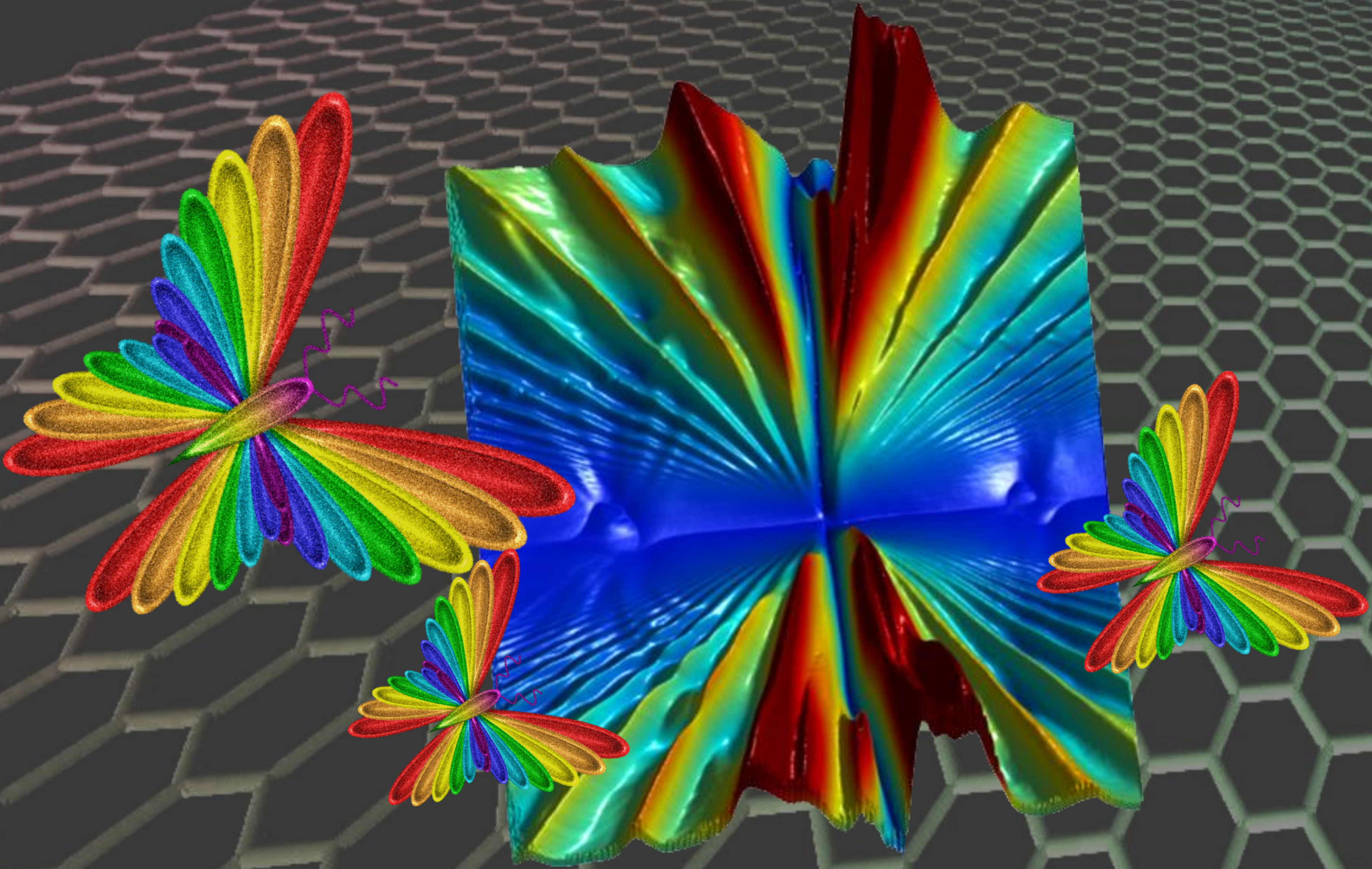
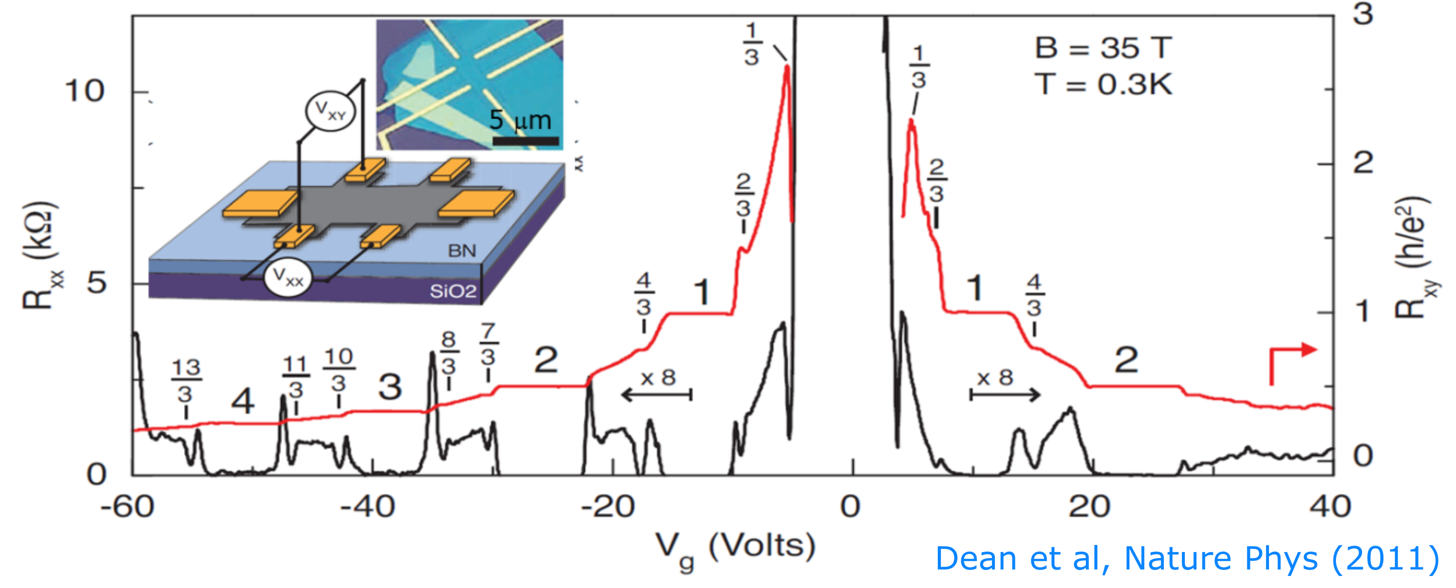
Bloch, Landau, and Dirac : Hofstadter’s Butterfly in Graphene

Philip Kim, Department of Physics, Columbia University

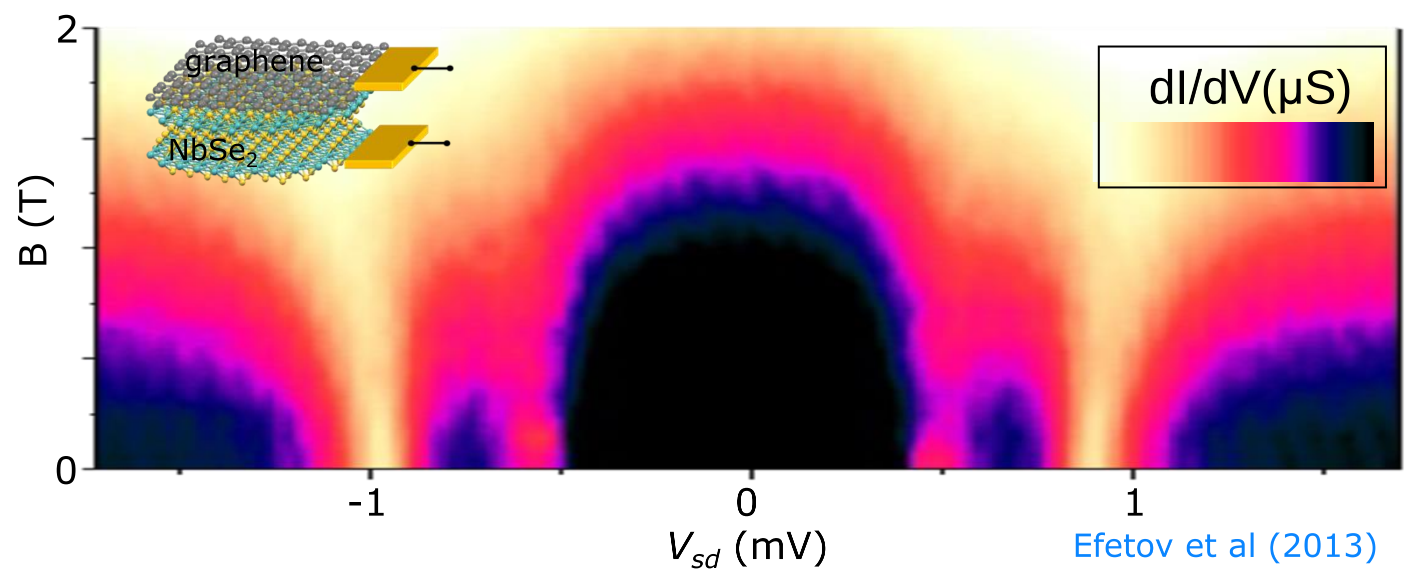


Fractional & Integer Quantum Hall effect in graphene: electron correlation



Dean et al, Nature Phys (2011)

Heteroepitaxy of Layered Materials: Andreev reflection between $NbSe_2$ and graphene



Efetov et al (2013)

Graphene Materials and Applications

Flexible/Transparent
Electrodes/Touch Panels

Transparent Electrodes

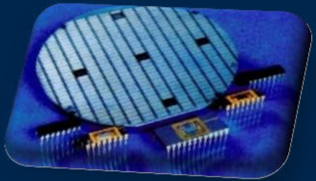
Printable Inks

Conductive Ink,
EMI shields

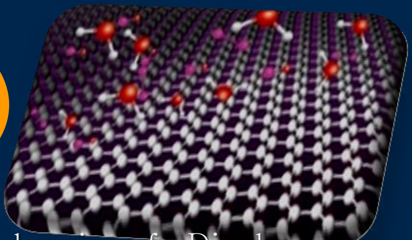
Large-Scale
CVD Graphene
+
Graphene
Nanoplatelet
Composites

Semi-conductors

Gas Barriers



Ultrafast Transistors,
RFIC,
Photo/Bio/Gas Sensors

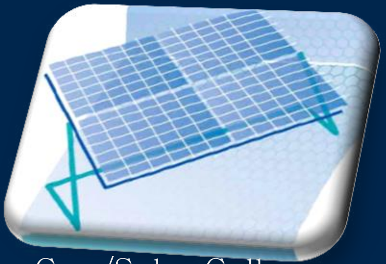


Gas barriers fo Displays,
Solar Cells

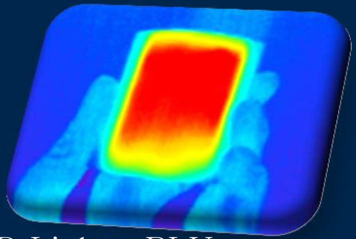
Energy Electrodes

Heat Dissipation

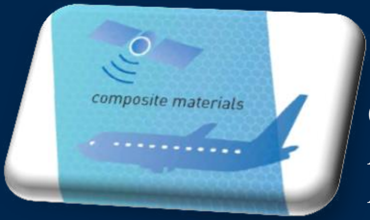
Composites



Super Cap./Solar Cells
Secondary Batteries
Fuel Cells



LED Lights, BLU
ECU, PC …



Cars,
Aerospace
Appliations

Images: Royal Swedish Academy

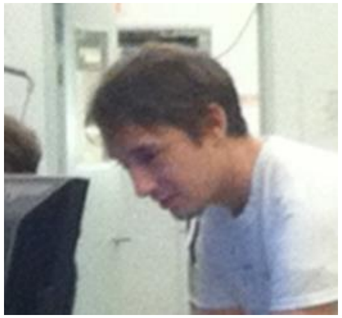
Courtesy: B. H. Hong

Will graphene appear in market soon?

Bloch, Landau, and Dirac : Hofstadter’s Butterfly in Graphene


Dr. Cory Dean

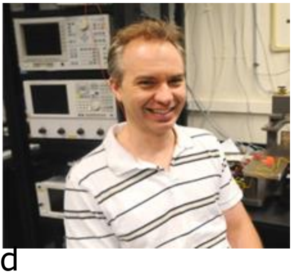

Lei Wang


Patrick Maher


Fereshte Ghahari


Carlos Forsythe

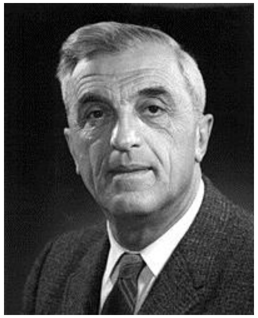

Prof. Jim Hone


Prof. Ken Shepard

Theory: P. Moon & M. Koshino (Tohoku)

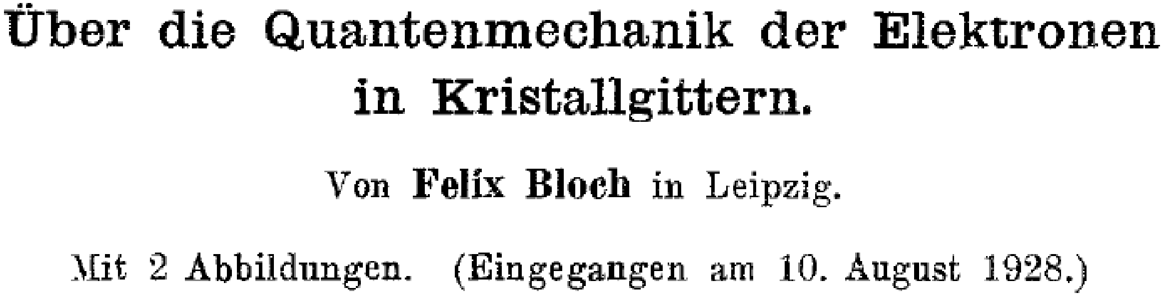
hBN samples: T. Taniguchi & K. Watanabe (NIMS)

Bloch Waves: Periodic Structure & Band Filling



Felix Bloch

Zeitschrift für Physik, 52, 555 (1929)



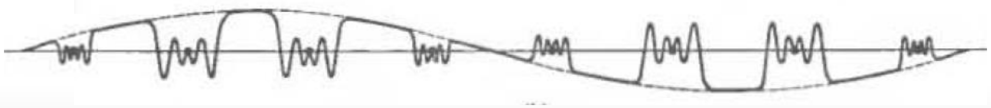
Über die Quantenmechanik der Elektronen in Kristallgittern.

Von **Felix Bloch** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 10. August 1928.)

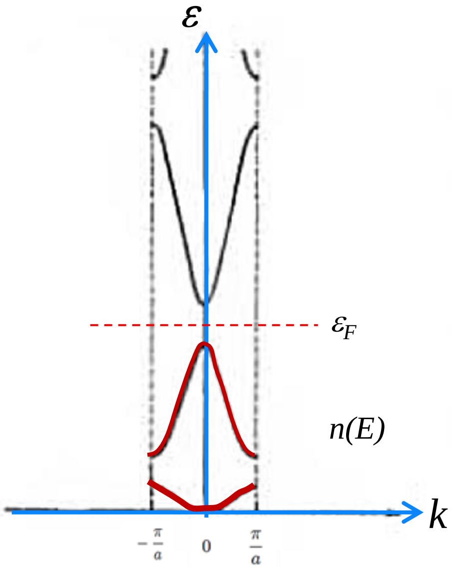
Periodic Lattice

$$\tilde{H} = \frac{\tilde{p}^2}{2m} + U(x), \qquad U(x) = U(x+a)$$



Block Waves:

$$\psi_{n,k}(x) = e^{ikx} u_{n,k}(x), \qquad u_{n,k}(x+a) = u_{n,k}(x)$$

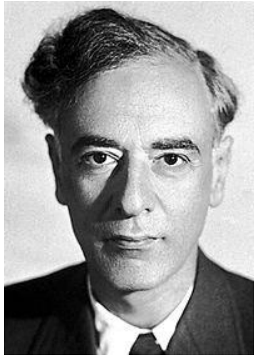


Band Filling factor

$$s = -A_0^2 \frac{\partial n(\varepsilon_F)}{\partial A_0}$$

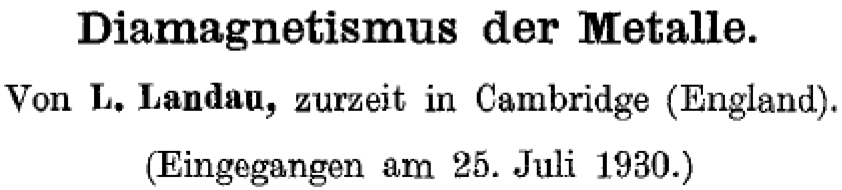
MacDonald (1983)

Landau Levels: Quantization of Cyclotron Orbits



Lev Landau

Zeitschrift für Physik, 64, 629 (1930)



Diamagnetismus der Metalle.

Von **L. Landau,** zurzeit in Cambridge (England).

(Eingegangen am 25. Juli 1930.)

Free electron under magnetic field

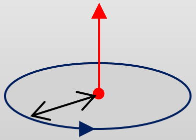
$$\tilde{H} = \frac{(\tilde{\mathrm{p}} - e\mathbf{A}/c)^2}{2m}$$

Energy and orbit are quantized:

$$\varepsilon_n = \hbar w_c(n + 1/2), \qquad w_c = eB/mc$$

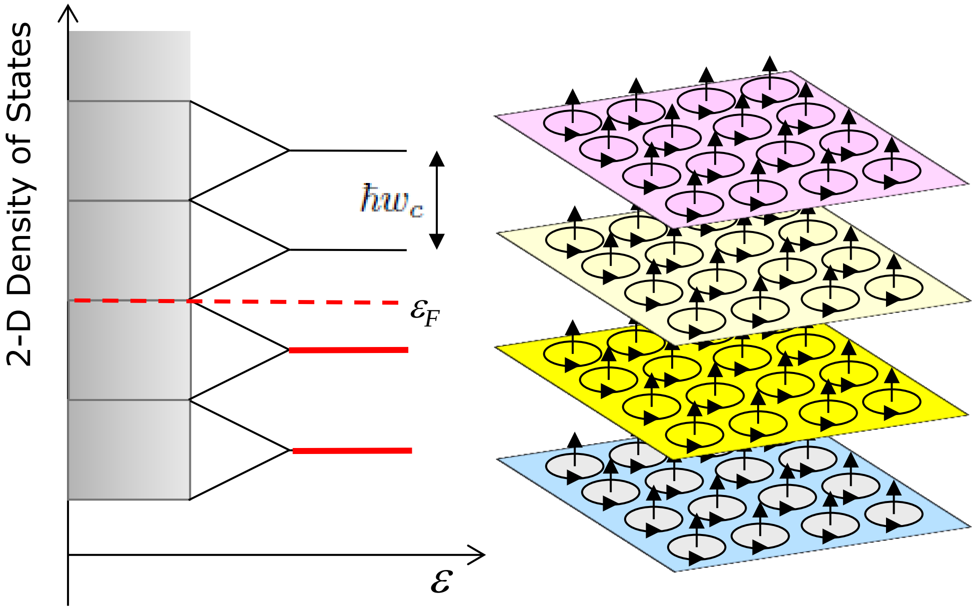
Each Landau orbit contains magnetic flux quanta

$$\phi_0 = \frac{hc}{e}$$



$$\ell_B = \sqrt{\hbar/eB}$$

2-dimensional electron systems

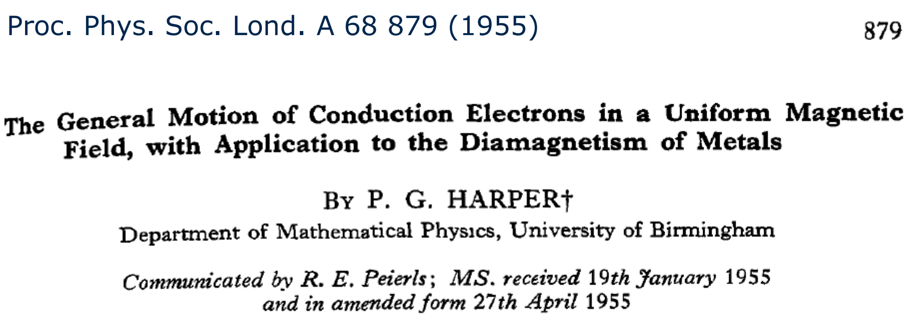


Massively degenerated energy level

Landau level filling fraction:

$$\nu = 2\pi\ell_B^2 n(\varepsilon_F)$$

Harper’s Equation: Competition of Two Length Scales



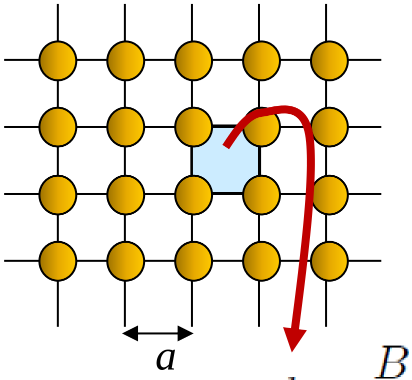
Proc. Phys. Soc. Lond. A 68 879 (1955)

879

The General Motion of Conduction Electrons in a Uniform Magnetic Field, with Application to the Diamagnetism of Metals

By P. G. HARPER†
Department of Mathematical Physics, University of Birmingham

Communicated by R. E. Peierls; MS. received 19th January 1955 and in amended form 27th April 1955



$$b = \frac{Ba^2}{\phi_0} = \frac{\phi}{\phi_0}$$

Tight binding on 2D Square lattice with magnetic field

$$\tilde{H} = \frac{(\tilde{\mathbf{p}} - e\mathbf{A}/c)^2}{2m} + U(\mathbf{r})$$

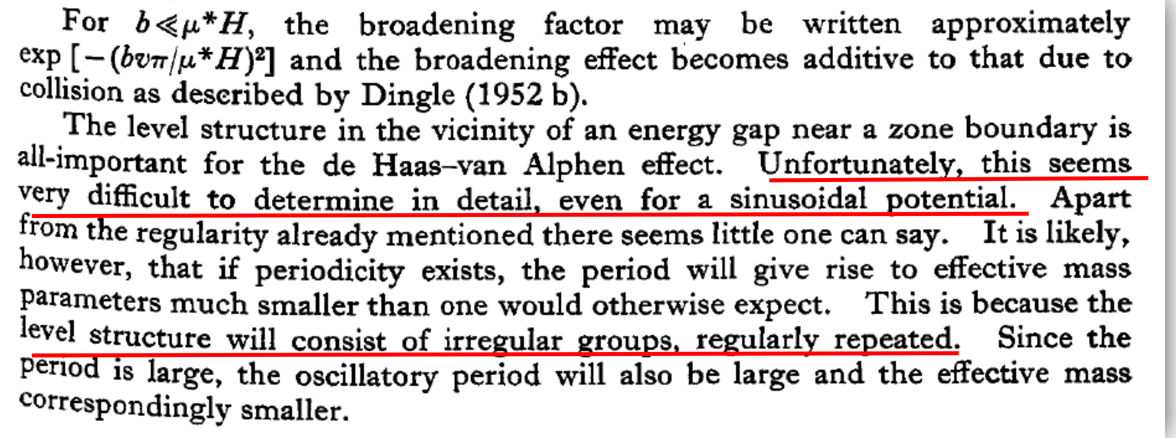
Two competing length scales:

a **: lattice periodicity**

$\boldsymbol{l_B}$ **: magnetic periodicity**

Harper’s Equation

$$2\psi_l \cos\left(2\pi l b - \kappa\right) + \psi_{l+1} + \psi_{l-1} = E\psi_l$$

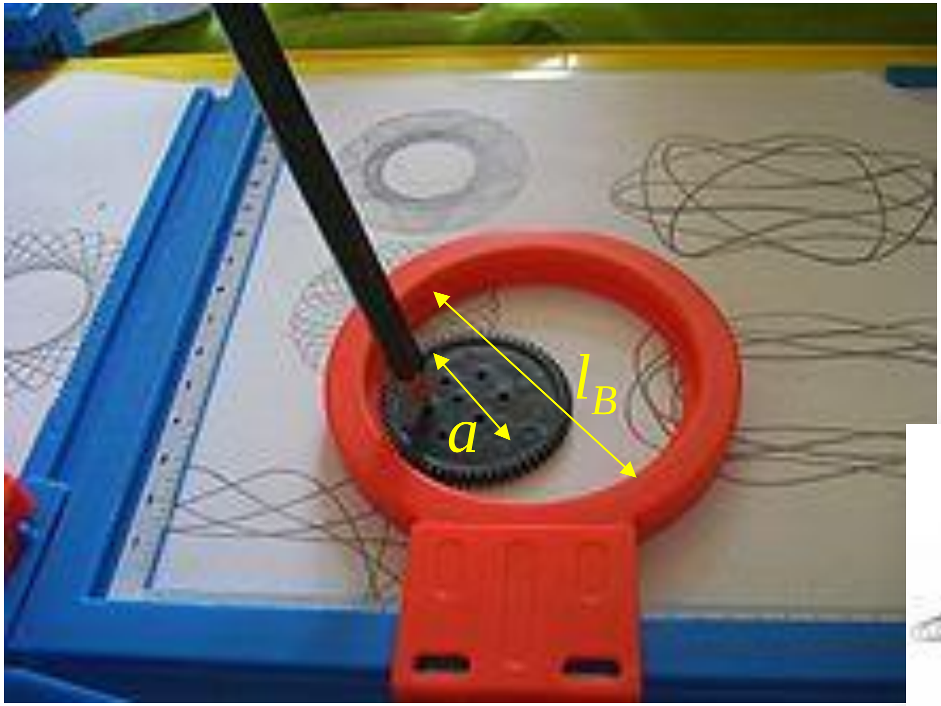


For $b \ll \mu^* H$, the broadening factor may be written approximately exp $[-(bv\pi/\mu^*H)^2]$ and the broadening effect becomes additive to that due to collision as described by Dingle (1952 b).

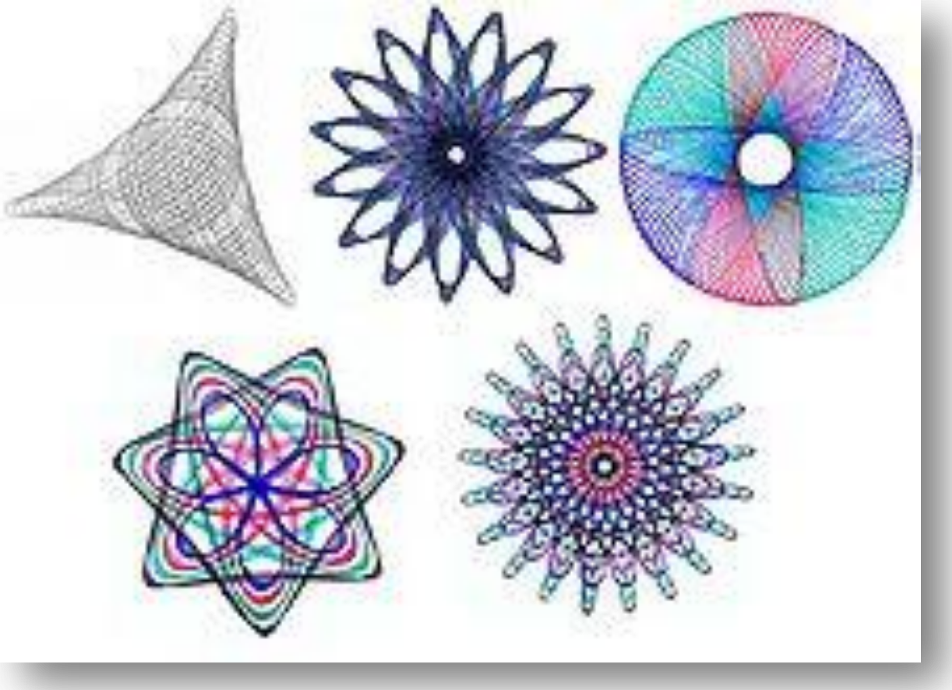
The level structure in the vicinity of an energy gap near a zone boundary is all-important for the de Haas–van Alphen effect. Unfortunately, this seems very difficult to determine in detail, even for a sinusoidal potential. Apart from the regularity already mentioned there seems little one can say. It is likely, however, that if periodicity exists, the period will give rise to effective mass parameters much smaller than one would otherwise expect. This is because the level structure will consist of irregular groups, regularly repeated. Since the period is large, the oscillatory period will also be large and the effective mass correspondingly smaller.

Commensuration / Incommensuration of Two Length Scales

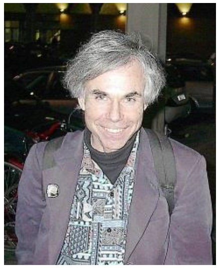
Spirograph

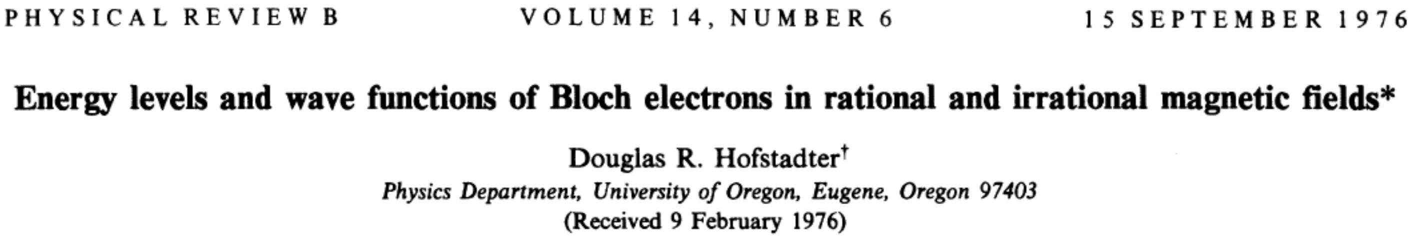


$$a \,/\, l_B = p/q$$



Hofstadter's Butterfly





PHYSICAL REVIEW B VOLUME 14, NUMBER 6 15 SEPTEMBER 1976

Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*

Douglas R. Hofstadter†

Physics Department, University of Oregon, Eugene, Oregon 97403

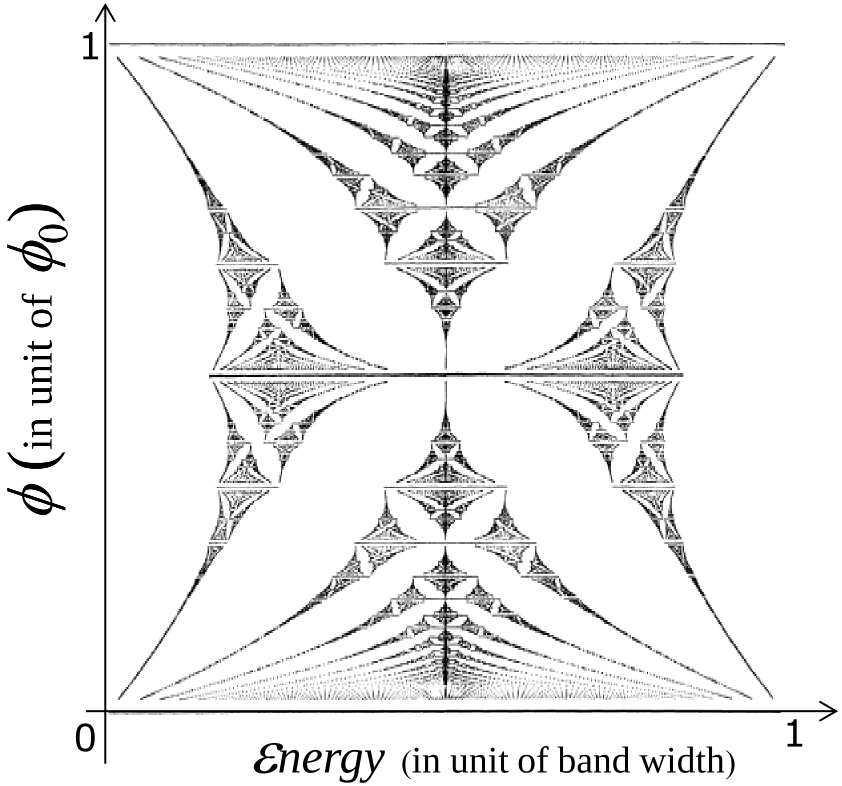
(Received 9 February 1976)

Harper's Equation

$$2\psi_l \cos\left(2\pi l b - \kappa\right) + \psi_{l+1} + \psi_{l-1} = E\psi_l$$

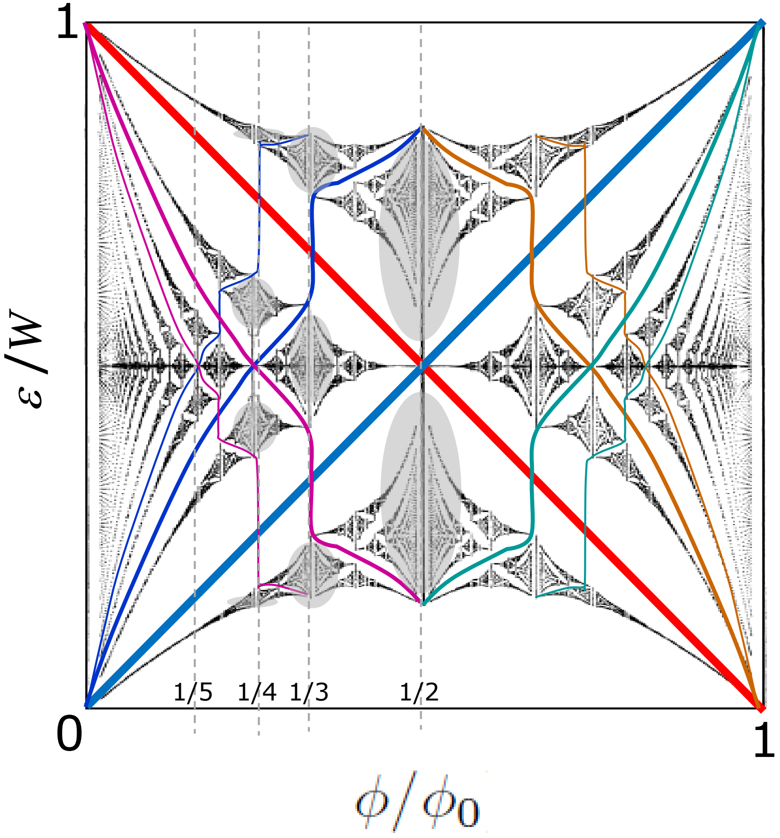
When *b*=*p*/*q*, where *p*, *q* are coprimes, each LL splits into ***q* sub-bands that are *p*-fold degenerate**

Energy bands develop ***fractal structure*** when magnetic length is of order the periodic unit cell



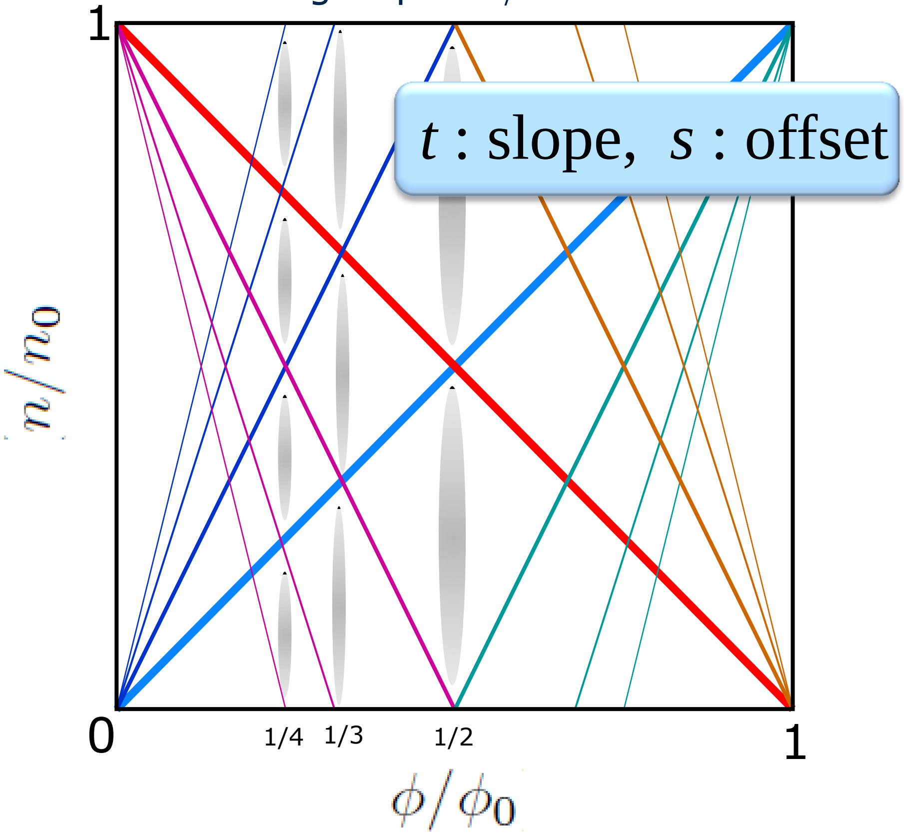
Energy Gaps in the Butterfly: Wannier Diagram

Hofstadter's Energy Spectrum



Wannier, *Phys. Status Solidi.* **88**, 757 (1978)

Tracing Gaps in ϕ and n



t : slope, s : offset

n_0: # of state per unit cell
ϕ : magnetic flux in unit cell
n : electron density

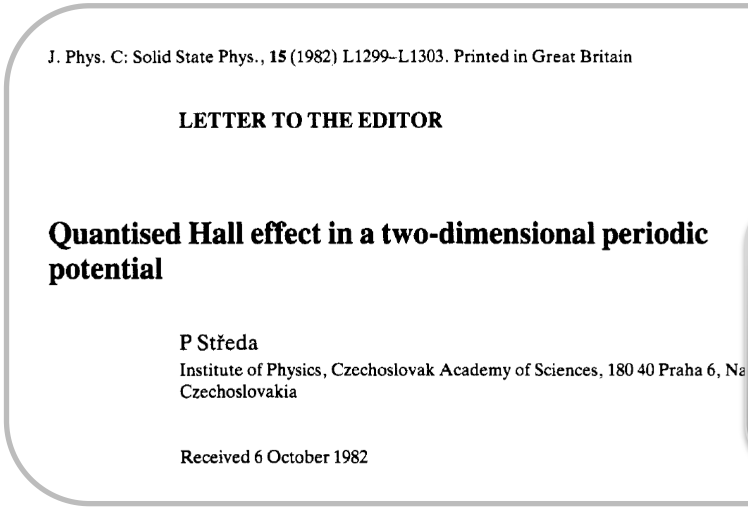
Diophantine equation for gaps

$$(n/n_0) = t(\phi/\phi_0) + s$$

$t, s \in \mathbb{Z}$

Streda Formula and TKNN Integers

What is the physical meaning of the integers *s* and *t* ?



J. Phys. C: Solid State Phys., **15** (1982) L1299–L1303. Printed in Great Britain

LETTER TO THE EDITOR

Quantised Hall effect in a two-dimensional periodic potential

P Středa
Institute of Physics, Czechoslovak Academy of Sciences, 180 40 Praha 6, Na
Czechoslovakia

Received 6 October 1982

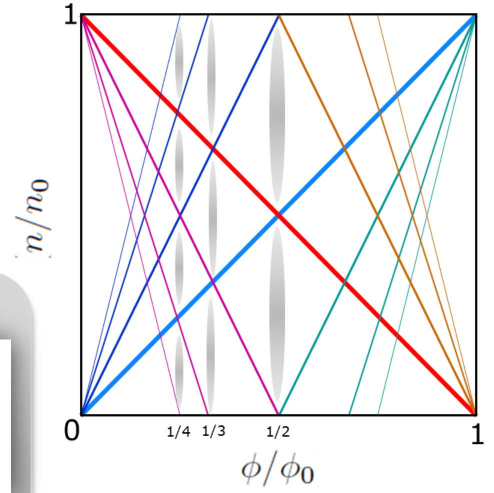
Band Filling factor

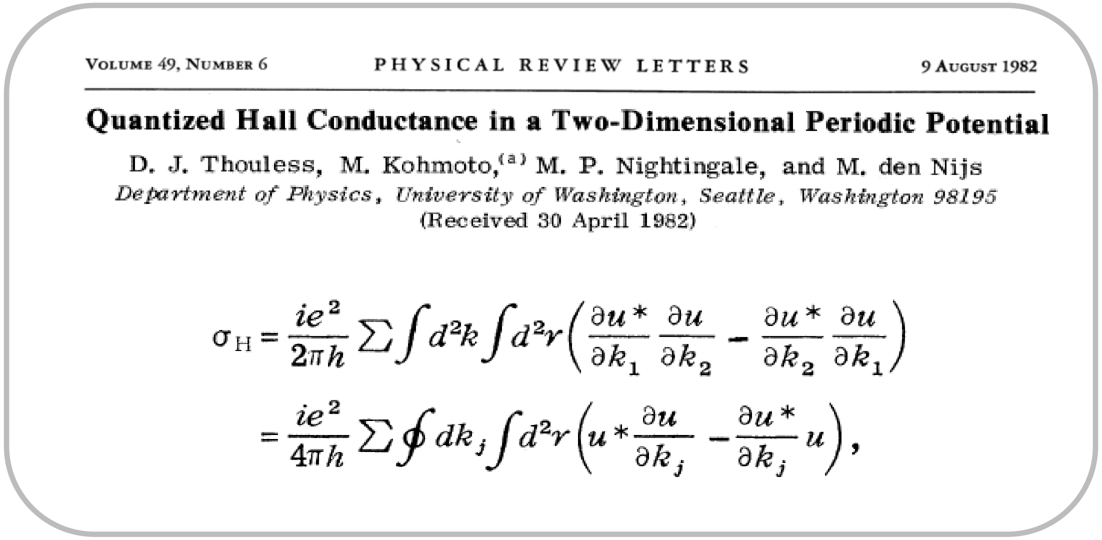
$$s = -A_0^2\frac{\partial n(\varepsilon_F)}{\partial A_0}$$

Quantum Hall Conductance

$$\sigma_{xy}^Q = ec\left.\frac{\partial n(E)}{\partial B}\right|_{E=E_F} = \frac{e^2}{h}t$$

$$(n/n_0) = t(\phi/\phi_0) + s$$





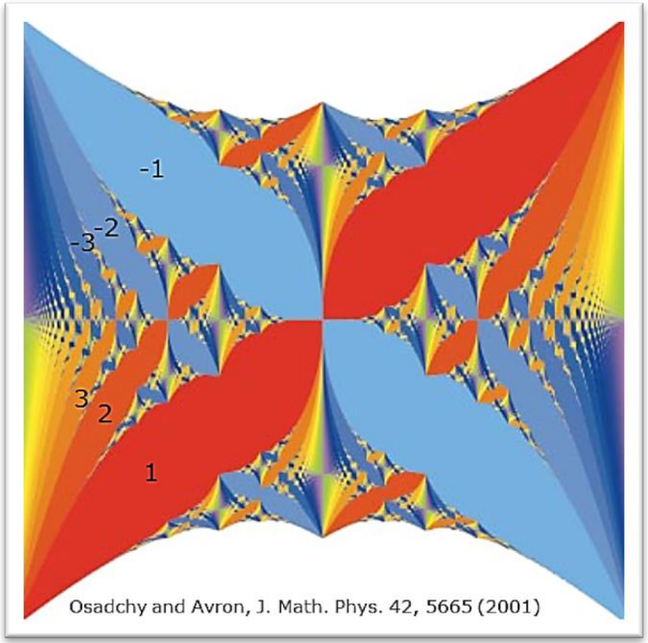
Volume 49, Number 6 PHYSICAL REVIEW LETTERS 9 August 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

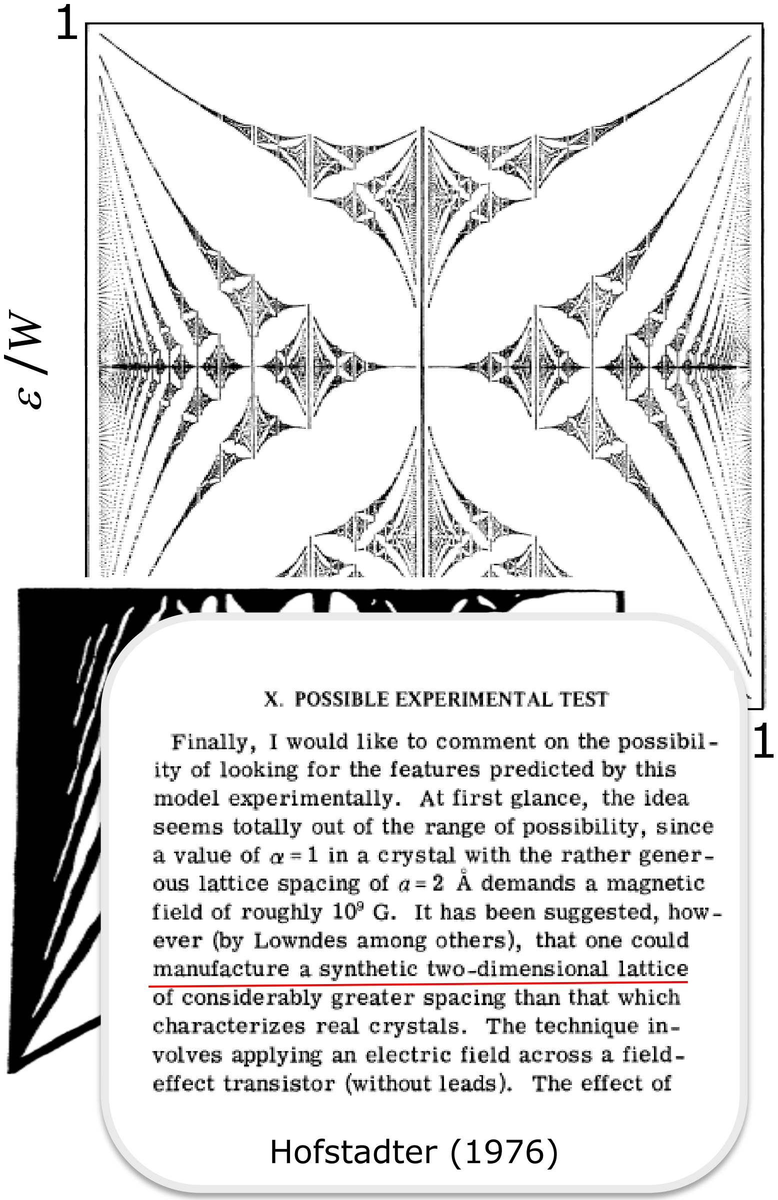
$$\sigma_H = \frac{ie^2}{2\pi h}\sum\int d^2k\int d^2r\left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1}\right)$$

$$= \frac{ie^2}{4\pi h}\sum\oint dk_j\int d^2r\left(u^*\frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j}u\right),$$



Osadchy and Avron, J. Math. Phys. 42, 5665 (2001)

Experimental Challenges



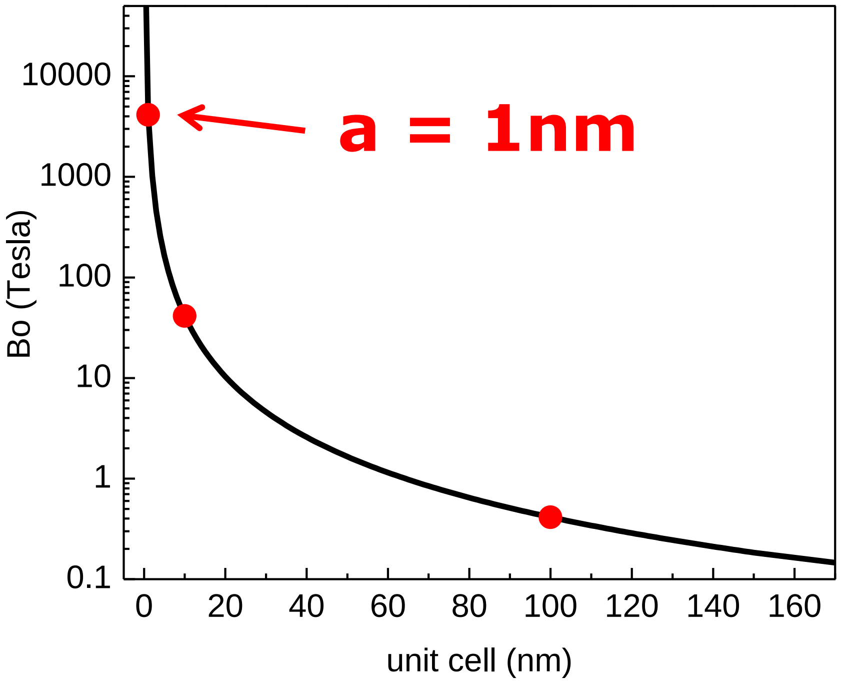
X. POSSIBLE EXPERIMENTAL TEST

Finally, I would like to comment on the possibility of looking for the features predicted by this model experimentally. At first glance, the idea seems totally out of the range of possibility, since a value of $\alpha = 1$ in a crystal with the rather generous lattice spacing of $a = 2$ Å demands a magnetic field of roughly 10^9 G. It has been suggested, however (by Lowndes among others), that one could manufacture a synthetic two-dimensional lattice of considerably greater spacing than that which characterizes real crystals. The technique involves applying an electric field across a field-effect transistor (without leads). The effect of

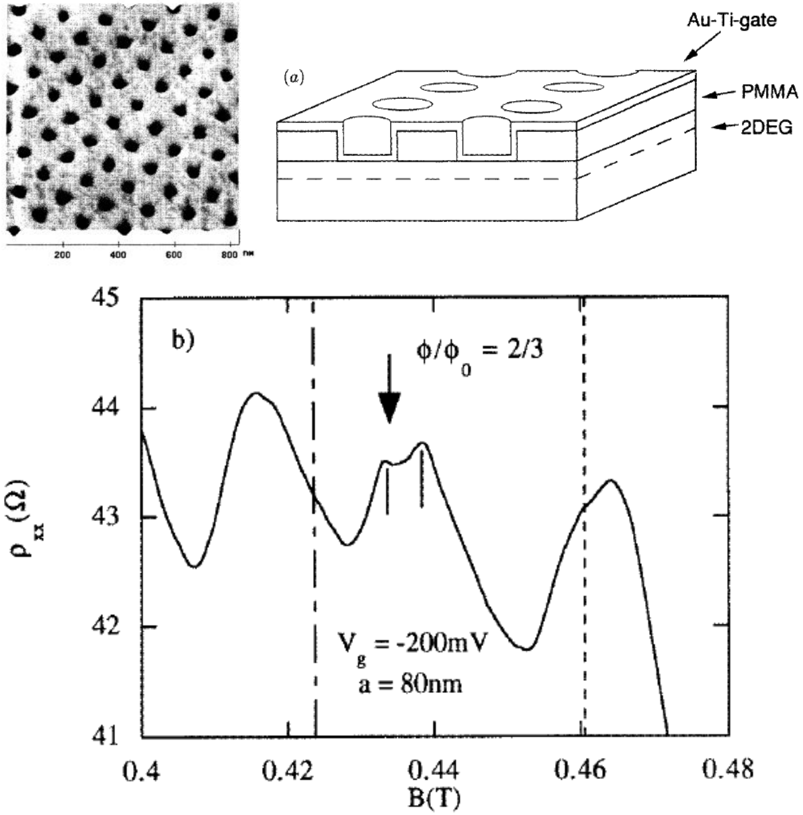
Hofstadter (1976)

Obvious technical challenge:

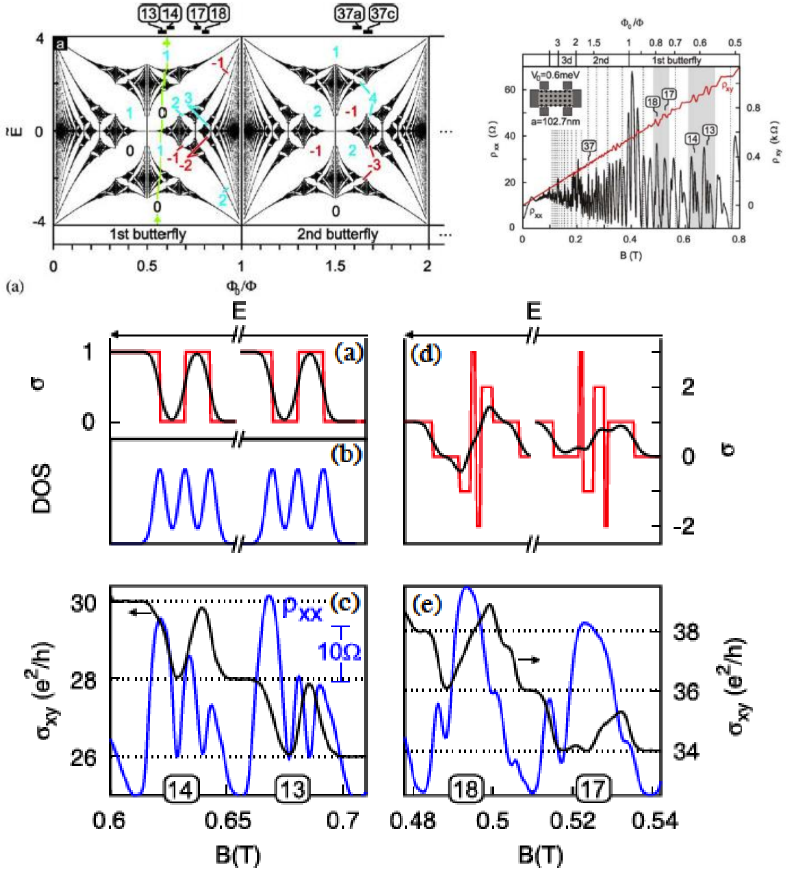
$$\frac{\phi}{\phi_o} = \frac{Ba^2}{h/e} \sim 1$$



Experimental Search For Butterfly



-Schlosser et al, Semicond. Sci. Technol. (1996)

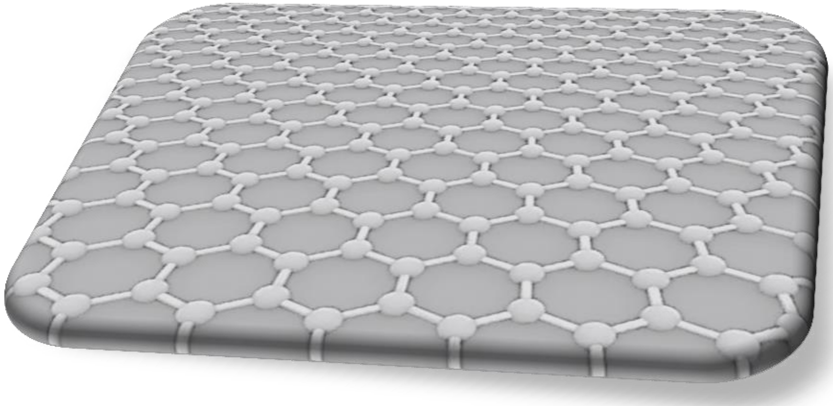


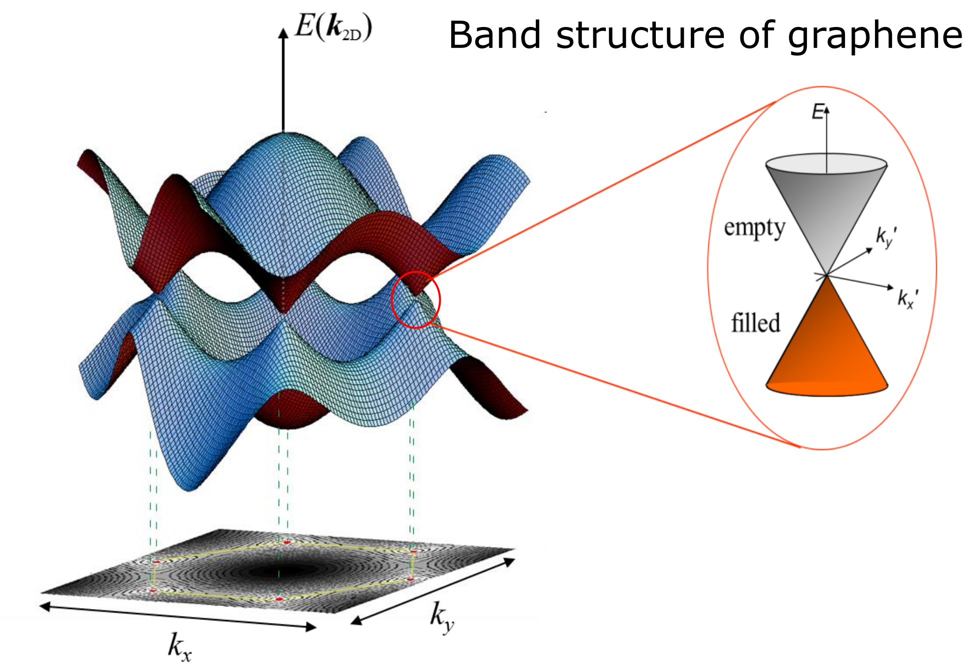
Albrecht et al, PRL. (2001);
Geisler et al, PRL (2004)

- Unit cell limited to ~100 nm
- limited field and density range accessible, weak perturbation
- Do not observe 'fully quantized' mingaps in fractal spectrum

Electrons in Graphene: Effective Dirac Fermions

Graphene,
ultimate 2-d conducting system



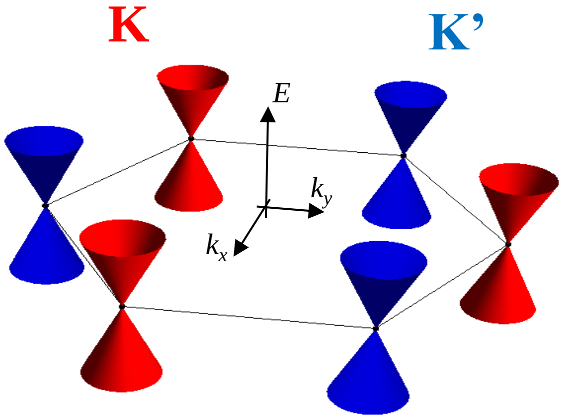


Band structure of graphene

Effective Dirac Equations

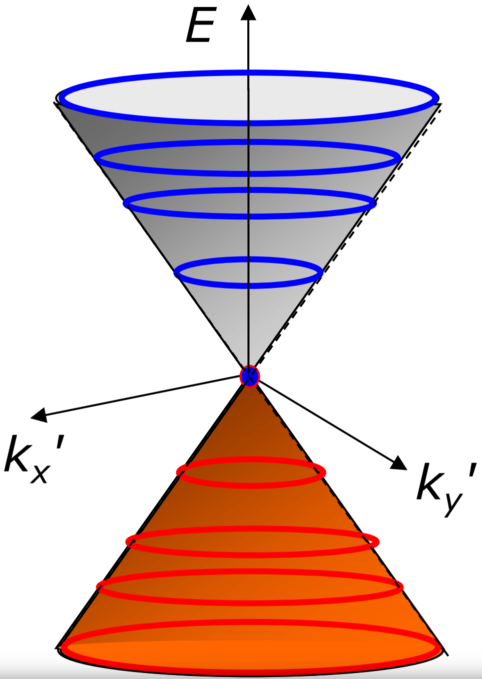
$$H_{eff} = \pm\hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix} = \pm\hbar v_F \vec{\sigma}\cdot\vec{k}_\perp$$

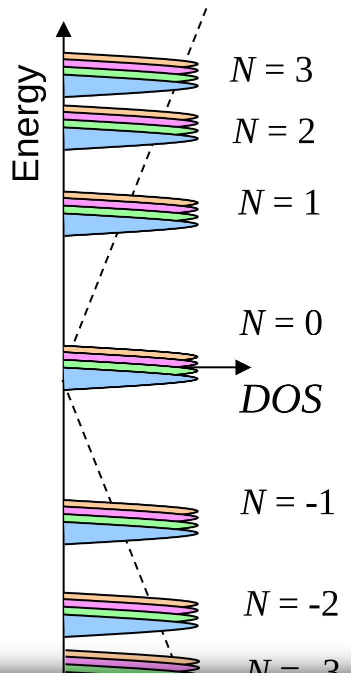
DiVincenzo and Mele, PRB (1984); Semenov, PRL (1984)

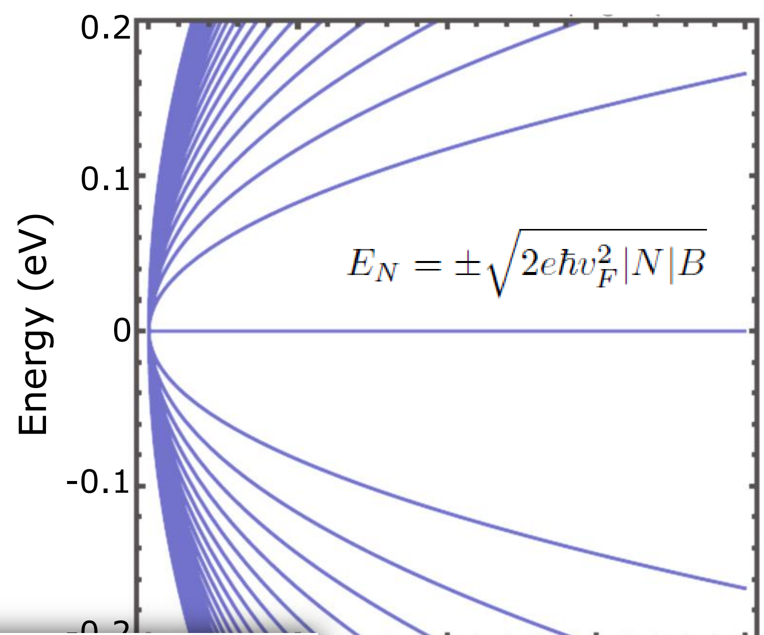


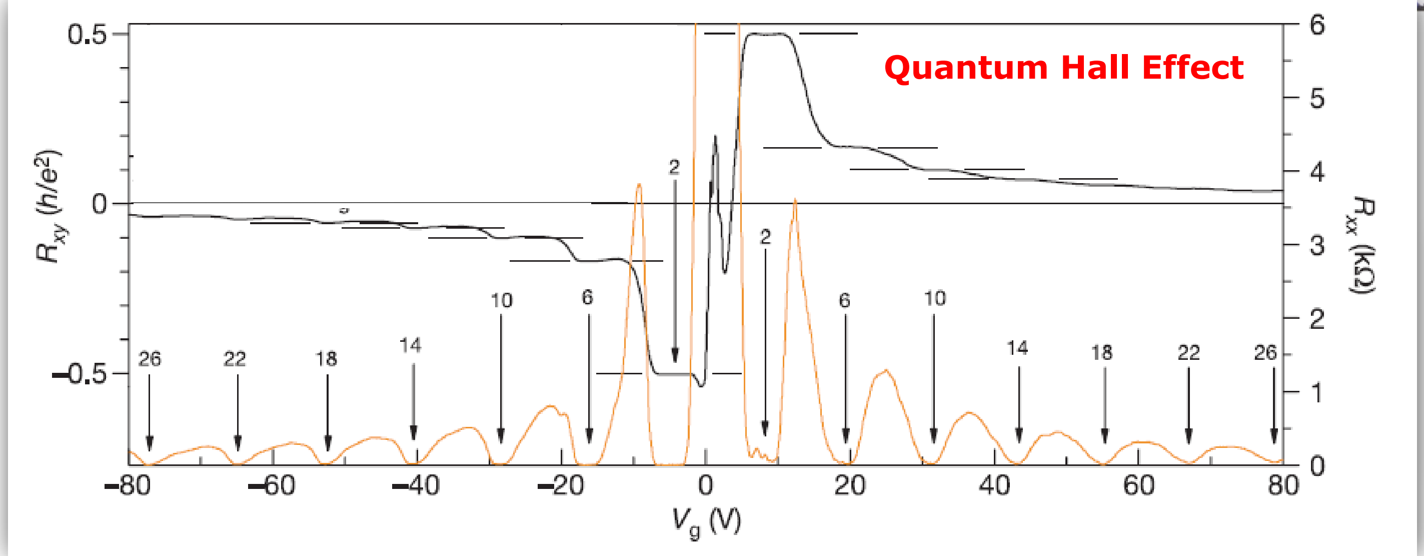
Paul Dirac

Graphene: Under Magnetic Fields









Quantization Condition

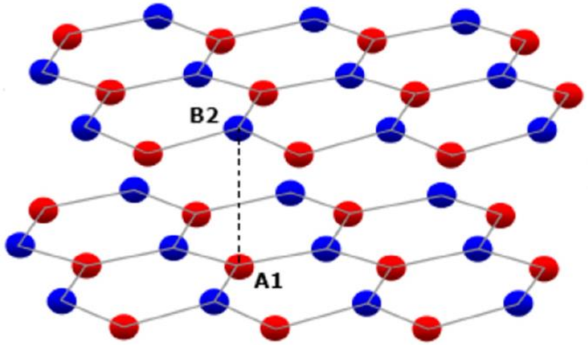
$$R_{xy}^{-1} = \frac{4e^2}{h}(N + \frac{1}{2})$$

$$\nu = \pm 2, \pm 6, \pm 10, \ldots$$

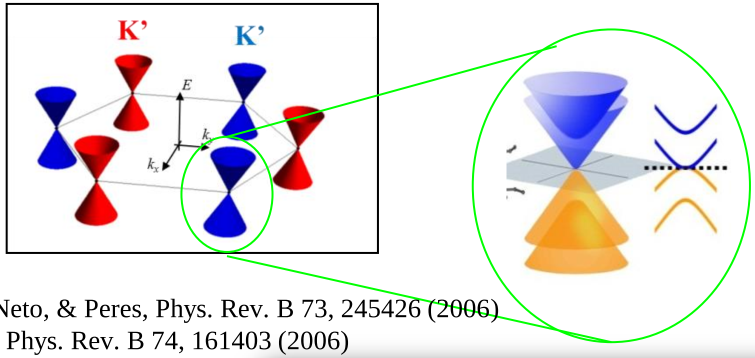
Novoselov et al (2005)
Zhang et al (2005)

Bilayer Graphene

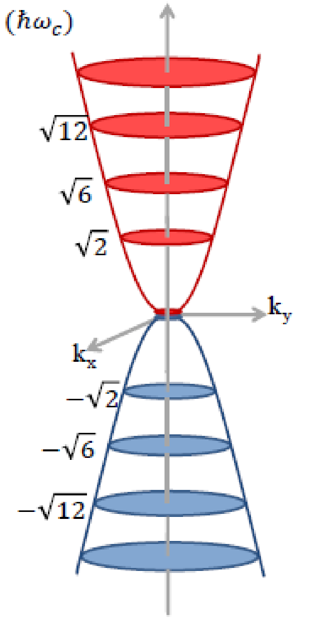
Bernal stacked bilayer graphene



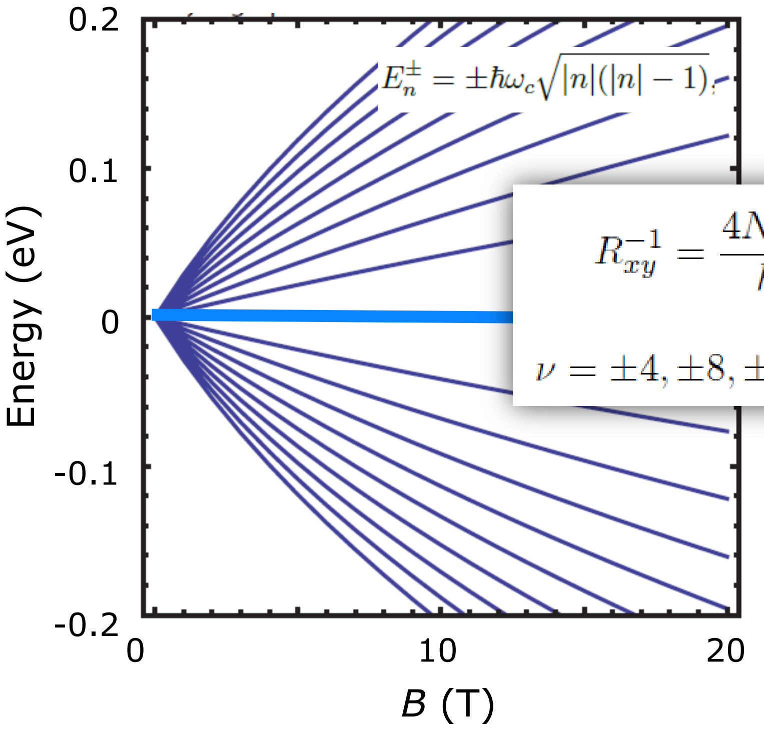
Low energy approximation in 1st BZ



Guinea, Neto, & Peres, Phys. Rev. B 73, 245426 (2006)
McCann, Phys. Rev. B 74, 161403 (2006)



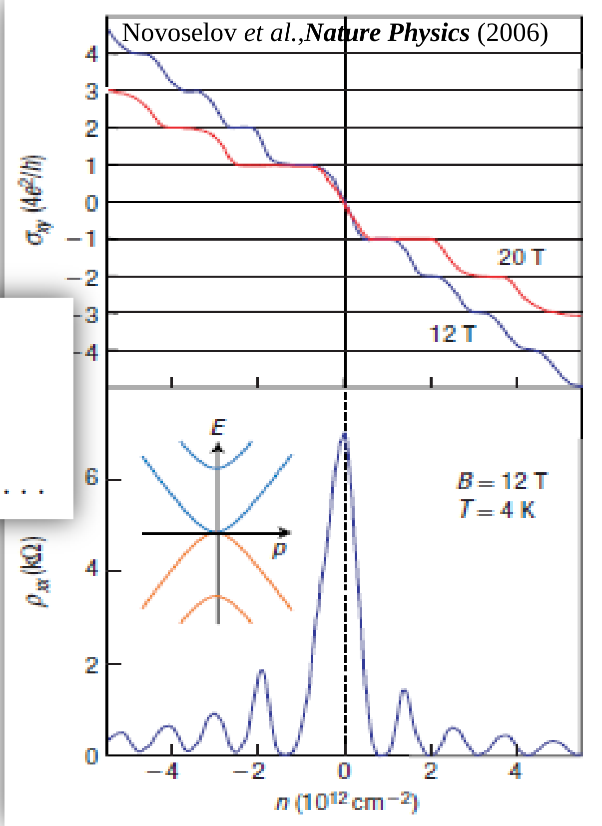
Bilayer LL spectrum



$$E_n^{\pm} = \pm\hbar\omega_c\sqrt{|n|(|n|-1)}$$

$$R_{xy}^{-1} = \frac{4Ne^2}{h}$$

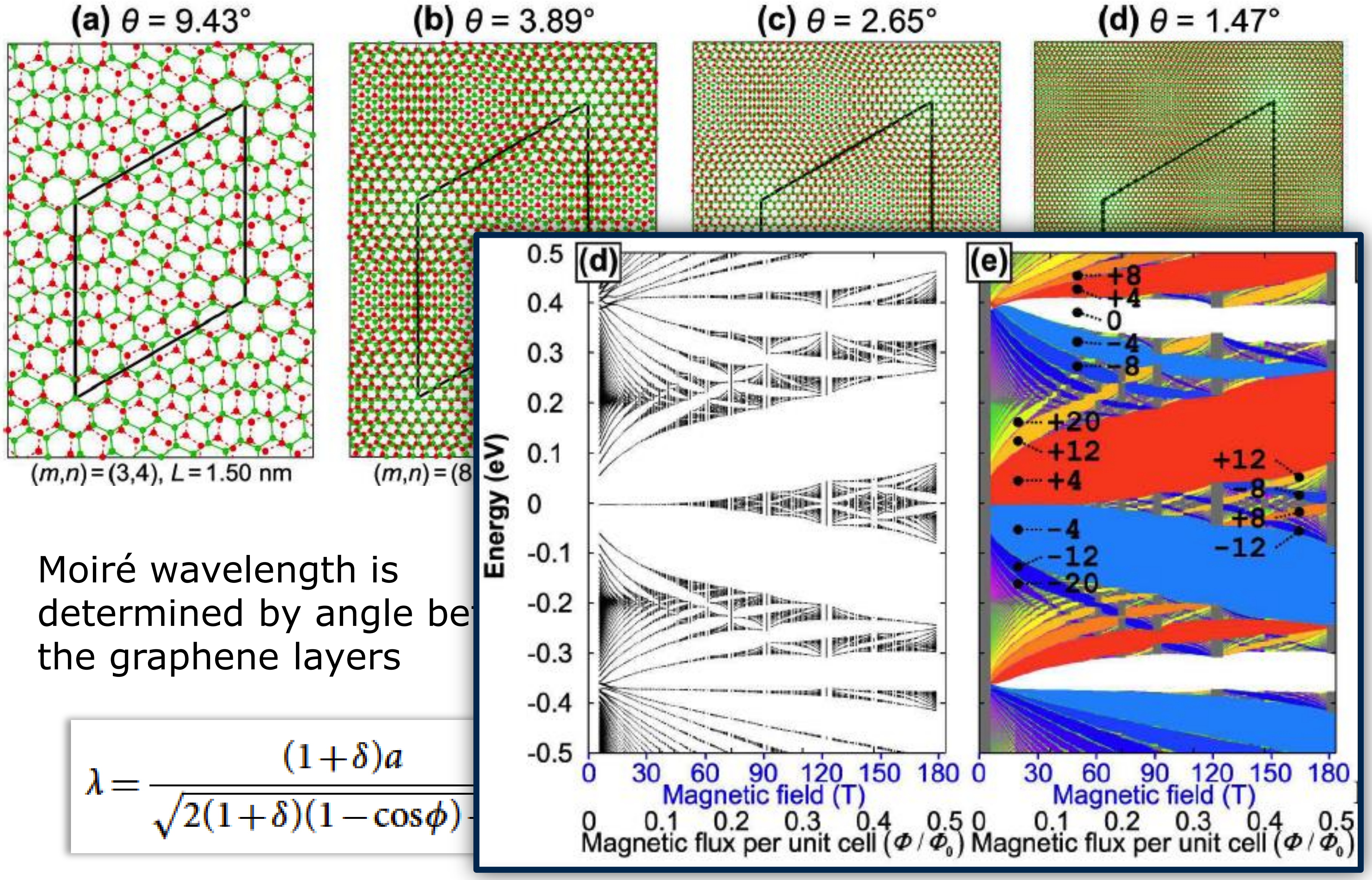
$$\nu = \pm 4, \pm 8, \pm 12, \ldots$$



Novoselov *et al.*,***Nature Physics*** (2006)

Hofstadter’s Butterfly in Twisted Graphene

-Moon and Koshino, PRB (2012); See also Bistrizer and MacDonald (2011)

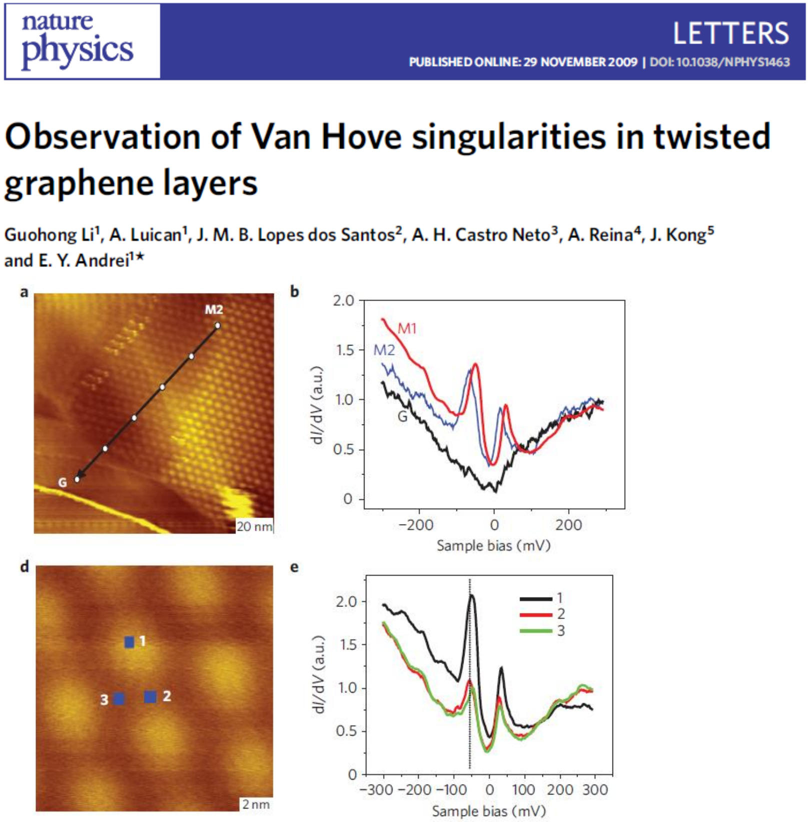


Moiré wavelength is determined by angle be

the graphene layers

$$\lambda = \frac{(1+\delta)a}{\sqrt{2(1+\delta)(1-\cos\phi)}}$$

Moire Pattern in Twisted Graphene Layers



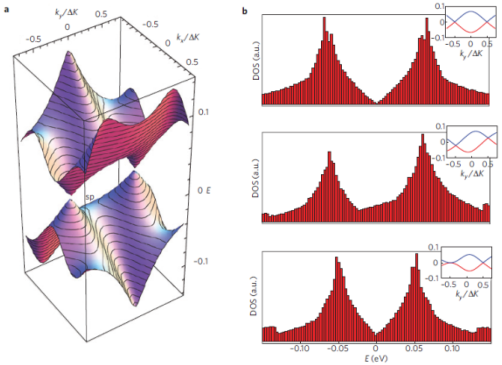
nature physics

LETTERS

PUBLISHED ONLINE: 29 NOVEMBER 2009 | DOI: 10.1038/NPHYS1463

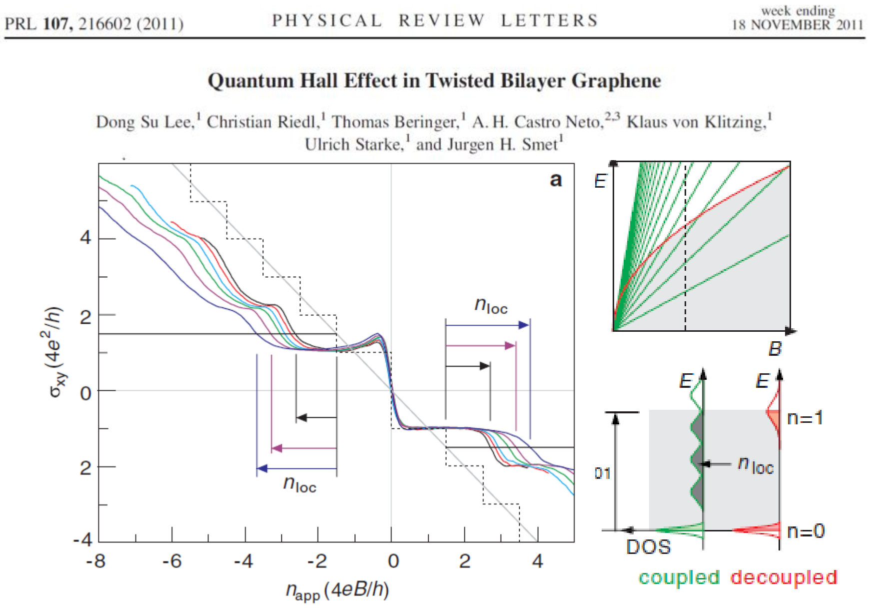
Observation of Van Hove singularities in twisted graphene layers

Guohong Li[1], A. Luican[1], J. M. B. Lopes dos Santos[2], A. H. Castro Neto[3], A. Reina[4], J. Kong[5] and E. Y. Andrei[1]*



STM observation
of Moire pattern

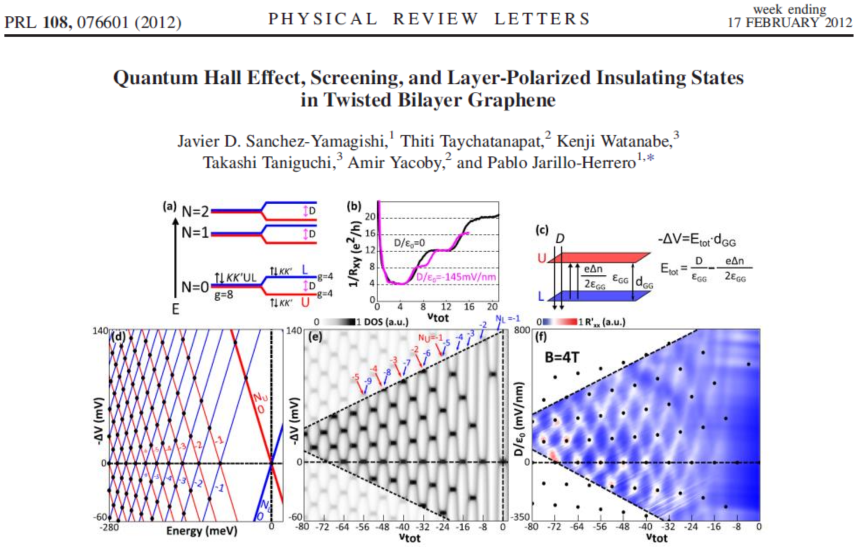
Twisting angle
~1.16,
a = 7.7 nm



PRL 107, 216602 (2011) PHYSICAL REVIEW LETTERS week ending 18 NOVEMBER 2011

Quantum Hall Effect in Twisted Bilayer Graphene

Dong Su Lee,[1] Christian Riedl,[1] Thomas Beringer,[1] A. H. Castro Neto,[2,3] Klaus von Klitzing,[1]
Ulrich Starke,[1] and Jurgen H. Smet[1]



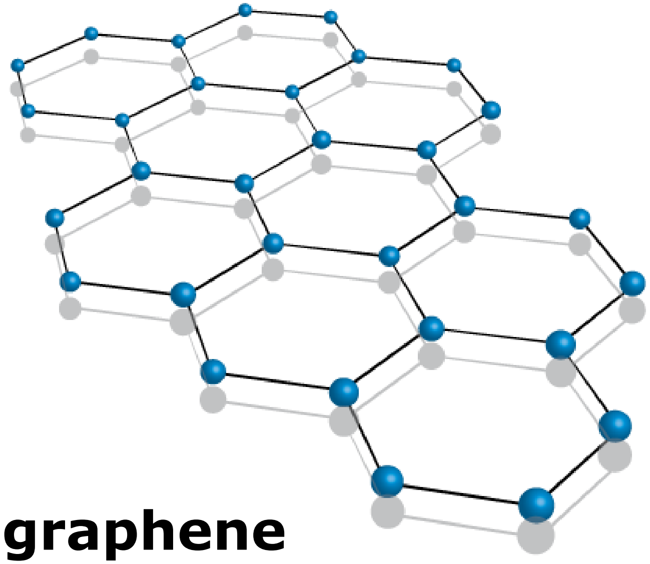
PRL 108, 076601 (2012) PHYSICAL REVIEW LETTERS week ending 17 FEBRUARY 2012

Quantum Hall Effect, Screening, and Layer-Polarized Insulating States in Twisted Bilayer Graphene

Javier D. Sanchez-Yamagishi,[1] Thiti Taychatanapat,[2] Kenji Watanabe,[3]
Takashi Taniguchi,[3] Amir Yacoby,[2] and Pablo Jarillo-Herrero[1,*]

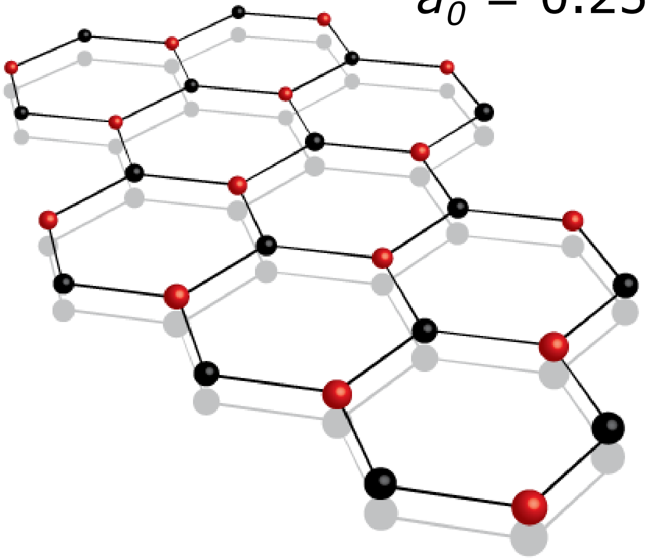
Hexa Boron Nitride: Polymorphic Graphene

a_0 = 0.246 nm



graphene

a_0 = 0.250 nm



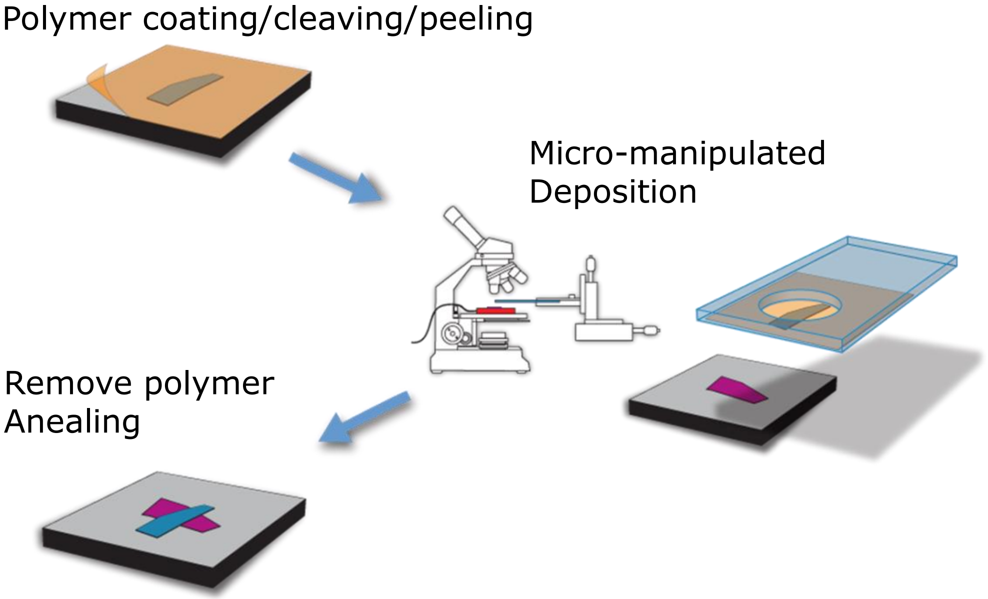
Boron Nitride

Comparison of h-BN and SiO_2

	Band Gap	Dielectric Constant	Optical Phonon Energy	Structure
BN	5.5 eV	~4	>150 meV	Layered crystal
SiO2	8.9 eV	3.9	59 meV	Amorphous

Stacking graphene on hBN

Dean et al. Nature Nano (2009)

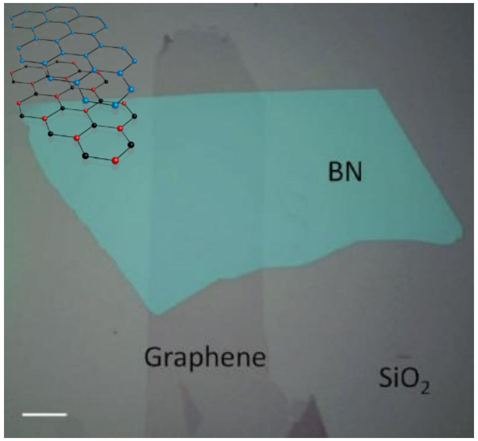


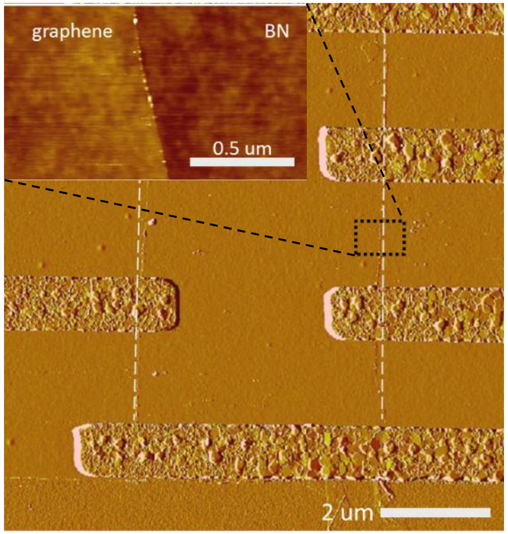
Polymer coating/cleaving/peeling

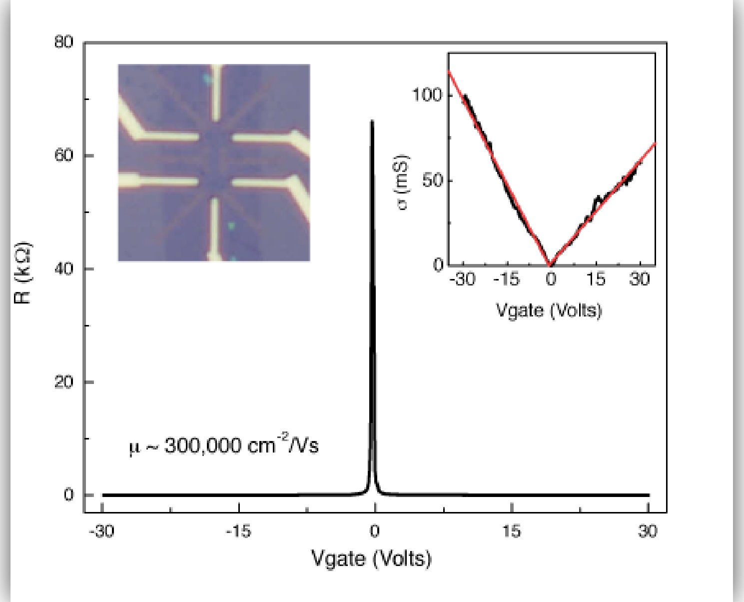
Micro-manipulated
Deposition

Remove polymer
Anealing

- Co-lamination techniques
- Submicron size precision
- Atomically smooth interface



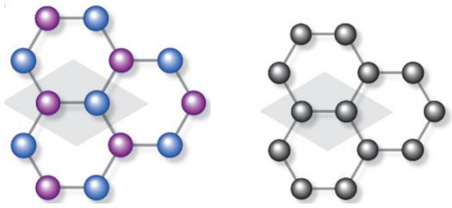


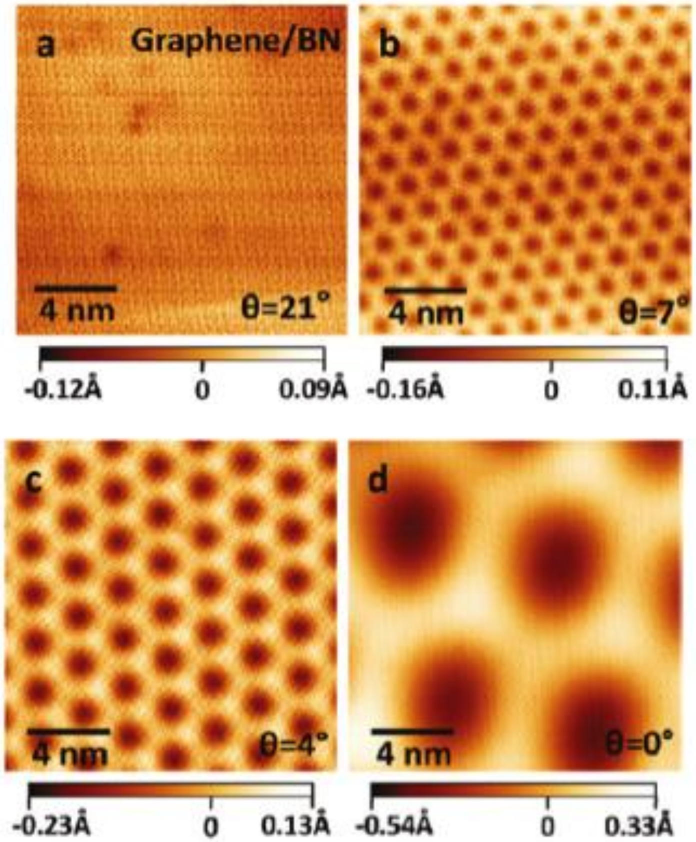


Mobility > 100,000 cm^{2}V^{-1}s^{-1}

Moire pattern in Graphene on hBN:
a new route to Hofstadter's butterfly?

Graphene on BN exhibits clear Moiré pattern





Xue et al, Nature Mater (2011);
Decker et al Nano Lett (2011)

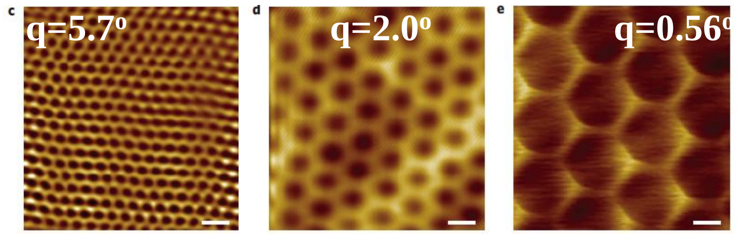
LETTERS

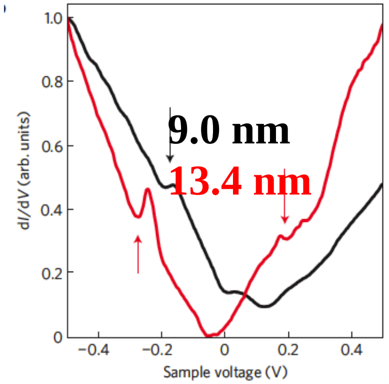
PUBLISHED ONLINE: 25 MARCH 2012 | DOI: 10.1038/NPHYS2272

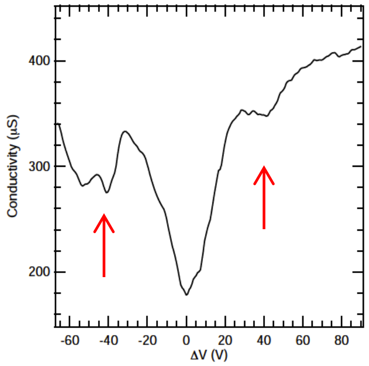
nature physics

Emergence of superlattice Dirac points in graphene on hexagonal boron nitride

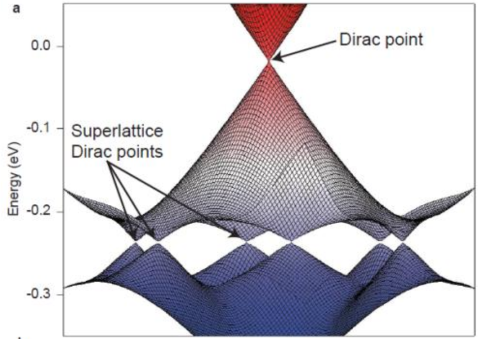
Matthew Yankowitz[1], Jiamin Xue[1], Daniel Cormode[1], Javier D. Sanchez-Yamagishi[2], K. Watanabe[3], T. Taniguchi[3], Pablo Jarillo-Herrero[2], Philippe Jacquod[1,4] and Brian J. LeRoy[1]*



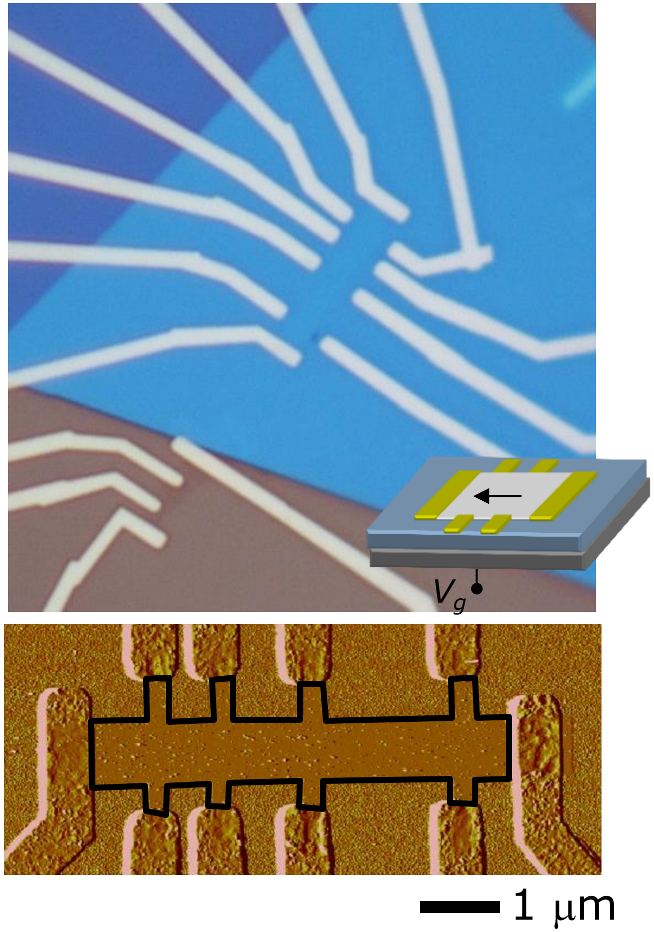


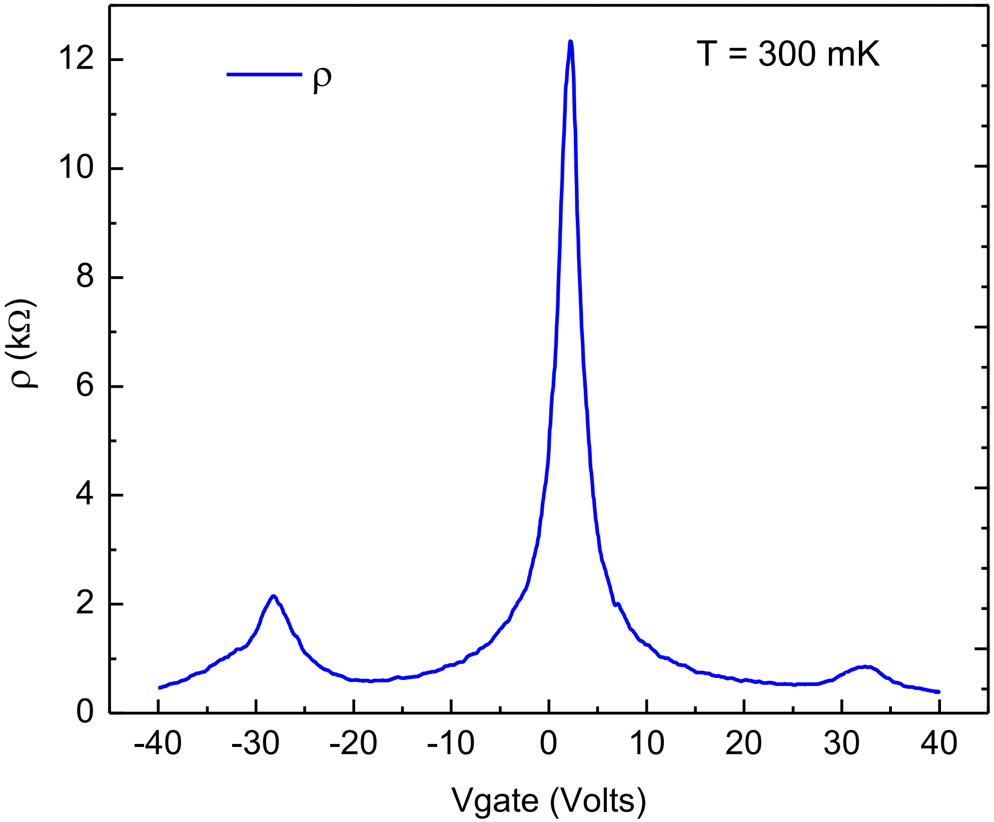


Minigap formation
near the Dirac point
due to Moire superlattice



Some Bilayer Graphene on hBN



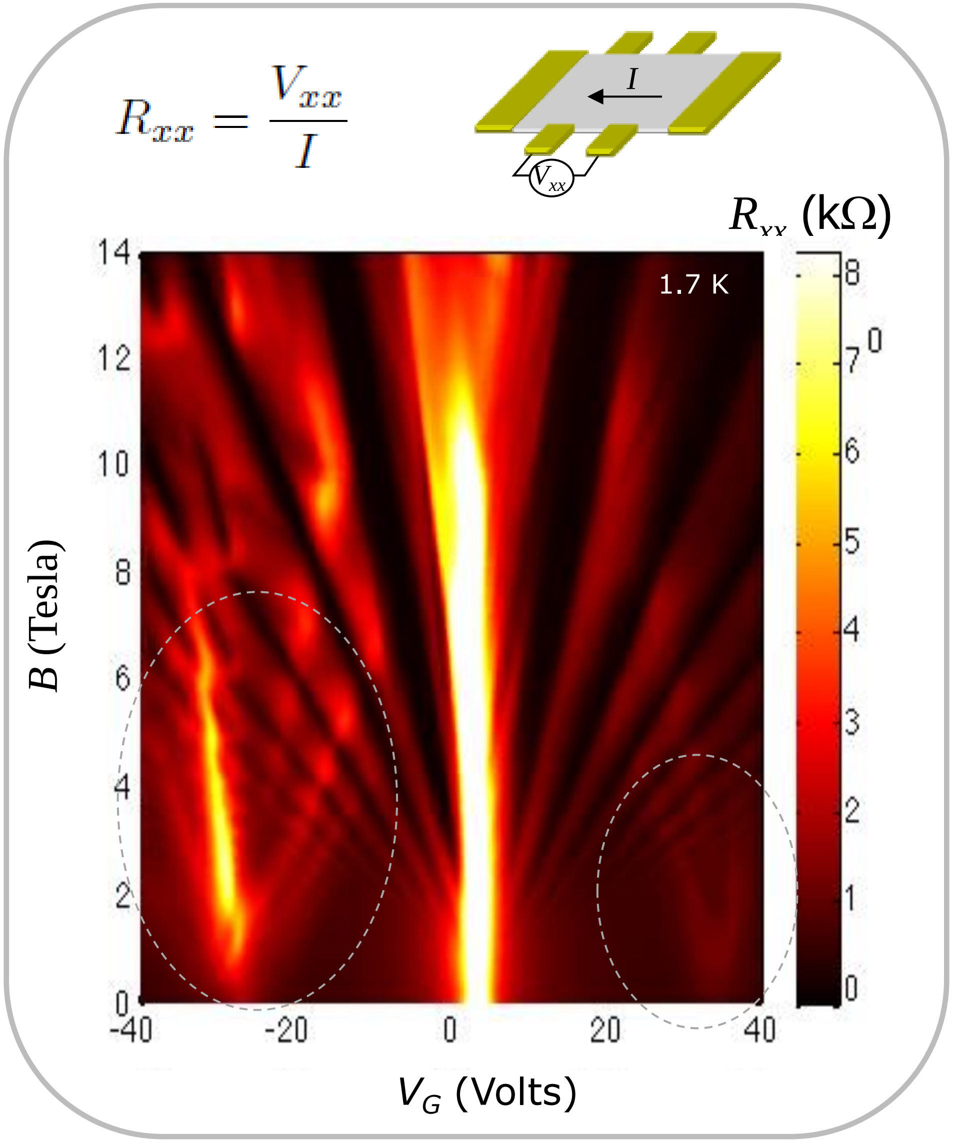


Bilayer graphene on BN substrates shows strong signature of satellite peaks…**some times… (~ 30%)**

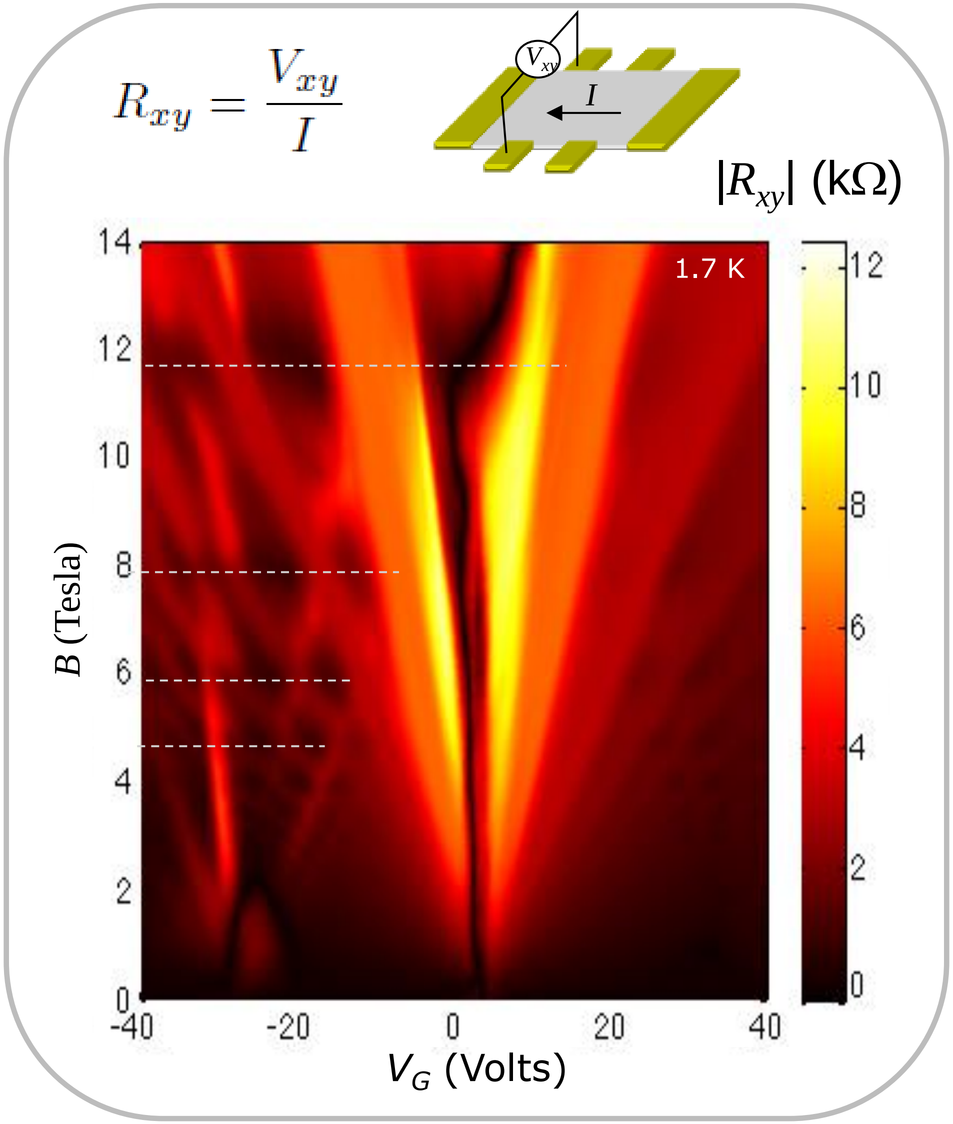
Abnormal Landau Fan Diagram in Bilayer on hBN

Special Samples with Large Moire Unit Cell

Low Magnetic field regime



$$R_{xx} = \frac{V_{xx}}{I}$$

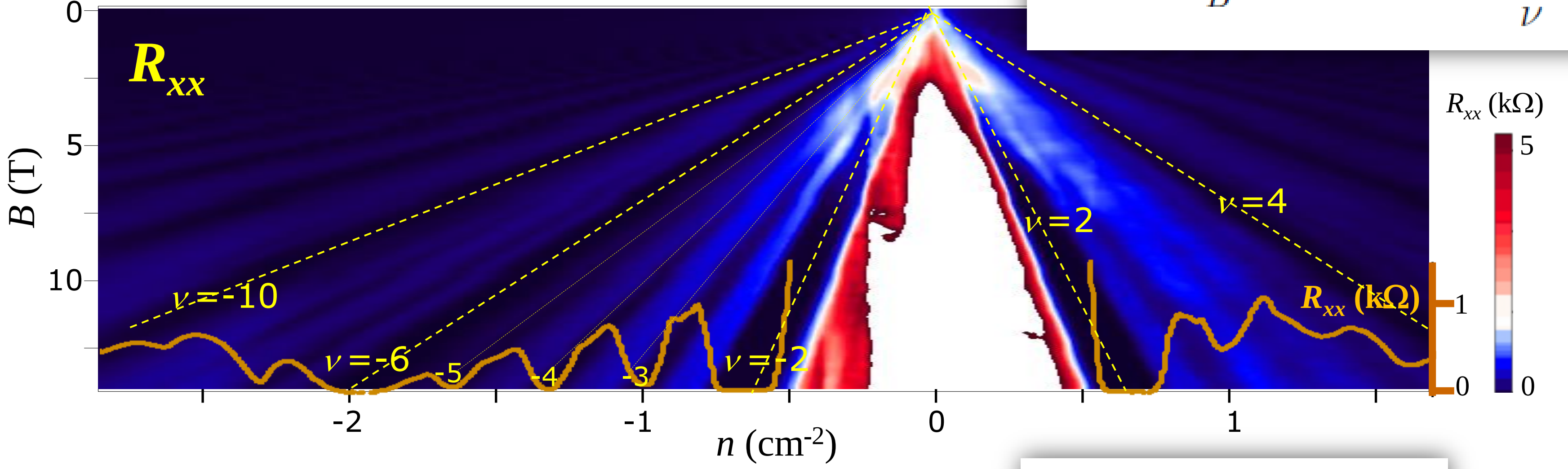


$$R_{xy} = \frac{V_{xy}}{I}$$

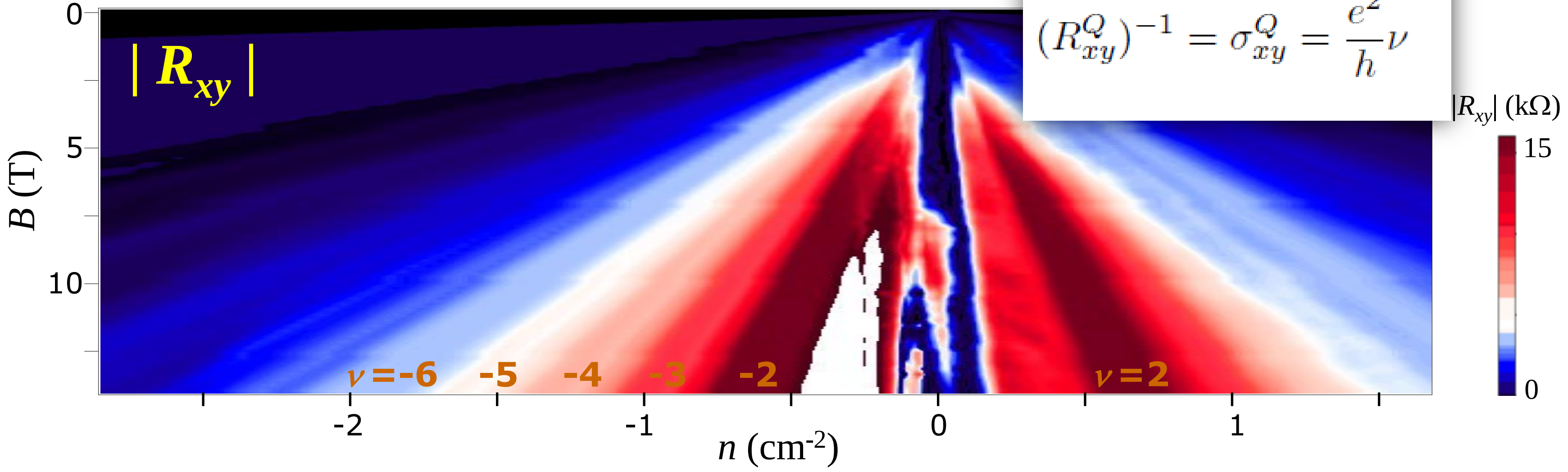
How to “Read” *Normal* Landau Fan Diagram?

Landau Fan Diagram for “typical” graphene

$$\nu = 2\pi\ell_B^2 n \quad \Rightarrow \quad B = \frac{\phi_0}{\nu} n$$



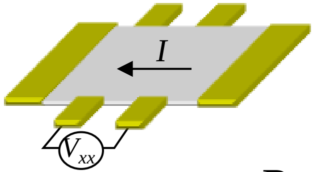
$$(R_{xy}^Q)^{-1} = \sigma_{xy}^Q = \frac{e^2}{h}\nu$$

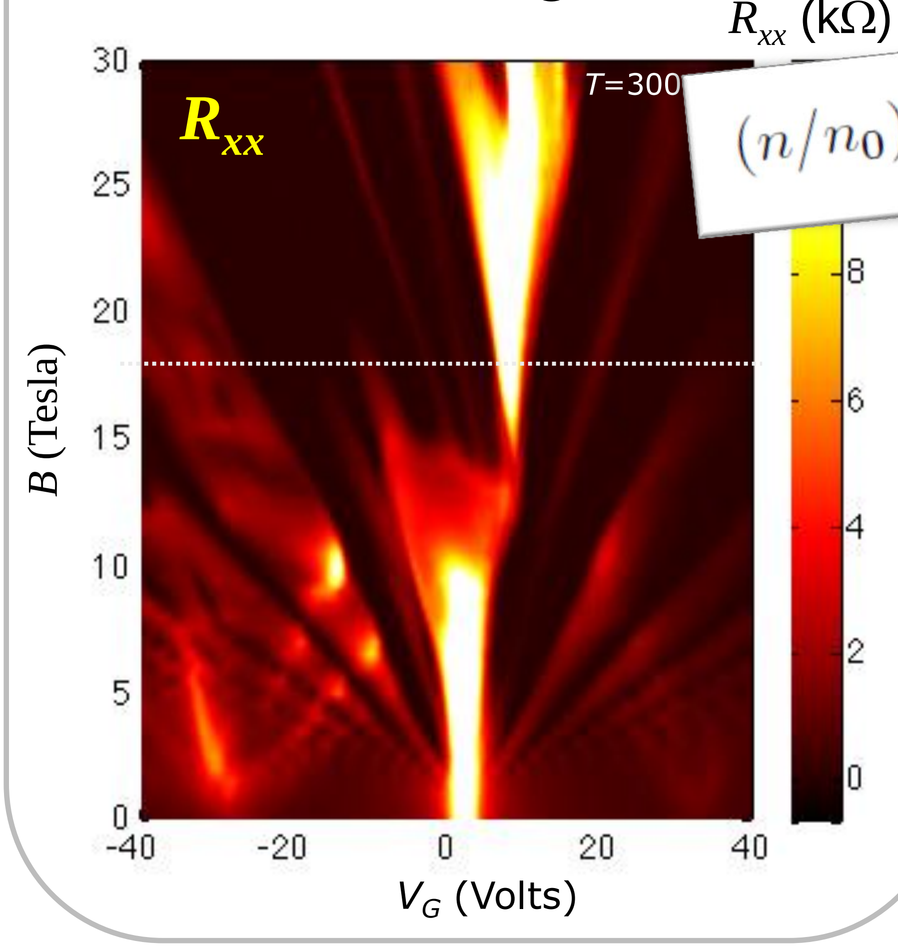


Abnormal Quantum Hall Effect

Quantum Hall-like Transport

$$R_{xx} = \frac{V_{xx}}{I}$$





$$(n/n_0) = t(\phi/\phi_0) + s$$

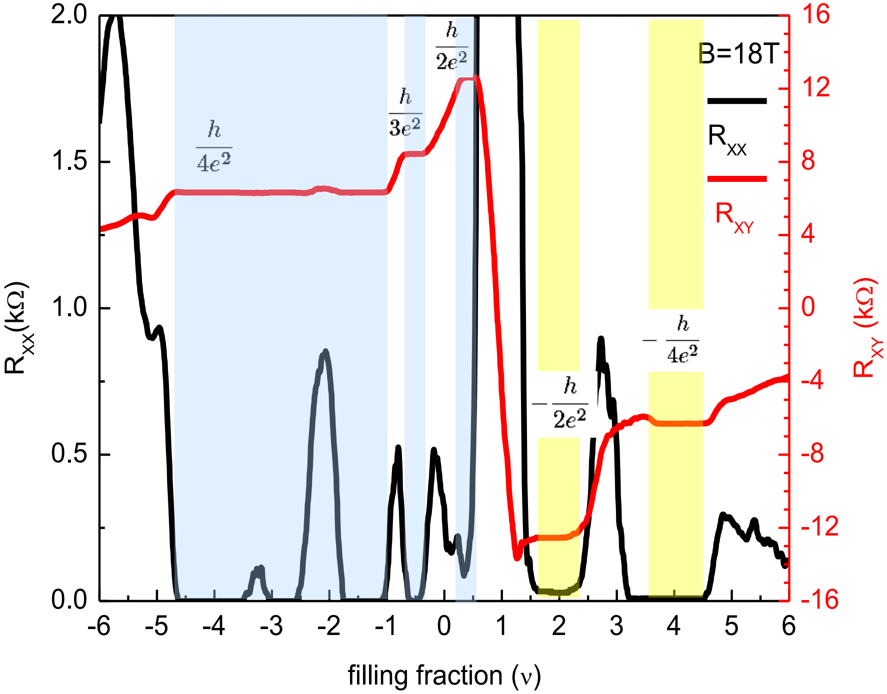
Landau level filling factor

$$\nu = \frac{\phi_0}{B} n$$

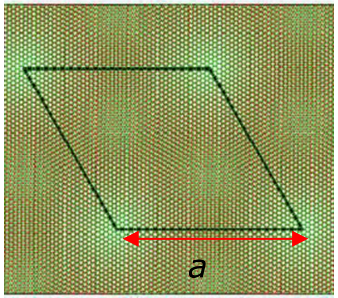
Quantum Hall conductance

$$R_{xy}^{-1} = \frac{e^2}{h} t$$

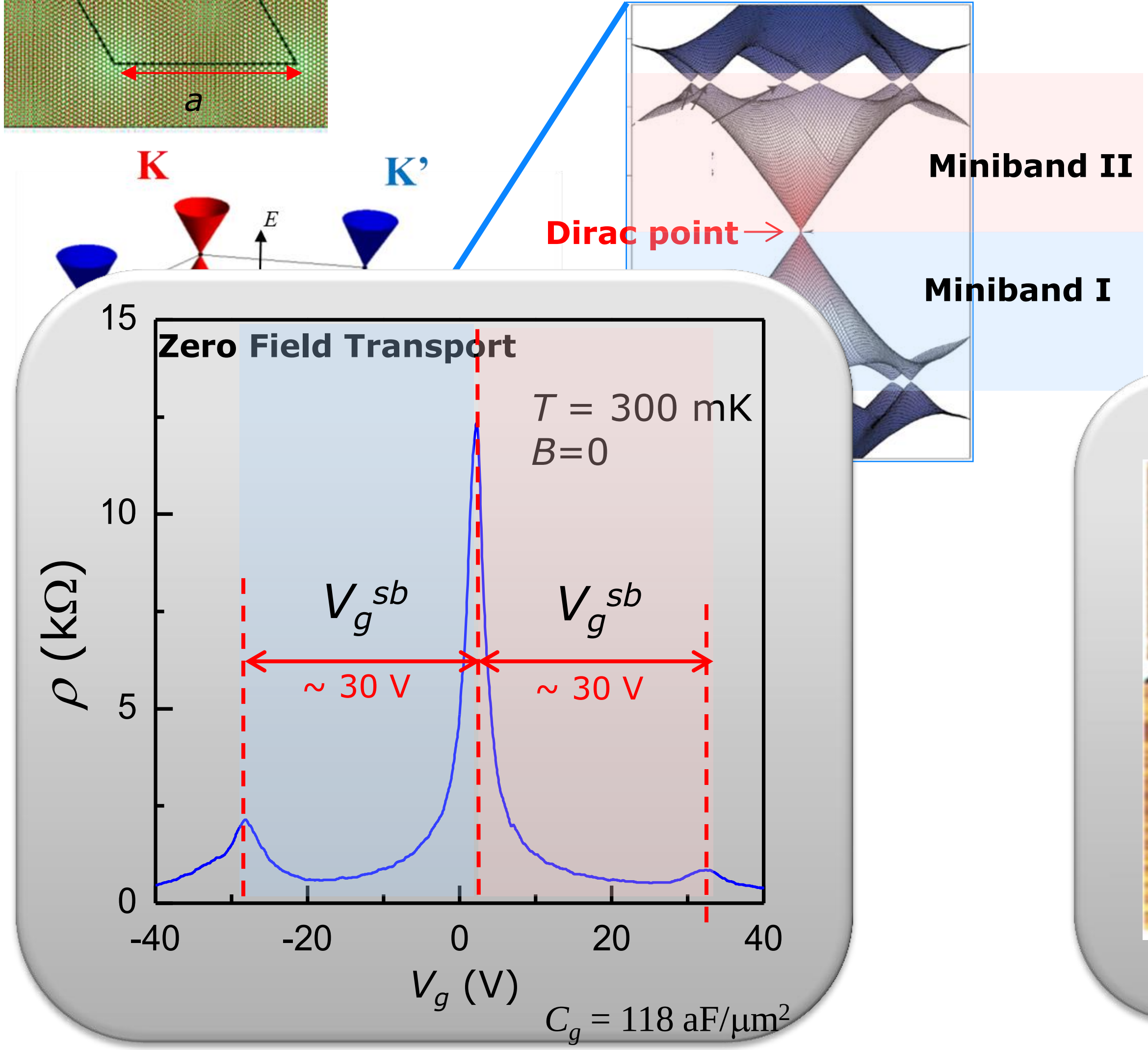
$$\cancel{\nu = t} \quad \in \mathbb{Z}$$



Size of the Moire Supper Lattice in Graphene



Yankowitz et al., (2012)



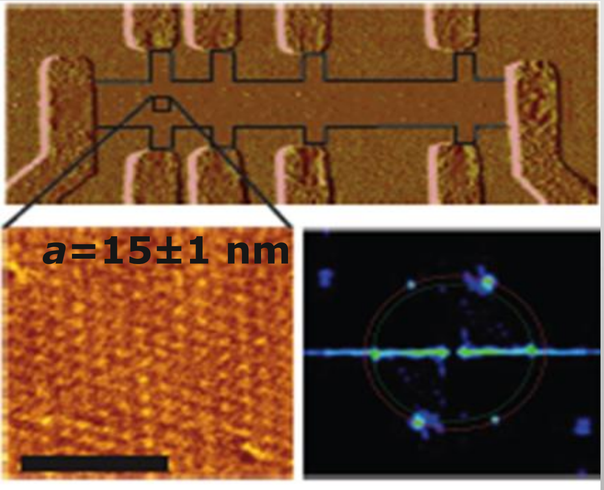
$$A_0^{-1} = n_0 = C_g V_g^{sb}/4$$

$$A_0 = \frac{\sqrt{3}a^2}{2}$$

$$\phi = BA_0$$

a = 14.3 nm

Confirmed by UHV AFM

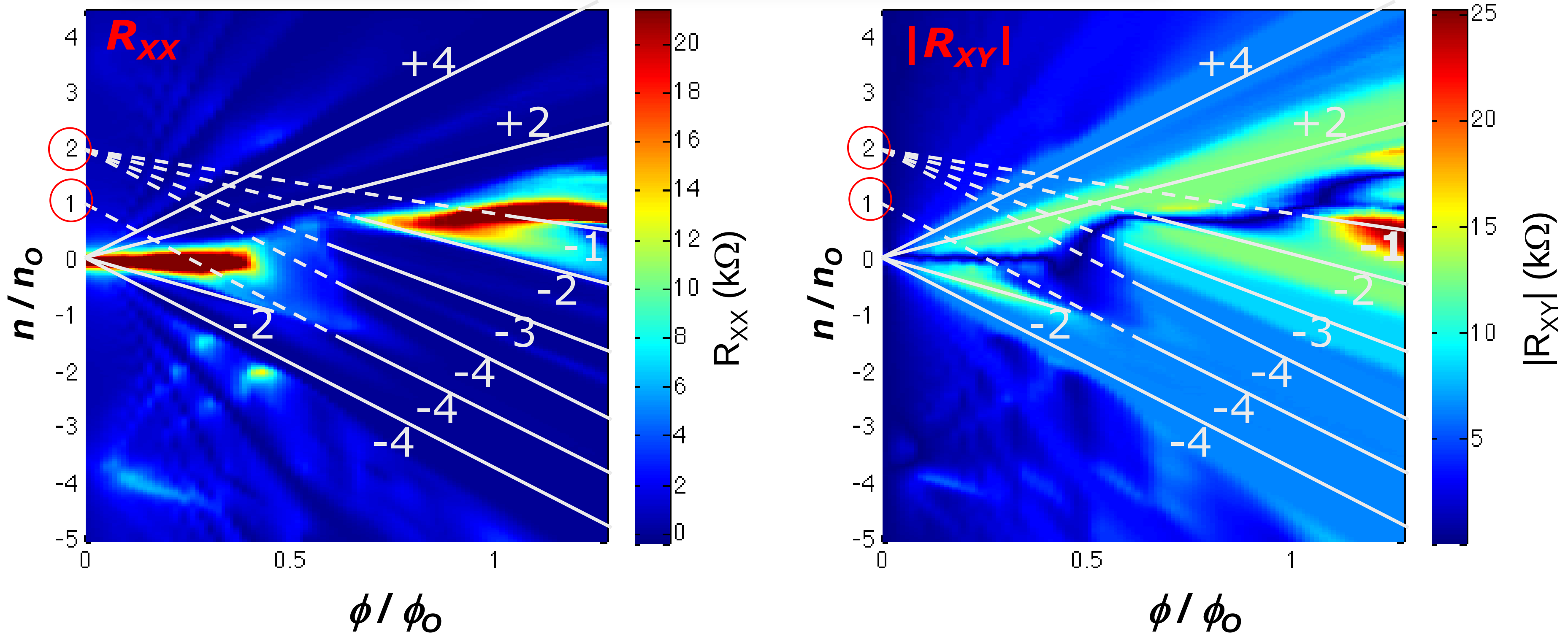


Ishigami group (UCF)

Normalized Fan Diagrams

$$n/n_0 = 4V_g/V_g^{sb}$$

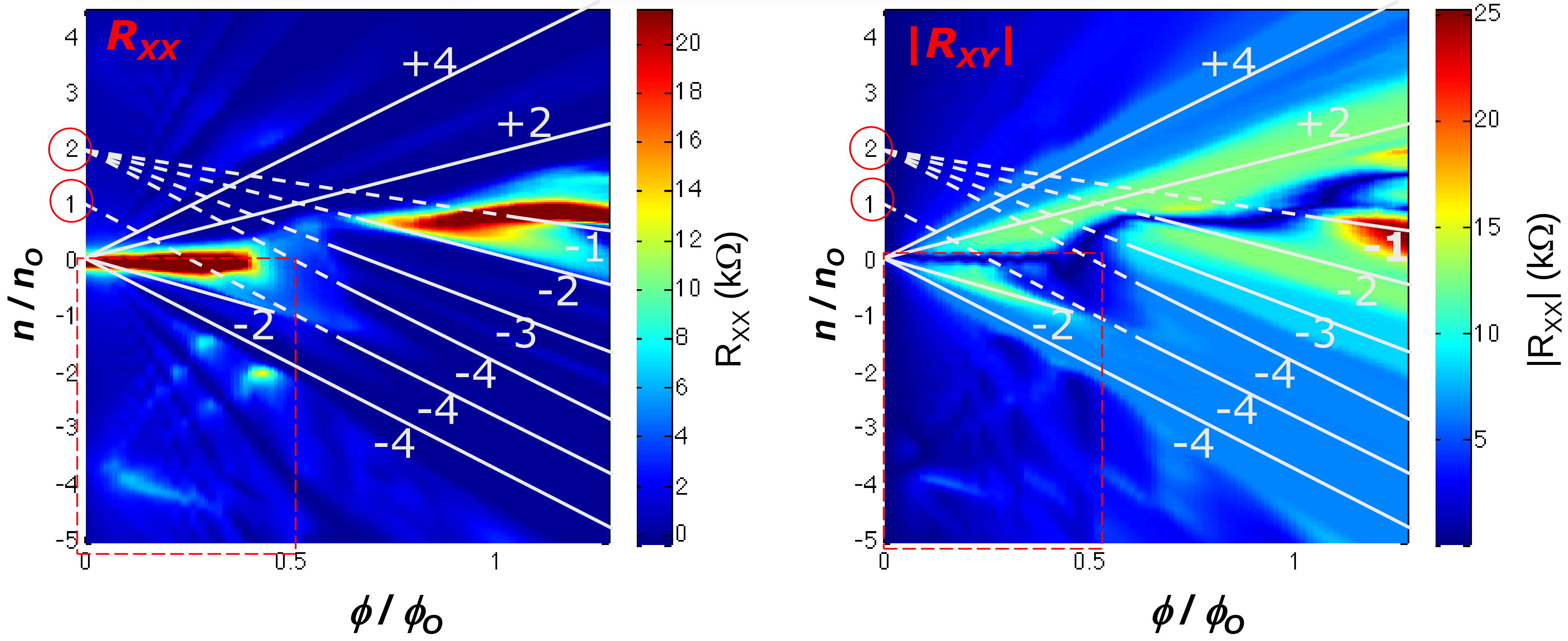
$$\phi/\phi_0 = B/B_0, \qquad B_0 = \phi_0/A_0$$



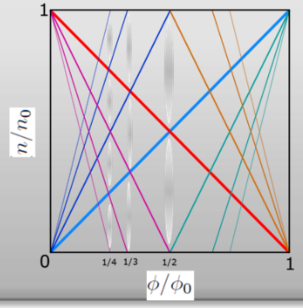
Normalized Fan Diagrams

$$n/n_0 = 4V_g/V_g^{sb}$$

$$\phi/\phi_0 = B/B_0, \qquad B_0 = \phi_0/A_0$$



Wannier diagram:
Tracing gaps in
Hofstadter's Butterfly



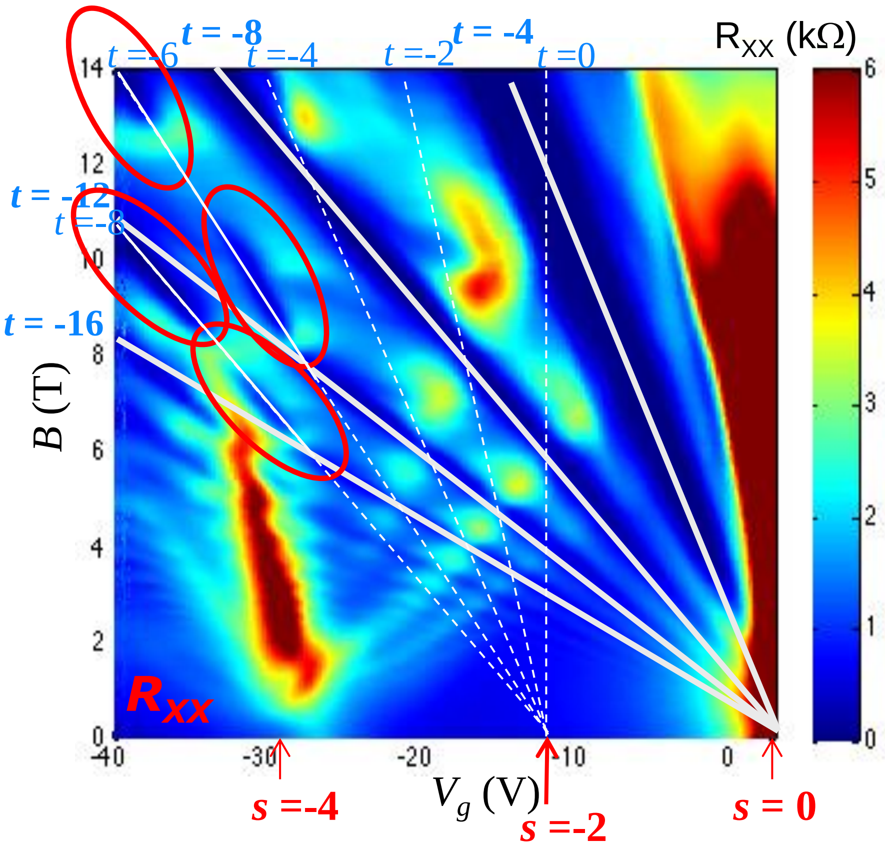
slope

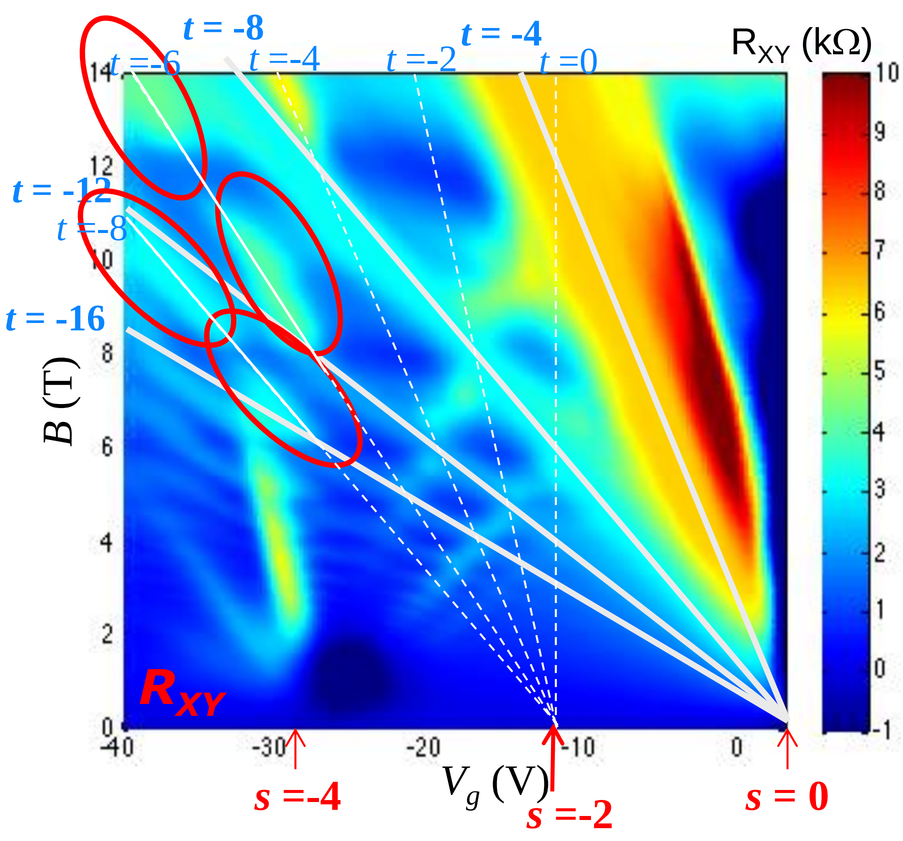
offset

$$(n/n_0) = t(\phi/\phi_0) + s$$

$$R_{xy}^{-1} = \frac{e^2}{h}t$$

Landau Fan in Low Magnetic Field Regime

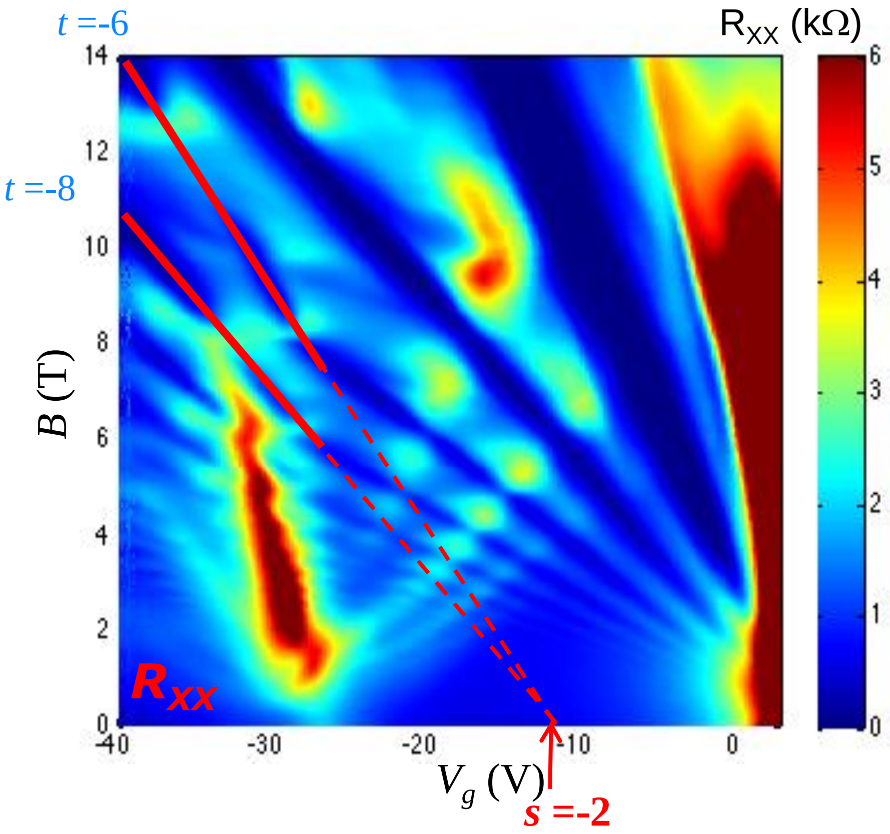


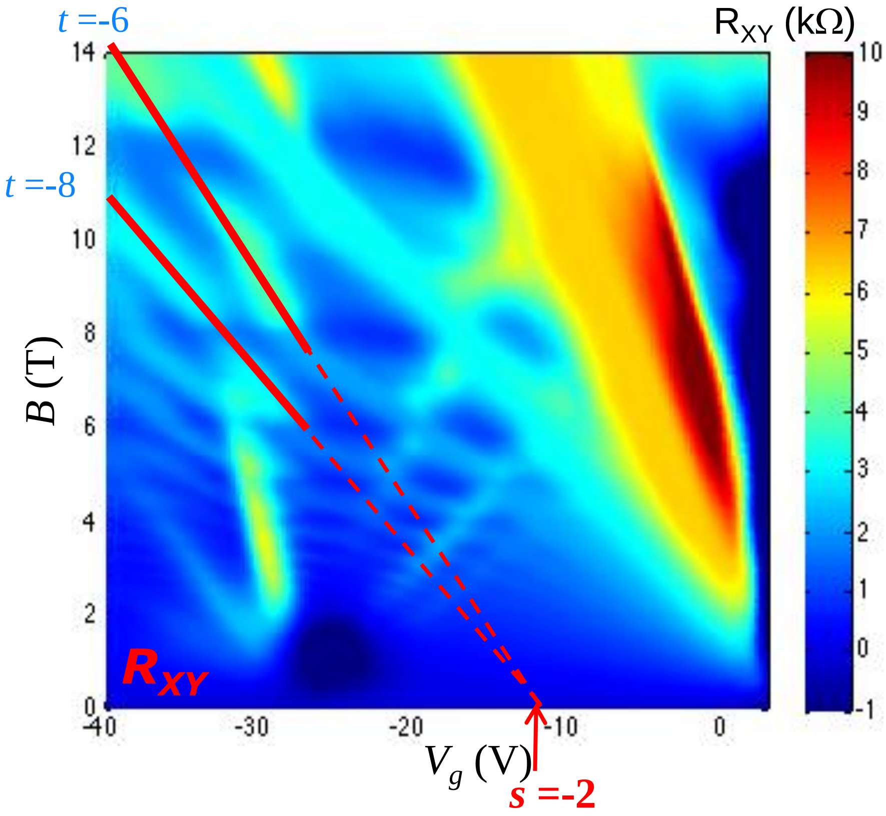


$$(n/n_0) = t(\phi/\phi_0) + s$$

$$R_{xy}^{-1} = \frac{e^2}{h} t$$

Landau Fan in Low Magnetic Field Regime





$$(n/n_0) = t(\phi/\phi_0) + s$$

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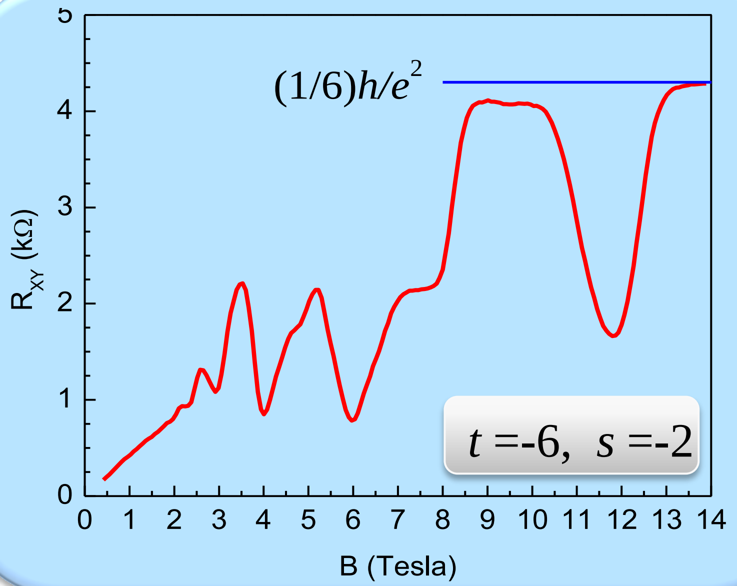
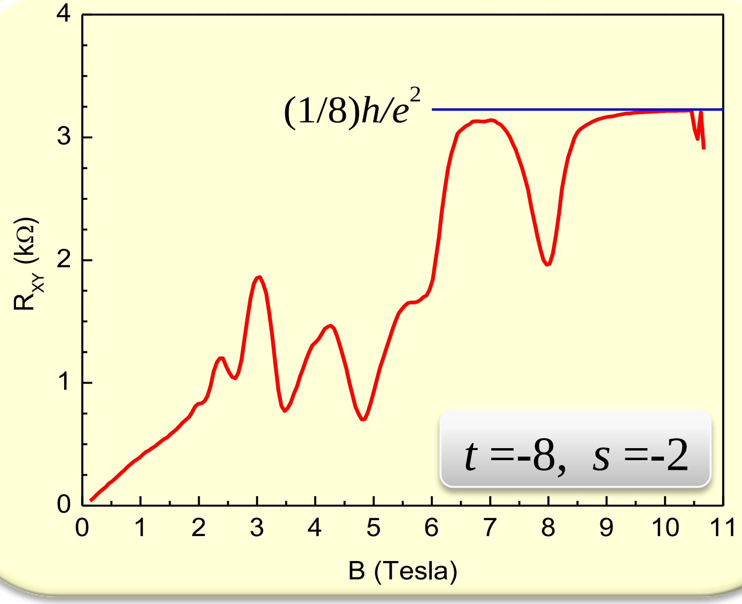
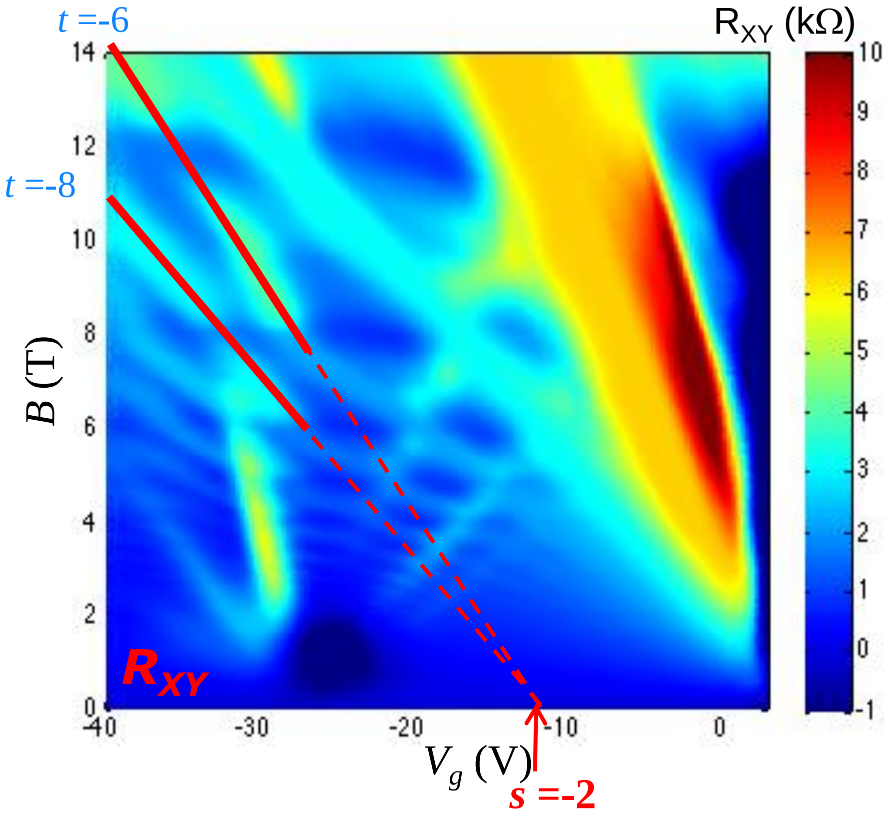
Landau Fan in Low Magnetic Field Regime



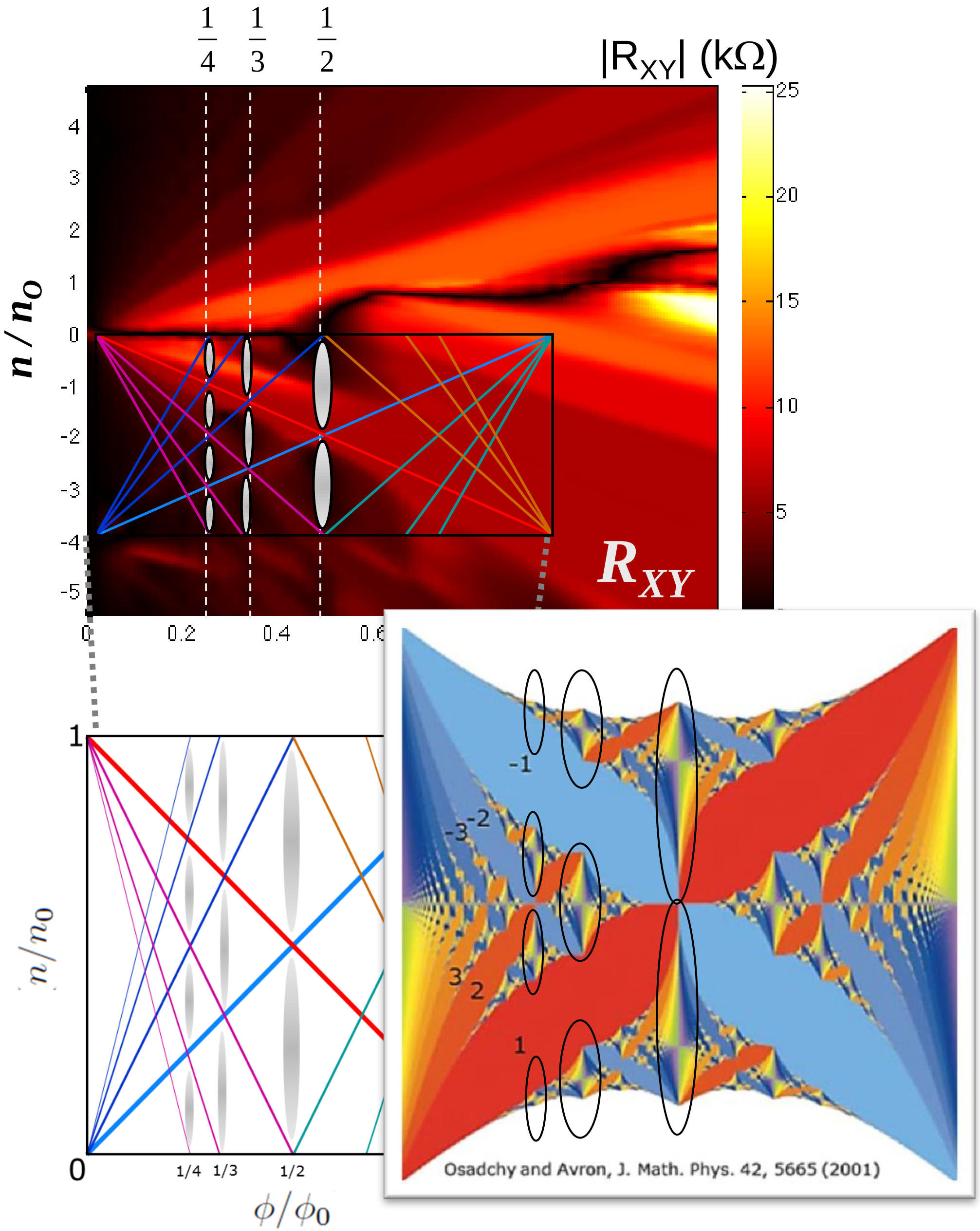




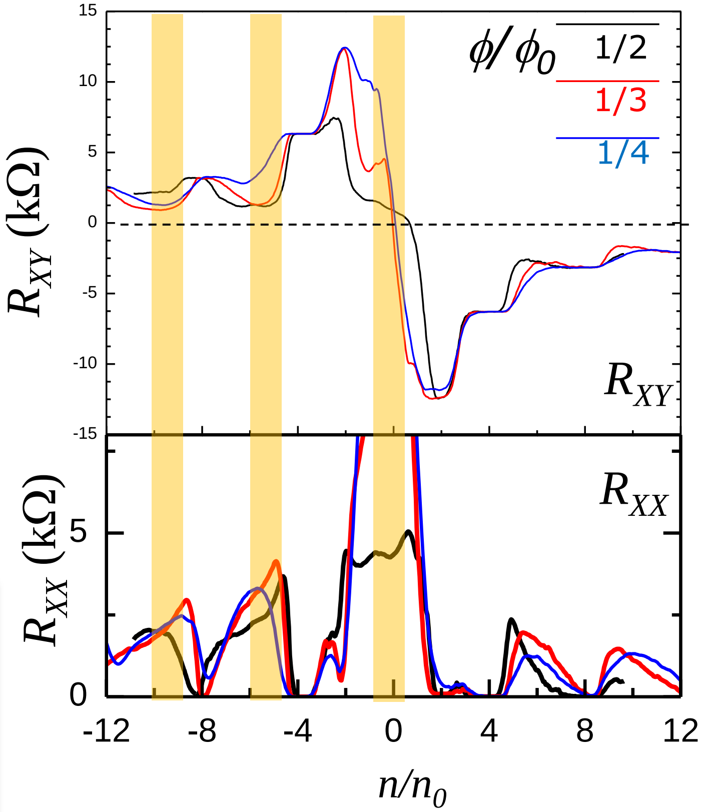
$$(n/n_0) = t(\phi/\phi_0) + s$$

$$R_{xy}^{-1} = \frac{e^2}{h}t$$

Hall Conductance in Fractal Band



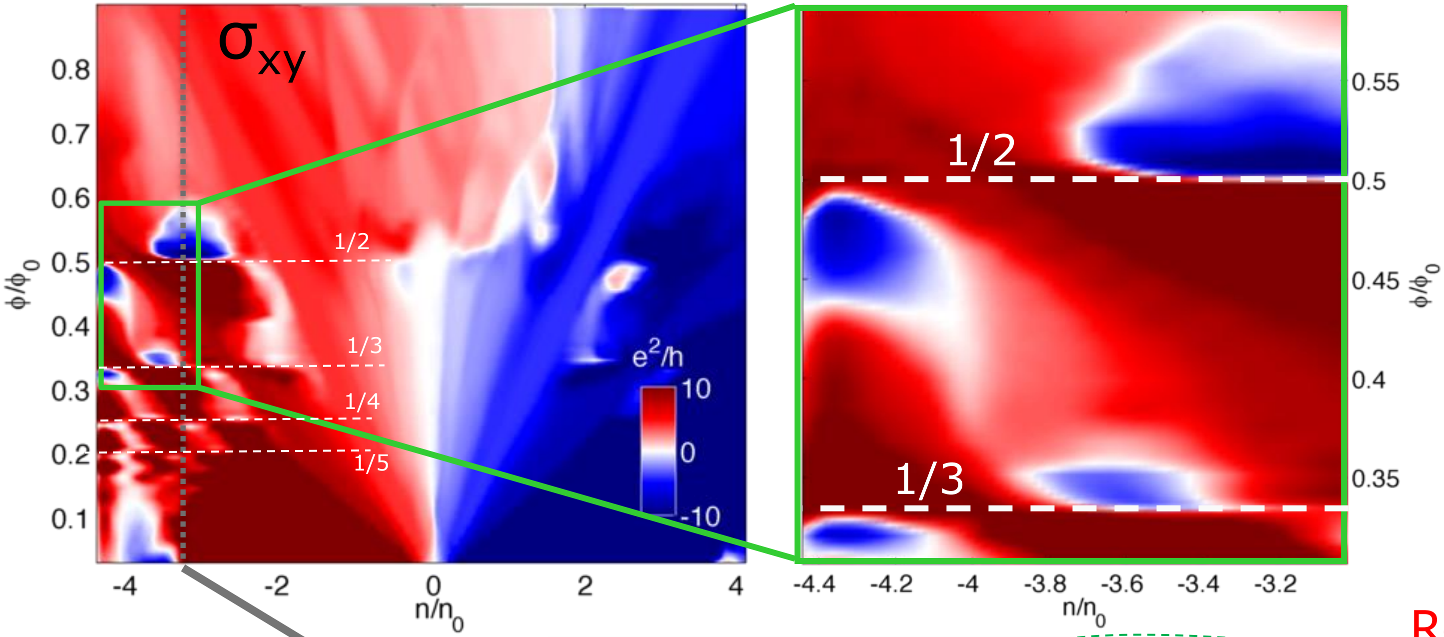
Osadchy and Avron, J. Math. Phys. 42, 5665 (2001)



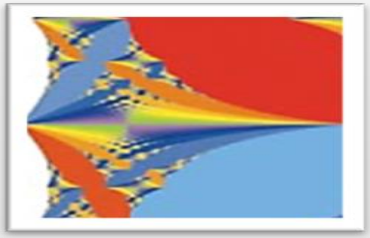
Strong suppression of R_{xy} while R_{xx} is finite!

Recursive QHE near the Fractal Bands

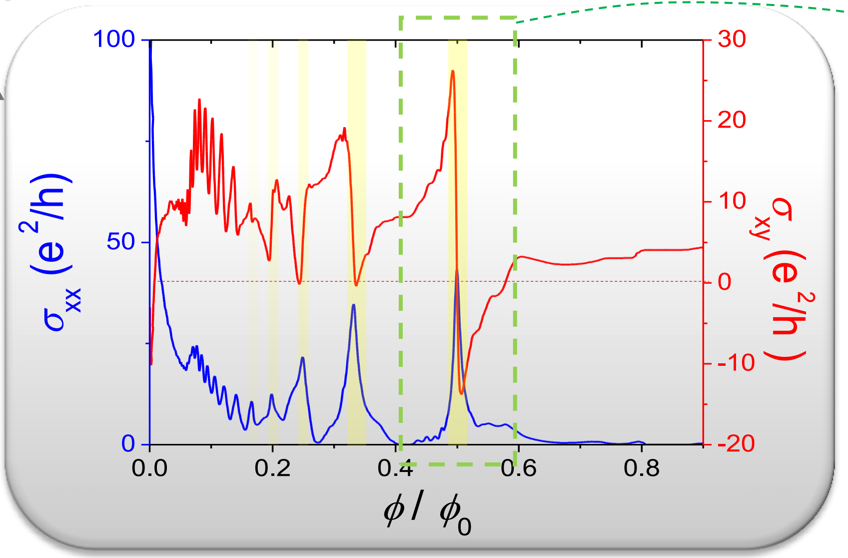
Higher quality sample with lower disorder

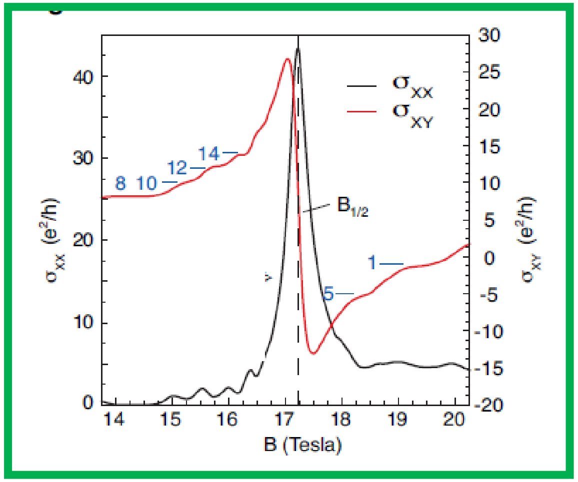


Hall conductivity across Fractal Band



Recursive QHE!





At the Fractal Bands

Sign reversal of σ_{xy}

Large enhancement of σ_{xx}

Summary

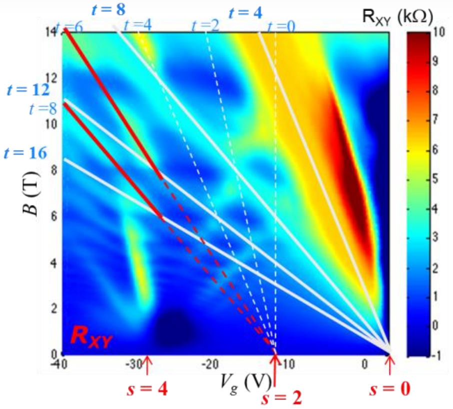
- Graphene on hBN with high quality interface created Moire pattern with supper lattice modulation

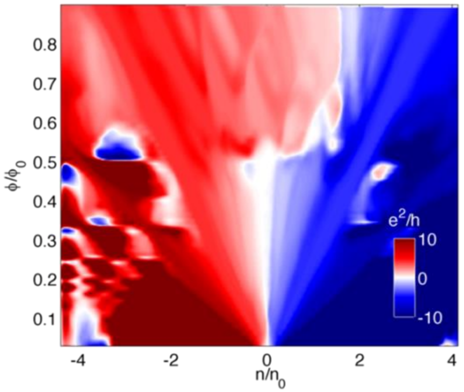
- Quantum Hall conductance are determined by two TKNN integers.

- Anomalous Hall conductance at the fractal bands

Open Questions:
- Elementary excitation of the fractal gaps?
- Role of interactions, Hofstadter Butterfly in FQHE?

$$(n/n_0) = t(\phi/\phi_0) + s$$

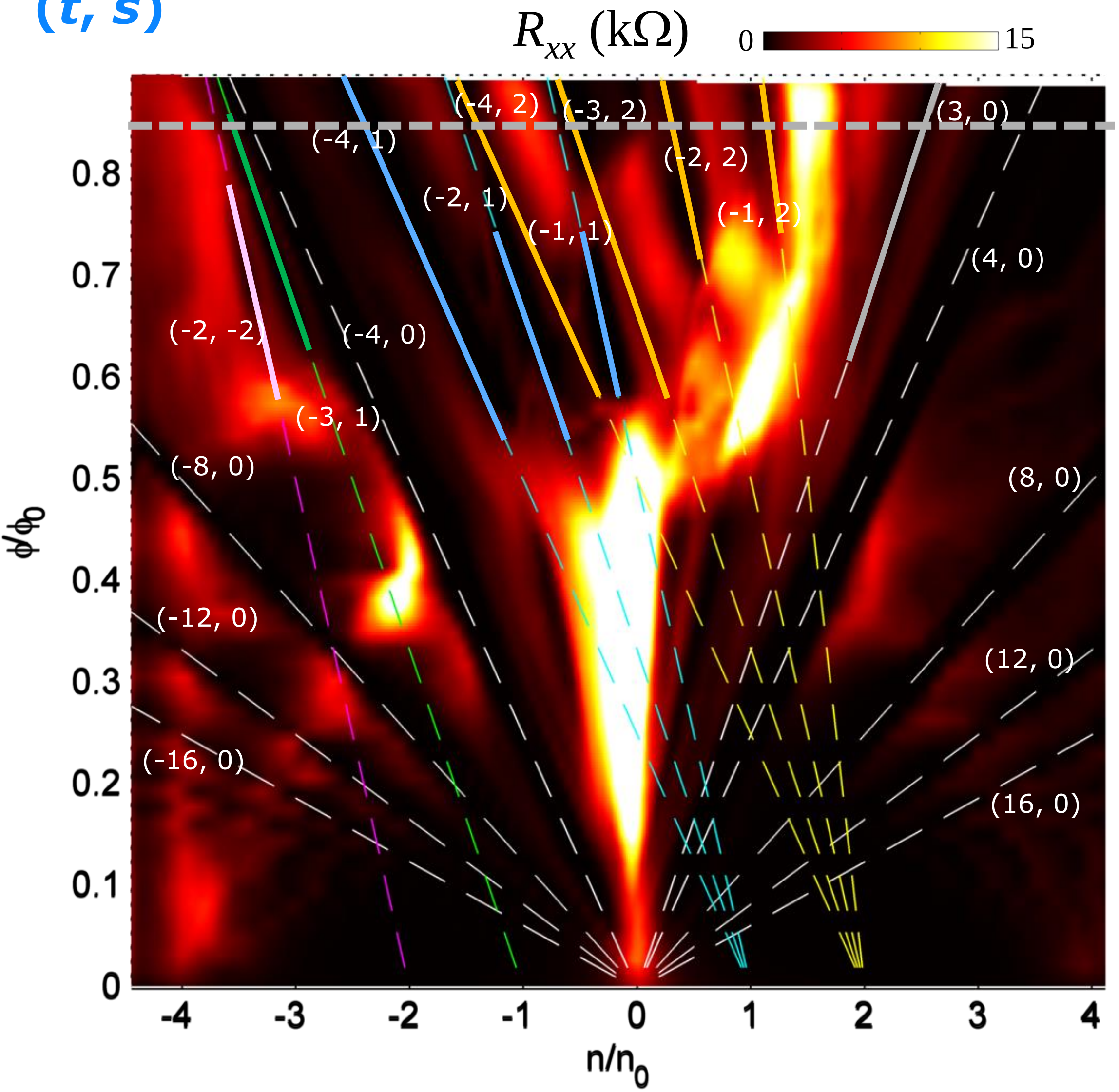




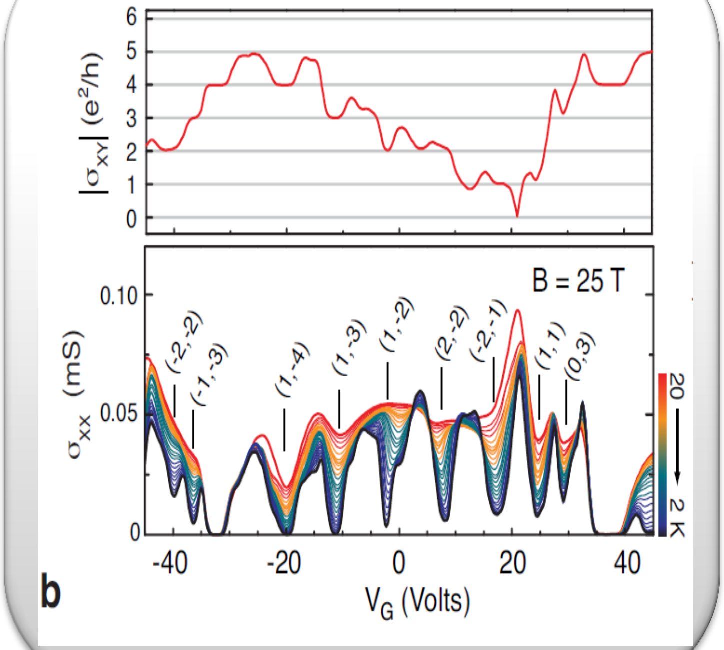
Fractal Gaps: Energy Scales

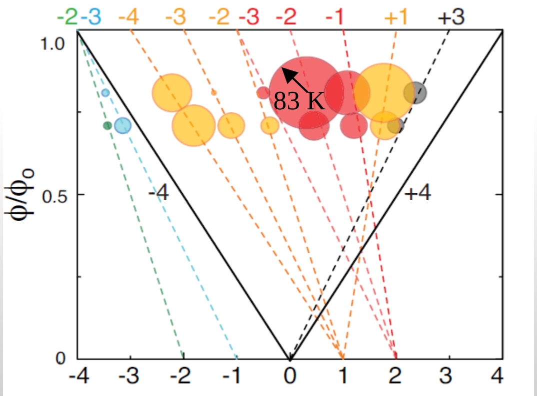
Fractal Quantum Hall Effect

(t, s)



Temperature Dependence

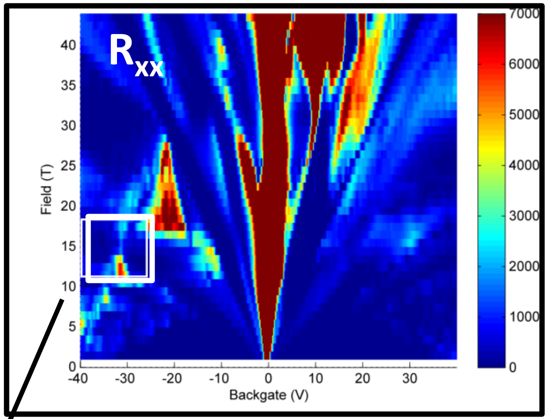


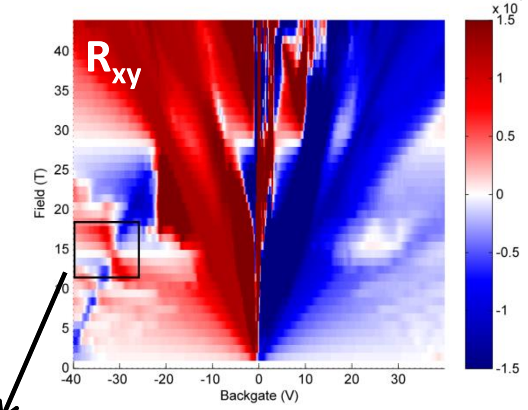


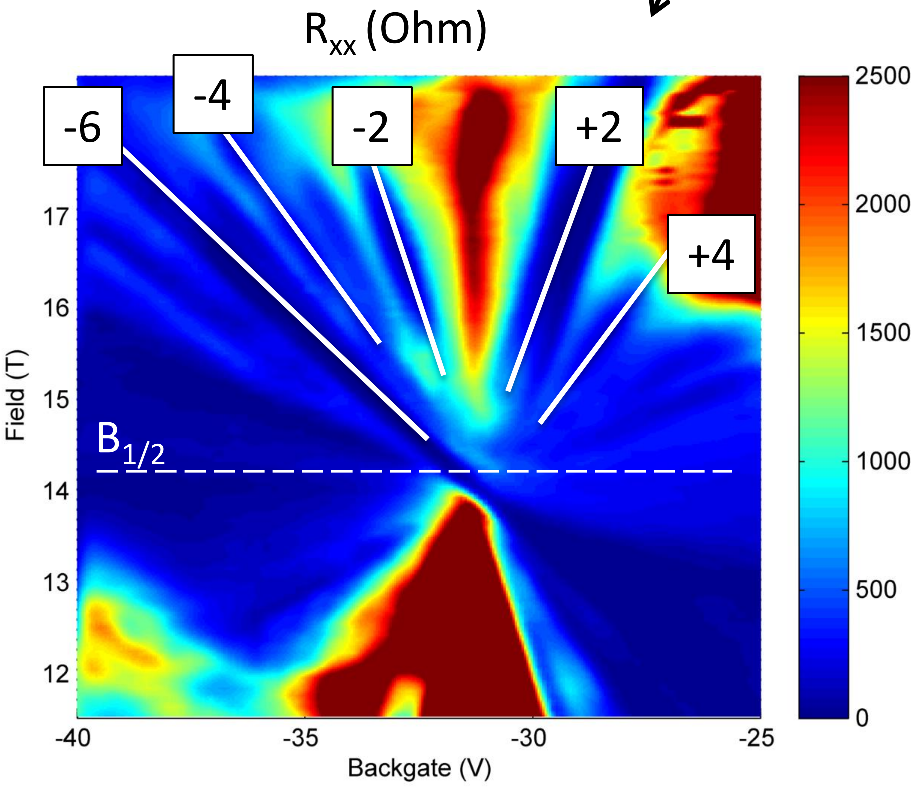
Fractal Gap Size

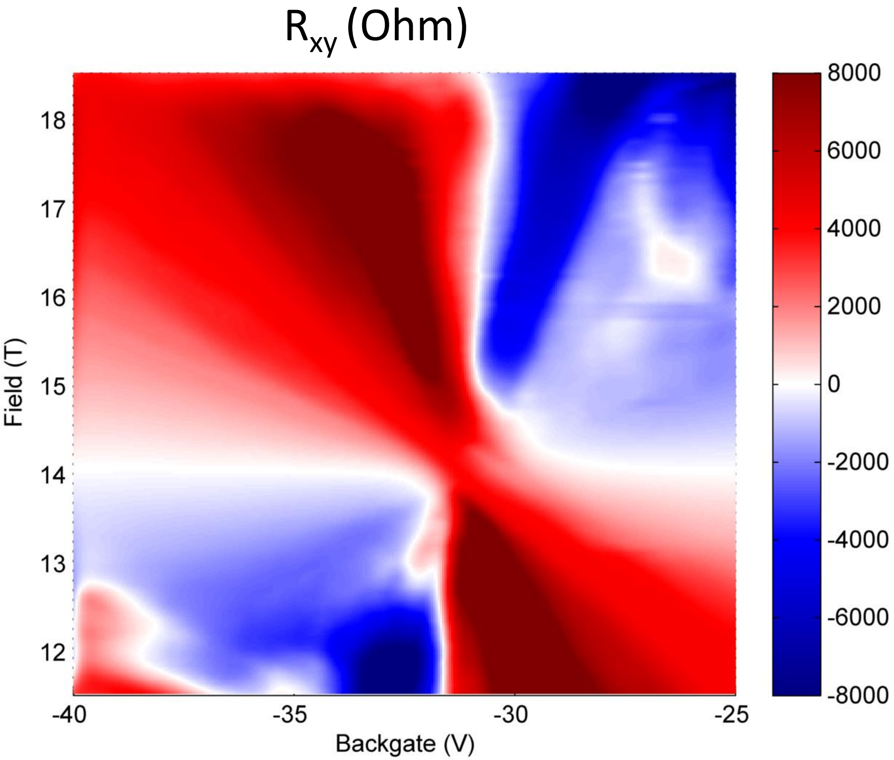
Single Layer Graphene Hofstadter's Butterfly

Similar physics is observed in single layer graphene on hBN





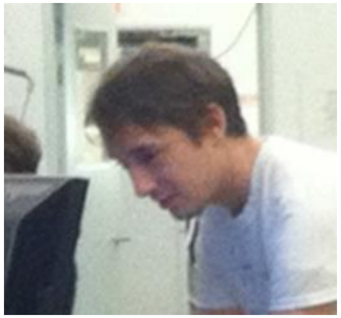

R_{xx} (Ohm)


R_{xy} (Ohm)

Acknowledgement

Dr. Cory Dean

Lei Wang

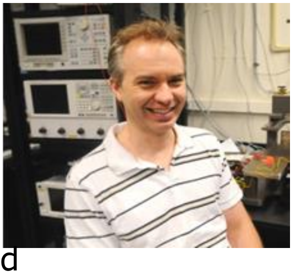


Patrick Maher

Fereshte Ghahari

Carlos Forsythe

Prof. Jim Hone



Prof. Ken Shepard

Theory: P. Moon & M. Koshino (Tohoku)

hBN samples: T. Taniguchi & K. Watanabe (NIMS)

Funding: