Projection Hamiltonians for clustered quantum Hall wavefunctions

T. S. Jackson, N. Read and S. H. Simon

Work in progress
(brief) Talk outline

- Introduction to the theory of the fractional quantum Hall effect
  - Trial wavefunctions, projection Hamiltonians and conformal field theories (CFTs)

- Relating CFTs to Hamiltonians
  - Our example: three-body Hamiltonian and supersymmetric CFTs
  - Progress on obtaining Hamiltonians for a specific theory
Background on the quantum Hall effect
QHE: motivation

• Nobel prizes:
  • von Klitzing, 1985 — integer effect
  • Laughlin, Störmer, Tsui, 1998 — fractional effect
• Topological order: no local order parameter
• Realization of extended spin-statistics in $d=2$
• Application: topologically protected quantum computing
**The quantum Hall effect**

Landau level filling fraction \( \nu = n_e \frac{\Phi_0}{B} \)

\[
\rho(E) \quad \begin{array}{c}
\hbar \omega_c = \frac{\hbar eB}{mc} \\
\{BA/\Phi_0
\end{array}
\]

From Stormer, RMP 71, 875
The quantum Hall effect

Landau level filling fraction $\nu = n_e \frac{\Phi_0}{B}$

From Stormer, RMP 71, 875
The quantum Hall effect

Landau level filling fraction $\nu = n_e \frac{\Phi_0}{B}$

\[ \rho(E) \]

\[ \hbar \omega_c = \frac{\hbar e B}{mc} \]

From Stormer, RMP 71, 875
The quantum Hall effect

Landau level filling fraction \( \nu = n_e \frac{\Phi_0}{B} \)

From Eisenstein and Stormer, Science 248, 1461

From Stormer, RMP 71, 875
FQHE trial wavefunctions

FQH state is an \textit{incompressible} electron fluid

FQH droplet (in plane)  
Quasi-hole excitation  
Quasi-electron excitation  
(chiral) edge excitation

Symmetric gauge & lowest Landau level  
⇒ wavefunctions ≈ \textit{analytic polynomials}

\[
\psi_m(z) \propto \frac{z^m}{\ell_B^{m+1}} e^{-\frac{1}{4} \frac{|z|^2}{\ell_B^2}} \\
\Psi(z_1, \ldots, z_N) \propto \det [\psi_m(z_n)]_{m,n}
\]

\textit{Problem}: Landau levels are macroscopically degenerate; can’t set up perturbation theory around non-interacting system!
FQHE trial wavefunctions

**Problem:** Landau levels are macroscopically degenerate; can’t set up perturbation theory!

**Laughlin:** Account for Coulomb repulsion by extra Jastrow factors

\[ \Psi_L(z_1, \ldots, z_N) \propto \prod_{i<j} (z_i - z_j)^m \cdot e^{-\frac{1}{4} \sum_i |z_i|^2} \quad \nu = \frac{1}{m} \]

(Validity established by exact diagonalization)

**Quasihole:** \( \Psi_{L,w} \propto \prod_i (z_i - w) \cdot \prod_{i<j} (z_i - z_j)^m \cdot e^{-\frac{1}{4} \sum_i |z_i|^2} \)

Quasiholes have *anyonic statistics:*  
braiding phase of \( \theta = \pi/m \)
Projection Hamiltonians

Rest of talk: FQHE of bosonic particles (w/log) ⇒ LLL Hilbert space ≡ symmetric polynomials

Haldane: Laughlin state is unique, exact highest-density eigenstate of projection Hamiltonian

\[ \mathcal{H} = \sum_{i<j} \sum_{\ell} \frac{1}{\nu} V_\ell \mathcal{P}_{i,j}[\ell] \]

\[ \Psi(z_1, \ldots, z_N) = \sum_a \psi_a(z_1, z_2) \Psi'_a(z_3, \ldots, z_N) \]

\[ \psi_a(z_1, z_2) = \sum_{b,c} C_{a}^{b,c} \psi_b^{CM} \left( \frac{1}{2}(z_1 + z_2) \right) \psi_c^{rel}(z_1 - z_2) e^{\frac{1}{4}(|z_1|^2 + |z_2|^2)} \]

Trial wavefunctions ↔ Projection Hamiltonians
**FQHE and CFT**

Moore & Read: FQHE trial wavefunctions from “conformal blocks” of conformal field theory

**Why?** Fractional quasiparticle statistics

⇒ Chern-Simons TQFT

⇒ wavefunctions are CFT amplitudes (Witten)

(Read: $d=1+1$ edge excitation CFT same as $d=2+0$ bulk wavefunction CFT)

Opens door for *non-Abelian* statistics!

(Willett, Pfeiffer & West): Experimentally observed?
"CFT for pedestrians"

Infinite number of local conformal transformations in $d=2 \Rightarrow$ finite amount of data to specify theory

Virasoro algebra $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$

Rational theories have a finite number of primary fields (descendants may be singular)

\[
\begin{align*}
|\phi\rangle & \quad L_{-1}|\phi\rangle & \quad L_{-2}|\phi\rangle, L_{-1}^2|\phi\rangle & \quad L_{-3}|\phi\rangle, L_{-2}L_{-1}|\phi\rangle, L_{-1}^3|\phi\rangle & \quad \cdots
\end{align*}
\]

Operator product expansion (OPE)

\[
T(z)\phi(w, \bar{w}) \sim \frac{h\phi(w, \bar{w})}{(z - w)^2} + \frac{\partial\phi(w, \bar{w})}{z - w}
\]

Ising CFT: $(c = 1/2)$

Fusion rules:

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$\psi$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>$1/2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$1/16$</td>
<td>$\sigma$ $\sigma$ $1 + \psi$</td>
</tr>
</tbody>
</table>

$\Rightarrow$ vector space of conformal blocks

$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4) \rangle$
Steps towards relating the CFT and Hamiltonian descriptions
Projection Hamiltonians

Rest of talk: FQHE of bosonic particles \((w/log)\) \implies LLL Hilbert space \(\equiv\) symmetric polynomials

**Haldane:** Laughlin state is unique, exact highest-density eigenstate of projection Hamiltonian

\[
\mathcal{H} = \sum_{i<j} \sum_{\ell} \frac{1}{\nu} V_\ell P_{i,j}[\ell]
\]

\[
\Psi(z_1, \ldots, z_N) = \sum_a \psi_a(z_1, z_2) \Psi'(z_3, \ldots, z_N)
\]

\[
\psi_a(z_1, z_2) = \sum_{b,c} C_{a}^{b,c} \psi^{CM}_b \left( \frac{1}{2}(z_1 + z_2) \right) \psi^{rel}_c(z_1 - z_2) e^{-\frac{1}{4}(|z_1|^2 + |z_2|^2)}
\]

Trial wavefunctions \(\leftrightarrow\) Projection Hamiltonians
Few-body Hamiltonians

Simon, Rezayi & Cooper: Systematic study of multiparticle pseudopotential Hamiltonians

⇒ Basis: translationally-invariant symmetric polynomials of degree $r$ in $k+1$ variables ($\nu = \frac{k}{r}$)

$(k = 2, r = 2): \tilde{e}_2 = z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_1z_3$

$D(k,r) =$ Dimension of space spanned by these polynomials

<table>
<thead>
<tr>
<th>$r =$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
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</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>$k = 2$</td>
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<td>2</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
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<td>5</td>
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<td>7</td>
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<tr>
<td>$k = 5$</td>
<td>1</td>
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<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
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<td>14</td>
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</tbody>
</table>
Q: What if the Hamiltonian penalizes all but one $k+1$-particle behavior at given $r$?

Hamiltonian will contain continuous free parameters selecting direction in subspace

Why? Important limitation of existing methods!

- Thin torus limit (Seidel, Lee et. al., Bergholtz, Karlhede, Hansson, Hermanns et. al.; Ardonne): CDW orbital filling only specifies integer data; limit not unique

![Diagram](image)
Q: What if the Hamiltonian penalizes all but one $k+1$-particle behavior at given $r$?

Hamiltonian will contain continuous free parameters selecting direction in subspace.

Why? Important limitation of existing methods!

- Jack polynomials (Bernevig, Haldane et. al.): $(k,r)$ fix state for single Jacks; correspond to $M_k(k+1,k+r)$ CFTs (Estienne & Santachiara)

From Bernevig and Haldane, PRL 100, 246802
Q: What if the Hamiltonian penalizes all but one $k+1$-particle behavior at given $r$?

Hamiltonian will contain continuous free parameters selecting direction in subspace

Why? Important limitation of existing methods!

- “Pattern of zeros” (Wen, Wang et. al.): also discrete; not unique and sufficient conditions not known; (later papers) additional CFT data must be added a priori
Q: What if the Hamiltonian penalizes all but one \( k+1 \)-particle behavior at given \( r \)?

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>( r=0 )</td>
<td>1</td>
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<td>( k=3 )</td>
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<td>( k=4 )</td>
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This work

Two linearly-independent ways for wavefunction to vanish as \( r=6 \) powers as \( k+1=3 \) particles coincide:

\[
\tilde{e}_2(z_1, z_2, z_3)^3 \propto (z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_1z_3 - z_2z_3)^3
\]

\[
\tilde{e}_3(z_1, z_2, z_3)^2 \propto (z_1 + z_2 - 2z_3)^2 (z_1 - 2z_2 + z_3)^2 
\times (-2z_1 + z_2 + z_3)^2
\]
Connection to CFTs

<table>
<thead>
<tr>
<th>$\mathbf{D}(k,r)$</th>
<th>0</th>
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This work, SCFTs

Simon, Rezayi & Regnault, GS ↔ $S_3$ CFTs

Simon: Supercurrent amplitudes at arbitrary $c$

$$
\langle G(z_1) \cdots G(z_{2n}) \rangle \propto J_{2n}^{-3} S \left[ \prod_{1 \leq i < j \leq n} \chi_c(z_{2i-1}, z_{2i}; z_{2j-1}, z_{2j}) \right]
$$

$$
\chi_c(z_1, z_2; z_3, z_4) = 3z_{1,3}^4 z_{1,4}^2 z_{2,3}^2 z_{2,4}^4 + (c - 3)z_{1,3}^3 z_{1,4}^3 z_{2,3}^3 z_{2,4}^3
$$

Three-particle behavior:

$$
S \left[ \lim_{z_4 \to \infty} z_4^{-6} \chi_c \right] = -(6 + 5c) \tilde{e}_2(z_1, z_2, z_3)^3 + (-\frac{81}{2} + 27c) \tilde{e}_3(z_1, z_2, z_3)^2
$$

SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

Set of zero-energy edge excitations = polynomial ideal $I_N$

Clustering map $C_m$: make $m$ clusters of $k=2$ particles

$C_m \Psi(z_1, \ldots, z_N) = \Psi(Z_1, Z_1, Z_2, Z_2, \ldots, Z_m, Z_m, z_{2m+1}, \ldots, z_N)$
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$\text{Im } C_m \cap I_N \propto \prod_{2m<i,j\leq m} (z_i - Z_j)^6 \cdot \prod_{i<j\leq m} (Z_i - Z_j)^{12}$

$F_m = \ker C_m \cap I_N; \quad F_0 = 0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{N/2} = I_N$
SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

\[ F_m = \ker C_m \cap I_N; \quad F_0 = 0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{N/2} = I_N \]

\[
\text{Im } C_m|_{F_{m+1}} / \ker C_m \subseteq \prod_{2m<i,j \leq m} (z_i - Z_j)^6 \cdot \prod_{i<j \leq m} (Z_i - Z_j)^{12} \]

Cluster-cluster and cluster-particle factors

\[
\times \prod_{2m<i<j} (z_i - z_j)^2 \cdot I_{N-2m}^{MR} \otimes \Lambda_m
\]

Irreducibility of three-body interaction

Charge excitations

Show this is an equality by construction of basis
SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

\[
\Psi_{\text{SCFT}} = \prod_{\text{all entires}} \chi_c \prod_{\text{pairs of rows}} \chi_{c\alpha} \in I_{MRN-2m_1} \{\alpha\}
\]

\[
\Psi_{\text{SCFT}} = S \prod_{\text{all entires}} \chi_c \prod_{\text{pairs of rows}} \chi_{c\alpha} \in I_{MRN-2m_1} \{\alpha\}
\]

\[
\zeta_{i,j}^{(\alpha)}
\]

\[
\alpha = 1
\]

\[
\alpha = 2
\]

\[
\alpha = 3
\]

\[
m = m_1
\]

\[
m = m_2
\]

\[
m = m_3
\]
SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

$\Psi^{\text{SCFT}} = S \prod_{\text{alle}} \chi_c \prod_{\text{pairs of rows}} \bar{\chi}_c$
SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

\[
\Psi^{\text{SCFT}} = S \prod_{\text{paired particles}} \prod_{\text{pairs of rows}} \chi_c \in I_{N-2m_1}^{\text{MR}}
\]

\[
D_2 \times \prod_{\text{all entries}} \overline{\psi}_{\{1,1\}} \times \prod_{\text{pairs of rows}} \overline{\psi}_{\{1,1\}}
\]

\[
m = m_1
\]

\[
\alpha = 1 \quad \alpha = 2 \quad \alpha = 3
\]

\[
\begin{array}{c}
m_1 \\
\vdots \\
m_2 \\
\vdots \\
m_3 \\
\vdots \\
\end{array}
\]

\[
\begin{array}{c}
(\alpha) \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

\[
\begin{array}{c}
z_{i,j} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

\[
\alpha = 1 
\]

\[
\begin{array}{c}
m_1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

\[
\prod_{\text{paired particles}} \chi_c
\]

\[
\prod_{\text{pairs of rows}} \chi_c
\]

\[
\prod_{\text{all entries}} \overline{\psi}_{\{1,1\}}
\]

\[
\prod_{\text{pairs of rows}} \overline{\psi}_{\{1,1\}}
\]
SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

\[ \Psi_{\text{SCFT}} = S \]

\[ \alpha = 1 \quad \alpha = 2 \quad \alpha = 3 \]

\[ m = m_1 \quad m = m_1 \quad m = m_1 \]

\[ \begin{array}{c}
\alpha = 1 \\
m = m_1 \\
\alpha = 2 \\
m = m_2 \\
\alpha = 3 \\
m = m_3 \\
\end{array} \]

\[ \begin{array}{c}
\prod \chi_c \quad \text{all entries} \\
\prod \chi_c \quad \text{pairs of rows} \\
D_2 \quad \text{paired particles} \\
\prod \bar{\psi}_{\{1,1\}} \quad \text{“half-broken” pairs} \\
\prod \bar{\psi}_{\{1,1\}} \quad \text{unpaired particles} \\
\end{array} \]

\[ \begin{array}{c}
\prod \bar{\chi}_c \quad \text{unpaired particles} \\
\end{array} \]
SCFT wavefunctions

Obtain basis for all zero-energy edge excitations, explicitly, via filtration method (Ardonne, Kedem, Stone; Read)

\[ \Psi_{SCFT} = S \]

\[ m = m_1 \]

\[ \alpha = 1 \]

\[ \alpha = 2 \]

\[ \alpha = 3 \]

\[ m_2 \]

\[ m_3 \]

\[ z_{i,j}(\alpha) \]

\[ \prod \chi_c \text{ pairs of rows} \]

\[ \prod \vartheta_{\{1,1\}} \text{ pairs of rows} \]

\[ \prod \chi_c \text{ all entries} \]

\[ \prod \vartheta_{\{1,1\}} \text{ all entries} \]

\[ D_2 \]

\[ e_{\lambda(1)} \]

\[ e_{\lambda(2)} \]

\[ e_{\lambda(3)} \]

charge excitations

paired particles

“half-broken” pairs

unpaired particles

\[ \in I_{N-2m_1}^{MR} \]
Counting wavefunctions

- # edge excitations at given angular momentum = character of edge excitation CFT (Wen)
- State counting gives character for generic SCFT — independent of $c$!

$$q^{-\frac{3}{2}N(N-2)} \text{ch } I_N = \sum_{m_2,m_3 \geq 0: 2m_2 + m_3 \leq N, (-1)^m_3 = (-1)^N} \frac{q^{2m_2 + \frac{1}{2}m_3(m_3+2)}}{(q)^{\frac{1}{2}(N-2m_2-m_3)}(q)m_2(q)m_3}$$

$$\lim_{N \to \infty} \frac{1}{(q)_{\infty}} \chi_{Kac}^\pm = \frac{(1 - q)}{(1 \pm q^{1/2})} \prod_{r=1}^{\infty} (1 \pm q^{r-1/2}) \frac{(1-q^{2})}{(q)_{\infty}}$$
Counting wavefunctions

• # edge excitations at given angular momentum = character of edge excitation CFT (Wen)

• State counting gives character for generic SCFT — independent of $c$!

\[
q^{-\frac{3}{2}N(N-2)} \text{ch } I_N = \sum_{m_2,m_3 \geq 0: 2m_2+m_3 \leq N, \ (-1)^{m_3} = (-1)^N} \frac{q^{2m_2+\frac{1}{2}m_3(m_3+2)}}{(q)^{\frac{1}{2}(N-2m_2-m_3)}(q)m_2(q)m_3} \]

\[
\lim_{N \to \infty} \left. \frac{1}{(q)_{\infty}} \chi^\pm_{\text{Kac}} \right|_{(plane)} = \frac{(1-q)}{(1 \pm q^{1/2})} \prod_{r=1}^{\infty} \frac{(1 \pm q^{r-1/2})}{(q)^{2\infty}}
\]

• Generic SCFT nonrational $\Rightarrow$ Hamiltonian is gapless for all $c$
Recap

Hamiltonians

- \((k,r)\) do not uniquely specify clustered state
- Generated basis for edge excitations of \((k=2,r=6,c)\) Hamiltonian
- Hamiltonian is gapless for all \(c\)

CFTs

- SCFT blocks interpolate between \((k=2,r=6)\) behaviors as function of \(c\)
- Edge theory is generic SCFT, for any value of \(c\) in Hamiltonian

Want gapped (stable) state = unitary, rational CFT (Read); project out singular vectors
“Improved” Hamiltonians and rational SCFTs
“Improving” the Hamiltonian

Obtain (unitary) minimal SCFTs by projecting out additional states: How many? Which ones?

- Use results of Feigin, Jimbo & Miwa for Virasoro M(3,ρ), a.k.a. \( k = 2 \) series of Jacks
  - Only three-body constraints (interactions) required
  - Recursive structure: polynomial ideal of zero-energy wavefunctions for M(3,ρ) related to that of M(3,ρ-3)
- Completely solved instance: SM(2,8) = M(3,8)
  - Project out additional three-particle state at degree 8
  - Manifest as extra couplings between “half-broken” excited pairs (built from Gaffnian wavefunctions)
SM(2, 8) wavefunctions

$\Psi^{SCFT} = S$

$\alpha = 1$

$\alpha = 2$

$\alpha = 3$

$D_2 \times \prod \bar{\vartheta}_{\{1,1\}} \times \prod \vartheta_{\{1,1\}} \times \prod \bar{\vartheta}_{\{1,1\}}$

paired particles

“half-broken” pairs

unpaired particles

charge excitations
SM(2,8) wavefunctions

\[ \psi^{M(3,8)} = S \]

\[ m = m_1 \]

\[ m_2 \]

\[ m_3 \]

\[ \alpha = 1 \]

\[ \alpha = 2 \]

\[ \alpha = 3 \]

\[ D_2 \]

\[ \prod \chi_c \text{ pairs of rows} \]

\[ \prod \bar{\vartheta}_{\{2,1\}} \text{ pairs of rows} \]

\[ \prod \bar{\vartheta}_{\{1,1\}} \text{ pairs of rows} \]

\[ \prod \chi_c \text{ all entries} \]

\[ \prod \vartheta_{\{2,1\}} \text{ all entries} \]

\[ \prod \vartheta_{\{1,1\}} \text{ all entries} \]

\[ z_{i,j}(\alpha) \]

\[ \Psi_{M(3,8)} \]

\[ e_{\lambda(1)} \]

\[ e_{\lambda(2)} \]

\[ e_{\lambda(3)} \]

paired particles

“half-broken” pairs

unpaired particles

charge excitations

\[ \prod \text{pairs of rows} \]

\[ \prod \text{pairs of rows} \]

\[ \prod \text{pairs of rows} \]

\[ \prod \text{all entries} \]

\[ \prod \text{all entries} \]

\[ \prod \text{all entries} \]
SM(2,8) wavefunctions

\( \frac{1}{(q)^\infty} \hat{\chi}_{Kac}^\pm = \sum_{m_2,m_3 \geq 0: (-1)^{m_3} = \pm 1} \frac{q^{2m_2 + \frac{1}{2} m_3 (m_3 + 2)}}{(q)^\infty (q) m_2 (q) m_3} \)

Confirm basis is correct: recover known character for SM(2,8)

\( \frac{1}{(q)^\infty} \hat{\chi}^{[2,8]}_{1,1} = \sum_{m_2,m_3 \geq 0} \frac{q^{m_2^2 + m_2 m_3 + \frac{1}{2} m_3^2 + m_2 + m_3}}{(q)^\infty (q) m_2 (q) m_3} \)

Hamiltonian: need to project out one additional behavior at degree eight (geometry-dependent)

Keep behavior \( \propto 9 \tilde{e}_3(z_1, z_2, z_3)^2 \tilde{e}_2(z_1, z_2, z_3) - \tilde{e}_2(z_1, z_2, z_3)^4 \)

In plane, remove \( \propto 54 \tilde{e}_3(z_1, z_2, z_3)^2 \tilde{e}_2(z_1, z_2, z_3) + 11 \tilde{e}_2(z_1, z_2, z_3)^4 \) behavior
Other minimal SCFTs?

- Would like a unitary minimal SCFT, i.e. gapped, stable FQH state
- These appear to require significant modifications to our formalism!
  - Simplest ex: Tricritical Ising model, $\text{SM}(3,5) = \text{M}(4,5)$
  - Manual construction of Verma module $\Rightarrow$ must have seven-particle interaction: clusters of clusters?

\[
\frac{1}{(q)_{\infty}} \hat{\chi}_{[3,5]}^{1,1} = \sum_{\vec{m} \geq 0} \frac{q^{\vec{m} A \vec{m}}}{\prod_{i=1}^{7} (q)_{m_i}}
\]

\[
\frac{1}{(q)_{\infty}} \hat{\chi}_{[3,5]}^{1,1} = \sum_{n_1, n_2 \geq 0: n_2 \geq n_1} \frac{q^{\frac{1}{2} n_1^2 + 2n_2^2 - n_1 n_2} (q)_{n_2}}{(q)_{\infty} (q)_{2n_2} (q)_{n_1} (q)_{n_2 - n_1}}
\]
Summary

• Take-home points:
  • \((k,r)\) alone are insufficient to uniquely specify a clustered quantum Hall state
  • Must consider entire zero-energy edge excitation spectrum before identifying eigenspace of Hamiltonian with CFT, not just densest (ground) state

• This work (so far):
  • Constructed and counted complete set of zero-energy eigenstates of projection Hamiltonian with continuous free parameter
  • Found complete set of states and modified Hamiltonian corresponding to SM(2,8) minimal SCFT, a.k.a. \((k=2,r=6)\) Jack state