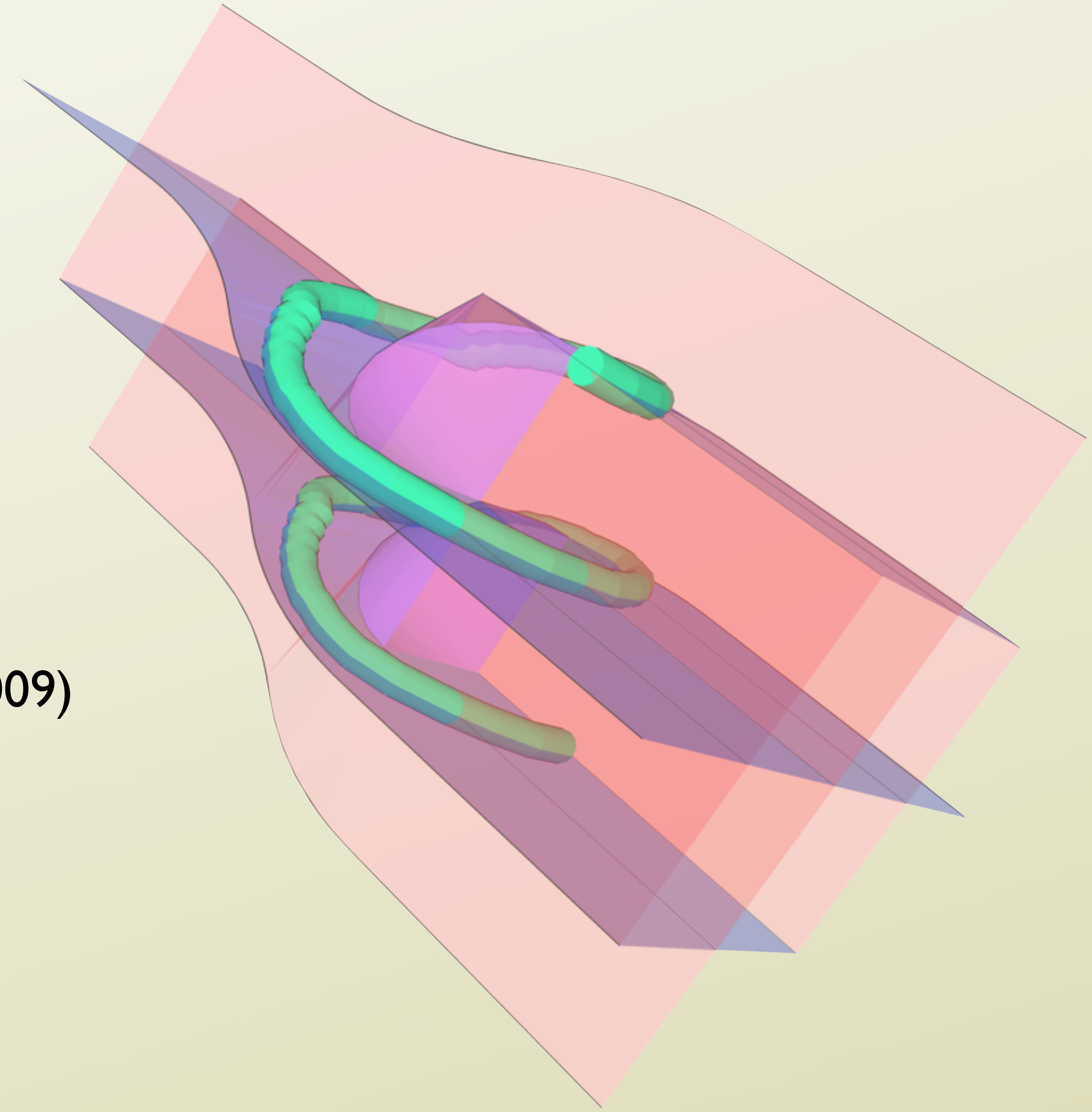
Thanks!



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Chen, Alexander, Kamien, *PNAS* **106**, 15577-15582 (2009)

Alexander, Chen, Matsumoto, Kamien, (2010)

We are grateful to Gary Gibbons, Jim Halverson, Juan Maldacena, Carl Modes and Ari Turner for stimulating discussions

Funding

NSF *DMR05-20020 (PENN MRSEC)*
DMR05-47230
Gifts from L. J. Bernstein and H. H. Coburn

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