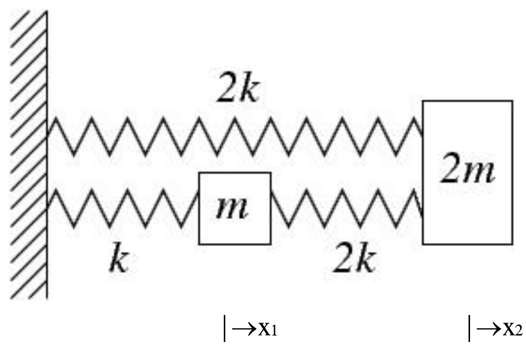
CLASSICAL MECHANICS

UPDATED: OCTOBER 2008

1. Consider the system shown below of two masses, a fixed wall, and three massless, ideal springs. Motion is restricted to one dimension (the horizontal direction). The two masses are $m_1 = m$ and $m_2 = 2m$, one spring has spring constant k, and the other two springs have spring constants $k' = 2k$, as denoted in the figure. Let x_i denote the displacement of mass i from its equilibrium position. At $t = 0$, the masses start at rest with $x_1 = 0$, and $x_2 = a$. Find the motions $x_1(t)$ and $x_2(t)$ of the masses at later times.
(Hint: Assume that the masses never hit each other or the wall. The answer does not depend on the equilibrium lengths of the springs).

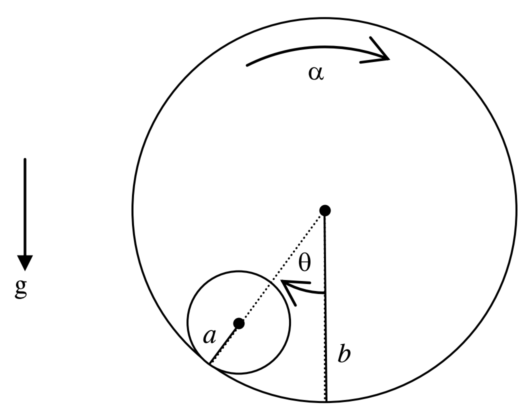


Problem #1

2. A uniform cylinder of mass m and radius a rolls without slipping inside a tube of radius b. The tube axis is at rest and horizontal in a uniform gravitational field g, but the tube is rotating about its axis with constant angular **acceleration** α. The position of the cylinder within the tube is described by θ as shown.

   (a) Using any method you wish, determine the equilibrium value of θ such that the position of the cylinder is constant in time.

   (b) If the coefficient of static friction between the cylinder and the tube is μ_s, how large can α be for the above equilibrium to be maintained without slipping?



Problem #2

3. A particle of mass m and energy E moving in one dimension comes in from $-\infty$ and encounters the repulsive potential:

$$U(x) = \frac{U_0}{\cosh^2 \alpha x}$$

where U_0 and α are given parameters, and the particle energy $E > U_0 > 0$.

Compute the delay time for this potential. In other words, how much longer will it take for the particle to travel from $x = -\infty$ to $x = +\infty$ as compared to the travel time for free motion at the same energy? (The travel time for finite L may be defined as the time it takes to travel from $-L$ to $+L$. Find an expression for this with and without the potential, and then find the limit of the time *difference* as $L \to \infty$.)

State what happens to the delay time as the particle energy E approaches (i) U_0 and (ii) infinity, and explain the physics behind these two types of behavior.

(You might need the following integral:

$$\int \frac{1}{\sqrt{x^2+bx+c}}\,dx = \ln\left|2x+b+2\sqrt{x^2+bx+c}\right| + \text{const} \;)$$

4. A particle of mass m is attached to one end of an ideal massless spring with spring constant k and relaxed length ℓ. The other end of the spring is attached to a fixed support, and the assembly hangs in a uniform gravitational field of acceleration g.

(a) Write down the Lagrangian and the equations of motion for the particle in three dimensions.

(b) Find the length of the spring and the angle from vertical that are required in order for the particle to execute uniform circular motion about the vertical axis at frequency Ω.

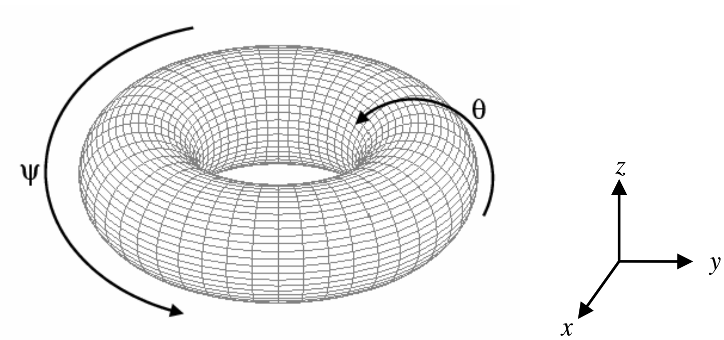
(c) Determine the range of frequencies Ω that can be physically supported, and qualitatively describe the motion of the system as Ω approaches the extremes of this range.

(d) If the particle is displaced slightly from the equilibrium of (b), it will undergo small oscillations. How many distinct frequencies will normally be observed in this motion? No calculation is required here, but do explain the reasoning for your answer.

5. A particle of mass m is confined to the surface of a torus as shown but otherwise is free. The points on the torus surface can be represented by a pair of angles ψ and θ such that

$$x = (a + b\cos\theta)\cos\psi$$
$$y = (a + b\cos\theta)\sin\psi$$
$$z = b\sin\theta$$

Here a and b are parameters, $a > b$, such that $a - b$ and $a + b$ are the radii of the innermost and outermost equatorial circles, respectively. The angles ψ and θ are oriented as shown, with $\theta = 0$ and $\theta = \pi$ corresponding to the outermost and innermost equatorial circles, respectively.



Problem #5

(a) Using ψ and θ as generalized coordinates, write down the Lagrangian and find all the constants of motion.

(b) Given the particle energy E and the z-component of its angular momentum M_z, find a condition for the motion in the variable θ to be bounded. Explain what happens when the condition under question does not hold.

(c) When the motion in the variable θ is bounded, provide explicit expressions for its turning points. Explain qualitatively how the motion changes as the energy increases from small to large values (for fixed M_z).

(d) Find the $\theta(t)$ dependence as well as the equation of trajectory, $\psi(\theta)$. Your answers should be of the form $t = \int$ of some function of θ and $\psi = \int$ of some function of θ, respectively. **Do not attempt to explicitly evaluate these integrals.**

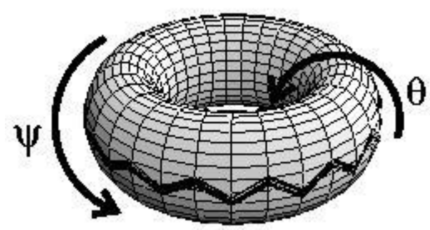
6. A particle of mass m is constrained to move on the surface of a torus (solid thick ring), shown below, but otherwise moves freely. (There is no gravity in this problem. The torus is floating somewhere in intergalactic space.) Points on the surface can be represented by a pair of angles (ψ, θ) such that

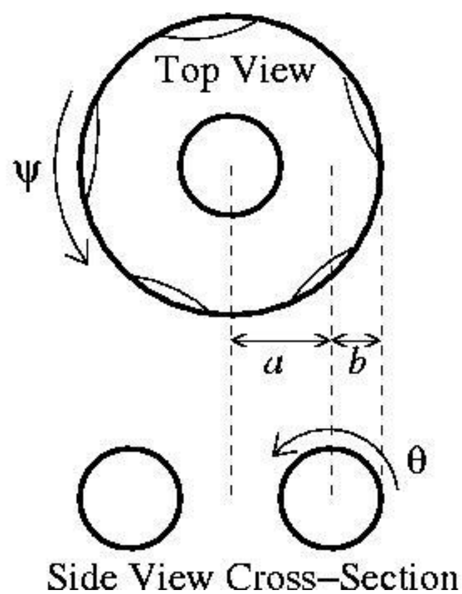
$$x = (a + b\cos\theta)\cos\psi$$
$$y = (a + b\cos\theta)\sin\psi$$
$$z = b\sin\theta$$

where $a > b$.

The particle is initially traveling around the outermost equatorial circle ($\theta = 0$) with velocity v. But then it is given a very, very tiny impulse in the θ-direction, so that it starts to oscillate while it continues to transverse the equatorial circle (as shown by the wiggling line on the torus below). Find the frequency of these oscillations in θ, treating θ as small.

[*Hint helpful for certain derivations*: If you find any equation of motion of the form d["something"]/dt = 0, then note that "something" is a constant of the motion.]

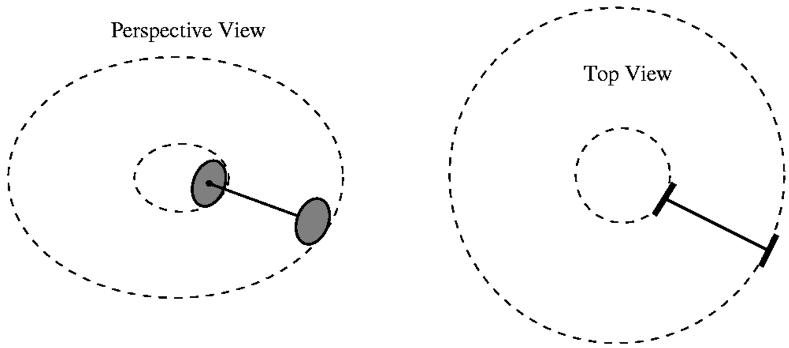




Problem #6

7. Two thin, identical wheels are joined by an axle of length L. The wheels can spin *independently* of each other but are always perpendicular to the axle. Each wheel has mass M and radius a. The axle has negligible mass. The two wheels roll without slipping, but one happens to be rolling faster than the other, so that the system travels in a circle as shown below. Let α and β be the angular speeds of the two wheels about the axis defined by the axle. Let I_1 be the moment of inertia of a single wheel about any diameter of that wheel, and let I_3 be the moment of a single wheel about the axis defined by the axle (that

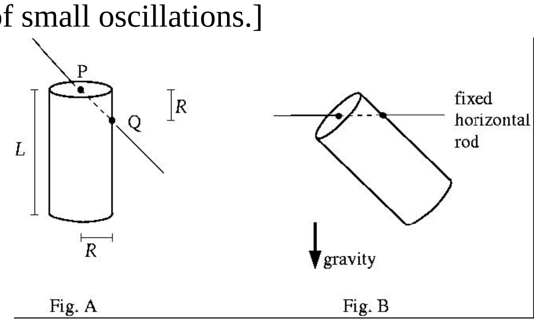
is, the axis perpendicular to the wheel). Give a formula for the total kinetic energy of the system in terms of α, β, M, a, L, I_1, and I_3.
Technical point: Assume that the wheels have negligible thickness, so that pivoting of the wheels' direction as they roll in the circles shown below dissipates negligible energy and does not count as "slipping" in this problem.



Problem #7

8. Consider a uniform solid cylinder of mass M, length L, and radius R, as shown in Fig. A. Let P denote the point at the center of the top face, and consider some point Q which is on the side of the cylinder a distance R from the top (where R is the radius of the cylinder and you may assume $L > R$). A narrow hole is drilled through the cylinder along the line PQ, and this hole is then slipped over a greased rod which is fixed to be horizontal, as shown in Fig. B below. Find the frequency of small oscillations of the cylinder about this rod, assuming it oscillates without friction, and expressing your answer in terms of g, M, L, and R.

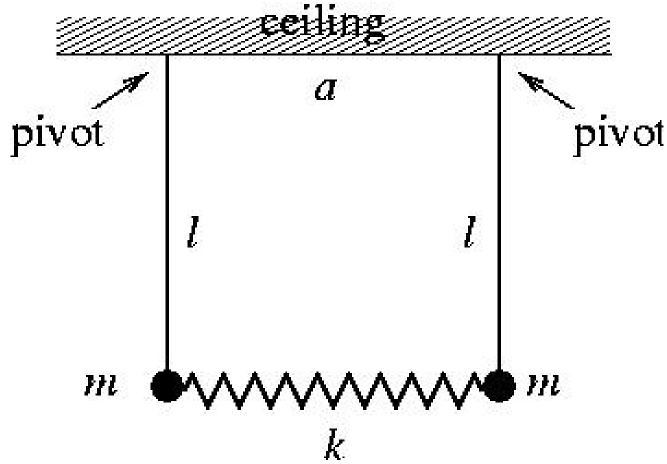
   [*Hint*: One way to do this problem is to first compute the components of the moment of inertia tensor about some conveniently chosen origin (state clearly what origin you choose) with some convenient choice of the three axis of your coordinate system. Then find the moment of inertia about the axis PQ that goes through P and Q. Then use that to find the frequency of small oscillations.]



Problem #8

9. Consider the system of two planar pendulums shown here, where the pendulums and all motion of the system lie within the same vertical plane—that of the paper in the figure below. Each pendulum consists of a massless rod of length ℓ and point-like bob of mass m, suspended from the horizontal ceiling. The pivot points of the rod are spaced a distance a

apart. The two bobs are connected by a massless spring whose equilibrium length is also *a* and whose spring constant is *k*. (a) Find all the frequencies of small oscillations of this system. (b) Describe clearly (in words or equations) the corresponding normal modes of the system, clearly identifying which mode corresponds to which frequency.



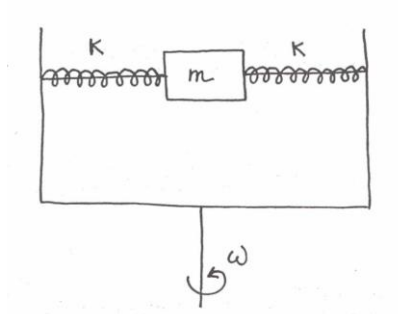
Problem #9

10. Protons, mass *m* and charge *e*, are given (nonrelativistic) energy *E* and sent as a beam to scatter from much heavier nuclei of charge *Ze*. The experiment shows that the differential cross section agrees with the Rutherford cross section for scattering angles less than some critical angle θ_c, but departs rapidly from it for larger angles. This is due to the presence of a strong force between the incoming proton and the nucleus from which it scatters; this force is of short range and effectively only comes into play when the proton touches the nucleus. Assuming that the nucleus is spherical, find its effective radius in terms of the given parameters. [*Hint*: The equation for the orbit of a particle of mass *m* and angular momentum *L* moving under the influence of a central force of magnitude k/r^2 is

$$\frac{1}{r} = \frac{mk}{L^2}\left(\sqrt{1+\frac{2EL^2}{mk^2}}\cos(\theta-\theta')+1\right)$$

where θ' is an arbitrary constant that specifies the orientation of the orbit.]

11. A bead of mass *m* can slide without friction along a horizontal rod fixed in place inside a large box. The bead is connected to the walls of the box by two large identical massless springs of spring constant *k* as sketched in the figure, and the entire box is rotated about a vertical axis through its center with angular speed ω.



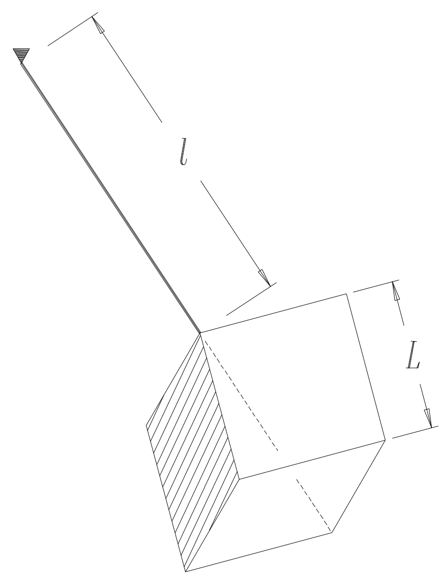
Problem #11

(a) Write down the Lagrangian using the distance r from the bead to the center of the rod as a generalized coordinate.

(b) What is the condition for the bead to be in equilibrium *off* the center of the rod? Please comment on whether this equilibrium is stable, neutral, or unstable.

(c) Compute the time dependence of the radial position of the bead $r(t)$ assuming that $r(0) = l$ (still within the box) and $dr(0)/dt = 0$ (the bead starts at a given value of r with no initial radial velocity). Note that there are two possible regimes, so please state precisely the corresponding conditions of validity.

12. A pendulum consists of a thin rod of length ℓ and mass m suspended from a pivot ▼ in the figure to the right. The bob is a cube of side L and mass M, attached to the rod so that the line of the rod extends through the center of the cube, from one corner to the diametrically opposite corner (dashed line).



Problem #14

(a) Locate the distance of the center of mass from the point of support.

(b) Find the moment of inertia I of the (entire) pendulum about the pivot point.

(*Hint*: obviously it is too hard to find the moment of inertia of a uniform cube about an arbitrary axis through its center of mass by integrating directly, so there must be some simple trick…)

(c) Write down the equation of motion in terms of I and any other relevant parameters.

(d) Find the frequency of small oscillations.

13. Under the influence of gravity, a bead of mass m slides without friction down a wire that has the form of a simple curve in a vertical plane – say the xz-plane. The bead starts at the point $(x_i, z_i) = (a, h)$ and ends at the point $(x_f, z_f) = (b, 0)$. Find the curve joining these points (brachistochrone) for which the bead reaches the end point in the least time, starting from rest at the initial point.

14. A space station orbits Earth on a circular trajectory. At some moment the captain decides to change the trajectory by turning on the rocket engine for a very short period of time. During the time the engine was on, it accelerated the station in its direction of motion. As a result, the station speed increased by a factor of α. Provide the conditions, in terms of α, that the new trajectory is elliptic, parabolic or hyperbolic. Justify your answers.

15. A particle of mass m, total energy E, and angular momentum L is moving in a central potential of the form

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2},$$

where α and β are positive constants. What is the condition for the particle motion to be bounded? For the case that the motion is bounded, compute the angular displacement $\Delta\varphi$ between two subsequent passages of the perihelion (the point $r = r_{min}$). What is the most general condition in terms of α, β, m, E, and L for the particle trajectory to be closed? (A trajectory is said to be closed if the radius-vector of the particle visits its original position more than once.)

16. An Earth satellite of mass m is placed in a circular orbit. Due to the fact that space is not an ideal vacuum, the satellite is subject to an extra frictional force **F**, which we assume is linear in the satellite velocity **v**, i.e. **F** = -*A***v** where A is a constant. This force dissipates the satellite energy so that eventually the spacecraft hits the ground, which determines its lifetime (in reality the drag constant A is a function of altitude and satellites often burn in upper layers of atmosphere, but we will ignore this). Assuming that the energy dissipated by friction during one full revolution is much smaller than the total energy, compute the lifetime of the satellite. Assume the Earth can be modeled by a sphere, the initial radius of the orbit is 10 times as big as the Earth radius, and the satellite is much lighter than the Earth.

17. Two identical point masses m are connected by a spring of constant k and unstretched/ uncompressed length a.

    (a) Write the Lagrangian of the system in terms of the 3-dimensional coordinates of the masses.

(b) Transform the Lagrangian to an appropriate set of generalized coordinates.

(c) List the conserved quantities, giving defining expressions.

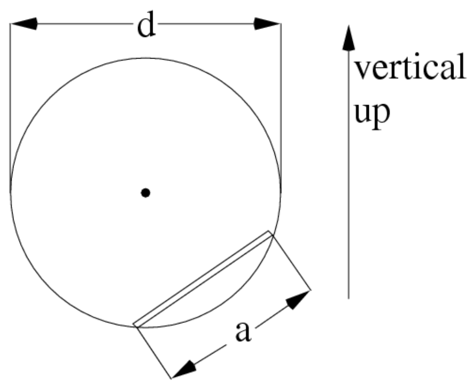
(d) If the system has large internal angular momentum, what is the equilibrium separation between the masses?

(e) Find the frequency of small oscillations about this equilibrium in the limit of large angular momentum.

18. A thin uniform rod of length a and mass m slides without friction with its two ends in contact with the inside of a vertical hoop of diameter d ($a < d$) in the gravitational field of the earth.

(a) Write down the Lagrangian.

(b) What is the angular frequency for small oscillations about equilibrium? How does it behave as $a/d \to 0$ and as $a/d \to 1$?



Problem #18

19. A free uniform disk lying on a frictionless horizontal surface is rotating about its (vertical) symmetry axis with angular velocity ω. Its center of mass is at rest. Suddenly a point on its circumference is fixed. Calculate the subsequent angular velocity.

20. A spacecraft is in a circular low-Earth orbit directly above the Equator, mean altitude 300 Km above the Earth's surface. The orbit must be transformed to a circular *geosynchronous* orbit (that is, one that keeps the spacecraft directly above the same point on the Equator).

(a) What is the radius of the geosynchronous orbit?

(b) The pilot wishes to attain the orbit change by *two* applications of the rockets (the minimum possible). What sort of intermediate orbit will the spacecraft be in, after the first period of acceleration?

(c) What is the change-of velocity that must be applied to transform the initial circular orbit to the intermediate orbit? That is, what change of speed is necessary and in what direction?

(d) What is the change-of-velocity necessary to transform the intermediate orbit into the final circular geosynchronous orbit? (Again specify change-of-speed and direction.)

You may, if you wish, assume the periods of acceleration have the form of instantaneous impulses.

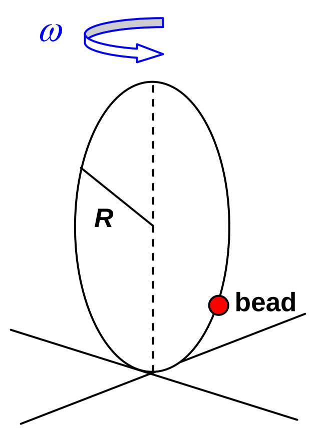
21. A spherical pendulum consists of a point mass m hung by a massless string of length R from a fixed point on a ceiling. θ is the angle the string makes from the vertical, where $\theta = 0$ is down, and angle ϕ is the azimuthal angle of the string about the vertical.

(a) Write the Lagrangian for the motion of the mass. You may assume that the string is always taut.

(b) With what velocity and in what direction must the mass be set in motion to make a circular orbit with $\theta = 30°$?

(c) If the mass is launched with a slight error and it oscillates about $\theta = 30°$, what is the angular frequency of the oscillation in θ?

22. A bead of mass m is free to slide along a smooth wire making a rigid circle of radius R (see the figure). The circle is oriented with its plane in the vertical and rotates with a constant angular velocity ω about the vertical diameter, and the bead can find a stable equilibrium position as a result.

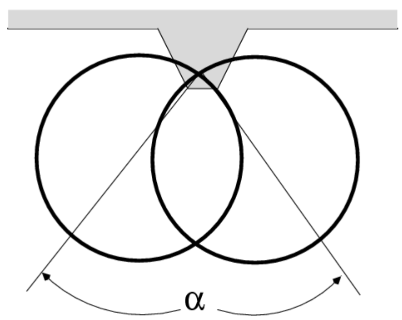


Problem #22

(a) For sufficiently large ω, the bead has a point of stable equilibrium that is *not* at the bottom. How big must ω be so that there is such a stable equilibrium point?
(b) The condition for an off-bottom stable equilibrium point is satisfied. Consider a small displacement from that position and obtain the condition for, and frequency of, simple harmonic motion about it.

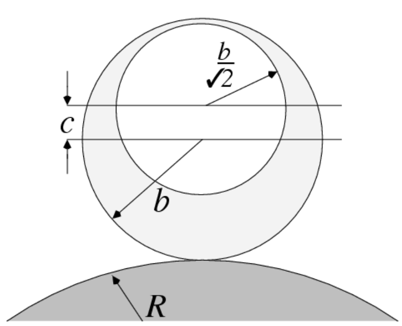
23. A square lamina, sides $2a$, mass m, is lying on a table when struck on a corner by a bullet of mass m with velocity v parallel to one of the edges of the lamina (inelastic). Find the subsequent angular velocity of the lamina.

24. Two rings of equal mass M and radius R are rigidly fastened together at a point on their periphery so that their diameters form an angle α. If they are free to swing as a pendulum in a vertical plane, find the torque τ_α tending to change the angle α for small motion about the position of equilibrium



Problem #24

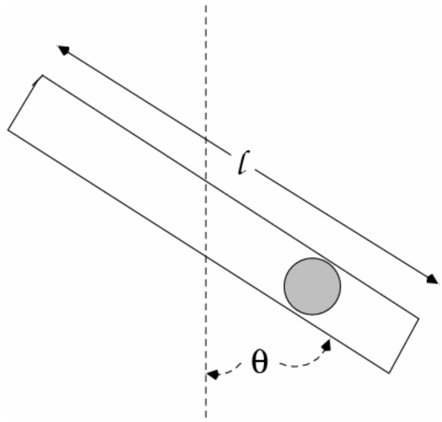
25. A solid cylinder of radius b has a cylindrical hole of radius $b/\sqrt{2}$ cut out of it. The hole is centered at a distance c from the center of the cylinder. This cylinder is at rest on top of a large perfectly rough, fixed cylinder of radius R as shown. For what values of R is the equilibrium position shown stable, and what will be the frequency of small oscillations about this equilibrium position?



Problem #25

26. A particle of mass m is placed in a smooth uniform tube of mass M and length ℓ. The tube is free to rotate about its center in a vertical plane. The system is started from rest with the tube horizontal and the particle a distance r_0 from the center of the tube.

For what length of the tube will the particle leave the tube when $\dot{\theta} = \omega$ is a maximum and $\theta = \theta_m$? Your answer should be in terms of ω and θ_m.



Problem #26

27. Two particles of equal mass m interact according to a (three-dimensional) spherical well potential,

$$V(r) = \begin{cases} 0, & r > a \\ -V_0, & r < a \end{cases}$$

where V_0 is positive. Initially the particles are separated by some distance greater than a, with one at rest and the other moving with speed v_0. Calculate the differential cross-section for scattering.

28. (a) State Euler's equations of motion, defining all terms precisely.

(b) Define Euler's angles for rigid body motion and express the body components of angular velocity in terms of them.

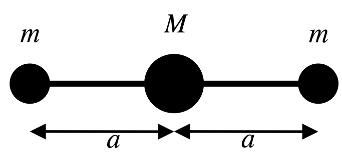
(c) A symmetrical top, (the moments of inertia are $I_1 = I_2, I_3 \neq I_1$) of mass M spins with one point fixed in the earth's gravitational field. Its center of mass is a distance b from the fixed point. Express Euler's equations for the top in terms of Euler's angles and impose the solution of a uniformly precessing top without nutation, i.e. the angle between the figure axis and the vertical direction remains constant. Substituting this solution into the equations of motion, obtain a condition between $I_1, I_3, M, b, \phi, \psi$ *and* θ for this solution to be valid.

29. A double pendulum has equal lengths, but the upper mass is much greater than the lower. Obtain the exact Lagrangian for motion in a vertical plane, and then make the approximation of small motion. What are the resonant frequencies of the system? What is the resultant motion if the system initially at rest is subjected at time $t = 0$ to a small impulsive force applied horizontally to the upper mass?

30. (a) In terms of its relationship to the kinetic energy, derive an expression for the inertia tensor of a rigid body relative to an arbitrary Cartesian coordinate system. Show how to diagonalize this matrix and prove the reality of the eigenvalues and eigenvectors obtained.

(b) What is the moment of inertia about the diagonal of a homogeneous cube?

31. Consider a classical treatment of the small oscillations of the atoms in a linear triatomic molecule. Two of the masses are equal and at equilibrium are located a distance a from the third. Making a reasonable assumption about the functional form of the potential for small motions, obtain the eigenfrequencies of vibration and describe the mode of vibration for each eigenfrequency. You need only consider motion along the molecular axis.



Problem #31

32. A particle of mass m moves in a central force field in a circular orbit of radius r_0 with angular speed ω.

(a) State the relation between m, r_0, ω, and $\partial V/\partial r$, where $V(r)$ is the potential.

(b) Consider small radial perturbations from the circular orbit, and describe them by defining variables ρ and ϕ so that the polar coordinates of the particle are $r = r_0 + \rho$ and $\theta = \omega t + \phi$. Expand the potential function in a Taylor series in ρ and write the Lagrangian, ignoring terms higher than second order in the perturbations ρ and ϕ and their time derivatives.

(c) Derive the equations of motion and make use of part (a) to show that stable oscillations in ρ will result if

$$\frac{3}{r_0}\frac{\partial V}{\partial r}\bigg|_{r_0} + \frac{\partial^2 V}{\partial r^2}\bigg|_{r_0} > 0 .$$

33. A rigid body rotates freely about its center of mass. There are no torques. Show by means of Euler's equations that if all three principal moments of inertia are different, then the body will rotate stably about either the axis of greatest moment of inertia or the axis of least moment of inertia, but that rotation about the axis of intermediate moment of inertia is unstable.

34. If a particle is projected vertically upward from a point on the earth's surface at northern latitude λ, show that it strikes the ground at a point $4/3\,\omega\cos\lambda\sqrt{8h^3/g}$ to the west (neglect air resistance and consider only small vertical heights).

35. A thin hoop of radius R and mass M is allowed to oscillate in its own plane (a vertical plane) with one point of the hoop fixed. Attached to the hoop is a small mass M which is constrained to move (in a frictionless manner) along the hoop. Consider only small oscillations and show that the eigen frequencies are

$$\omega_1 = \sqrt{2}\sqrt{g/R} \quad \omega_2 = \frac{\sqrt{2}}{2}\sqrt{g/R}$$

36. Show that the angular deviation ε of a plumb line from the true vertical at a point on the earth's surface at latitude λ is

$$\varepsilon = \frac{r_0\omega^2 \sin\lambda\cos\lambda}{g - r_0\omega^2\cos^2\lambda}$$

where r_0 is the radius of the earth.

37. Two discs of radius R and mass M are connected by an axle of radius r and mass m. A pendulum of length ℓ with a bob of mass μ is suspended from the midpoint of the axle. If the wheels and the axle can roll without changing the length of the pendulum, find the equations of motion of the system if it rolls down a slope of angle α.

38. Two bodies of equal mass m are connected by a smooth, fixed-length string which passes through a hole in a table. One body can slide without friction on the table; the other hangs below the hole and moves only along a vertical line through the hole. Using polar coordinates (r, θ) for the body on the table, write the Lagrange equations for the system, reduce them to a single second-order differential equation, and integrate this equation once to obtain an energy equation. Find the equation whose roots yield the maximum and minimum values of r. Imagine the string is long enough, and the legs of the table tall enough, so that the hanging body hits neither the hole nor the floor.

39. A bar of length $2L$, mass m, slides without friction on a horizontal plane. Its velocity is perpendicular to the axis of the bar. It makes a fully elastic impact (energy conserved) with a fixed peg at a distance a from the center of the bar. Using conservation of energy, momentum and angular momentum, find the final velocity of the center of mass.

40. A uniform hoop of mass M and radius a can roll without slipping on a horizontal floor. A small particle of mass m is constrained to slide without friction on the inside rim of the hoop.

(a) Introduce two coordinates that specify the instantaneous state of the system and calculate the Lagrangian in terms of those coordinates. One of the coordinates should be chosen so that it is zero when the particle is at floor level.

(b) Assuming that the particle never rises far off the floor, write down the equations of motion and show that the general motion consists of an oscillation of period

$$\tau = 2\pi \left(\frac{a}{g} \frac{2M}{2M+m} \right)^{1/2}$$

superimposed on a uniform translation.