Dynamical Electroweak Symmetry Breaking with a Heavy 4th Generation

Nov 10th, 2010, University of Virginia, Charlottesville

Chi Xiong

(based on arXiv:0911.3890, 0911.3892, 1011.xxxx) with P.Q. Hung

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Outlines

- Standard Model with four generations (SM4)
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- Renormalization group equation (RGE) (2-loop)
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- Schwinger-Dyson equation (SDE)
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- Renormalization group equation (RGE) (2-loop)
- Strong Yukawa couplings and Quasi-fixed point
- Bound states/condensates of the 4th generation
- Schwinger-Dyson equation (SDE)
- Implications of RGE+SDE
Why SM4?

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- A sequential family of heavy quarks and leptons has not been ruled out yet — G. D. Kribs, T. Plehn, M. Spannowsky and T. Tait, PRD 76, 075016(2007); P. Q. Hung and M. Sher, PRD 77, 037302 (2008); H. He, N. Polonsky and S. Su, PRD 64, 053004 (2001); M. Chanowitz, PRL 87, 231802 (2001) V. A. Novikov, L. B. Okun, A. N. Rozanov and M. I. Vysotsky, PLB 529 (2002); B. Holdom, PLB 686(2010); J. Erler, P. Langacker, PRL 105,031801 (2010), ...

- A heavy 4th generation can alleviate the naturalness (hierarchy) problem of SM3

- Similar to the top-quark condensation models, the 4th generation might trigger the dynamical electroweak symmetry breaking.
From our study with RGE+SDE, a heavy 4th generation drives the Yukawa couplings to the strong region $\Rightarrow$ binding force for condensates
How?

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- drives the Yukawa couplings to the strong region \( \Rightarrow \) binding force for condensates

- brings the cutoff scale down to \( \Lambda \sim \text{TeV} \) \( \Rightarrow \) hierarchy problem
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- suggests the restoration of scale invariance above $\Lambda \Rightarrow$ new conformal theories?
We begin with the RGE approach. The RG running of gauge couplings and Higgs couplings (quartic and Yukawa) in SM3 ($M_h = 120 GeV \sim 180 GeV$)
RGE in SM4

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e.g. the Higgs quartic coupling

$$\beta_\lambda = 24\lambda^2 + 4\lambda(3g_t^2 + 6g_q^2 + 2g_l^2 - 2.25g_2^2 - 0.45g_1^2)$$
$$-12(3g_t^4 + 6g_q^4 + 2g_l^4) + (16\pi^2)^{-1}[180g_t^6$$
$$+288g_q^6 + 96g_l^6 - (3g_t^4 + 6g_q^4 + 2g_l^4 - 80g_3^2(g_t^2$$
$$+2g_q^2))\lambda - 6\lambda^2(24g_t^2 + 48g_q^2 + 16g_l^2) - 312\lambda^3$$
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other couplings $\beta_{y_q}, \beta_{y_l}, \beta_{y_t}, \beta_{g_i}, i=1,2,3 \ldots$ (Machacek and Vaughn, 1983)
Zeros of $\beta$-functions

These RGEs can be integrated numerically, but first we search for roots of $\beta_{y_i} = 0$ with fixed gauge couplings.
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<table>
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<th>$g_3^2$</th>
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The fixed point values of the quartic and Yukawa couplings are approximately

\[ \frac{\lambda^*}{(4\pi)^2} \approx 0.11, \quad \frac{g_t^2}{(4\pi)^2} \approx 0.2, \quad \frac{g_q^2}{(4\pi)^2} \approx 0.33, \quad \frac{g_l^2}{(4\pi)^2} \approx 0.34 \]
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The gauge couplings contribute only small fluctuations. These correspond to naive $\overline{MS}$ masses (using

$$m_H = \sqrt{2\lambda}, \quad m_f = v g_f / \sqrt{2}, \quad v = 246 \text{ GeV}$$ )

$$m_H^* = 1.44 \text{ TeV}, \quad m_t^* = 0.97 \text{ TeV}, \quad m_q^* = 1.26 \text{ TeV}, \quad m_l^* = 1.28 \text{ TeV}.$$
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Questions:

- Can this (quasi)fixed point be reached?
- If yes, at what energy scale?
The RG running of Higgs couplings (quartic and Yukawa), light mass cases ($M_q = 120\text{GeV} \sim 250\text{GeV}$)
The RG running of Higgs couplings (quartic and Yukawa), heavy mass case ($M_q = 300\,\text{GeV} \sim 500\,\text{GeV}$)
Landau Pole vs. Fixed Point

Compare 1-loop and 2-loop results

\[ \alpha_l, \alpha_q, \alpha_t \]

\[ M_q = 500 \text{ GeV} \]
\[ M_l = 400 \text{ GeV} \]

\[ M_q = 120 \text{ GeV} \]
\[ M_l = 100 \text{ GeV} \]

\[ t = \log(E/91.2\text{GeV}) \]
From the numerical calculations, we see that

- The evolution of Higgs couplings run into a quasi-fixed point at some scale $\Lambda_{FP}$
- $\Lambda_{FP}$ decreases when the mass the 4th generation increases
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The existence of a quasi-fixed point $\Rightarrow$ the triviality problem;
The physical consequences of shifting the scale $\Lambda_{FP}$ down to TeV level $\Rightarrow$ provide an alternative solution to the hierarchy problem.
For the RGEs, the expansion parameters are
\[ \frac{g_t^2}{16\pi^2} \approx 0.2, \quad \frac{g_q^2}{16\pi^2} \approx 0.33, \quad \frac{g_q^2}{16\pi^2} \approx 0.34, \quad \frac{\lambda^*}{16\pi^2} \approx 0.11. \]
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\[ \frac{\alpha}{\pi} \text{ or } \frac{\alpha}{4\pi} \? \] An order of one expansion parameter? We have seen similar situations before, e.g., the Wilson-Fisher \( \epsilon \)-expansion, \( g_4 = 16\pi^2 \epsilon/3 \) for the physical value \( \epsilon = 1, \epsilon = 4 - d \).
Wilson-Fisher $\epsilon$-expansion

\[
\mu \frac{d}{d\mu} g_4(\mu) = -\epsilon g_4(\mu) + \frac{3g_4^2(\mu)}{16\pi^2} + \mathcal{O}(g_4^3(\mu))
\]
\[
\mu \frac{d}{d\mu} g_2(\mu) = g_2(\mu)[-2 + \frac{g_4(\mu)}{16\pi^2} + \mathcal{O}(g_4(\mu))]
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where $\epsilon = 4 - d$ and $g_2$ and $g_4$ come from terms $g_2\phi^2/2, g_4\phi^4/4!$ respectively.
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Solving $\beta(g_4), \beta(g_2) = 0$ equations, one finds a non-trivial fixed point at

$$g_4^* = \frac{16\pi^2 \epsilon}{3}, \quad g_2^* = 0.$$
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For $d=3$, it corresponds to $g_4^* = 16\pi^2/3 \approx 52.64$ or $g_4^*/16\pi^2 = 1/3$
Wilson-Fisher $\epsilon$-expansion

The critical exponent $\nu$ is then given by the $\epsilon$-expansion

$$\nu = 1/2 + \epsilon/12 + 7\epsilon^2/162 - 0.01904\epsilon^3 + O(\epsilon^4)$$
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$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + \frac{7\epsilon^2}{162} - 0.01904\epsilon^3 + O(\epsilon^4)$$

At 1-loop level, $\nu = 0.58$, at 2-loop level, $\nu = 0.63$, at 3-loop level, $\nu = 0.61$ while the experimental value is $\nu = 0.63$
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We do not expect such a precise calculation, but hopefully inclusion of higher order terms will not shift the location and the values of the quasi-fixed point by an order of magnitude.
Bound States/Condensates

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$$V(r) = -\alpha_Y(r) \frac{e^{-m_H(r)r}}{r}$$

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The possibility of forming bound states is characterized by

$$K_f = \frac{g_f^3}{16\pi \sqrt{\lambda}}.$$
Bound States/Condensates

The criteria is

- $K_f > 2$ (variational method)
- $K_f > 1.68$ (numerical method)
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If we use the fixed-point values of the quartic and Yukawa couplings, we find that the 4th generation may form loosely bound state, while the top quark cannot.

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An interesting region is around the “dip”, i.e. $\lambda \approx 0$, where the Yukawa potential becomes a strong Coulomb-like potential. → formation of condensates (Rafelski, Fulcher and Klein, 1978).
Region I: Condensates v.s. Region II: Fixed Point

Note: Neither technicolor nor other unknown interactions are introduced for condensates.
(\(m_q = 450\) GeV and \(m_l = 350\) GeV) \(K_f - K_0\) with \(K_f = \frac{g_f^3}{16\pi\sqrt{\lambda}}\) and \(K_0 = 1.68\). The horizontal dotted line indicates an estimate of \(K_f\) where the non-relativistic method is still applicable and the vertical dotted lines enclose the region where a fully relativistic approach is needed.
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- We use Schwinger-Dyson approach (Gap Equation, mean field theory, Hartree-Fock approximation,...)
Consider Yukawa couplings in SM4 (truncated to only the 4th generation)

\[ \mathcal{L}_Y = -g_{b'} \bar{q}_L \Phi b'_R - g_{t'} \bar{q}_L \tilde{\Phi} t'_R + h.c. \]

\[ \tilde{\Phi} = i\tau_2 \Phi^*, \quad q_L = (t', b')_L \] as usually defined in the SM.

Figure 1: Graphic representation of the Schwinger-Dyson equation for the quark self-energy (quenched approximation)
**Gap Equation**

For simplicity we only consider the 4th generation quarks. From the SDE the quark self energy satisfies

\[
\Sigma(p) = \frac{+2g^2}{(2\pi)^4} \int d^4q \frac{1}{(p - q)^2} \frac{\Sigma(q)}{q^2 + \Sigma^2(q)}
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which can be converted to a differential equation

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where \( \alpha_c = \pi/2 \)
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- compared with \( \alpha_c = \pi/3 \) in strong QED (Fukuda & Kugo, Bardeen, Leung & Love)
Gap Equation

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- **Early work:** K. Johnson, M. Baker and R. Willey. PR136(1964), 163(1967)
- The SDEs are similar, but the boundary conditions are different

\[
\lim_{p \to 0} p^4 \frac{d\Sigma}{dp^2} = 0
\]

\[
\lim_{p \to \Lambda} p^2 \frac{d\Sigma}{dp^2} + \Sigma(p) = 0
\]
asymptotic solutions in the weak and strong coupling regions:

\[ \Sigma(p) \sim p^{-1+\sqrt{1-\frac{\alpha}{\alpha_c}}}, \quad \text{for } \alpha \leq \alpha_c \]

\[ \Sigma(p) \sim p^{-1} \sin\left[ \sqrt{\frac{\alpha}{\alpha_c}} - 1(\ln p + \delta) \right], \quad \text{for } \alpha > \alpha_c \]
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Numerical Solutions
One can compute the condensates

\[ \langle t't' \rangle = -\frac{1}{4\pi^4} \int d^4q \frac{\Sigma(q)}{q^2 + \Sigma^2(q)} \]
Condensates

- One can compute the condensates
  \[
  \langle \bar{t}'t' \rangle = -\frac{1}{4\pi^4} \int d^4q \frac{\Sigma(q)}{q^2 + \Sigma^2(q)}
  \]

- Self energy, condensates and induced scalar mass depend on the cutoff and the Yukawa couplings as
  \[
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  \]
Condensates

One can compute the condensates

$$\langle \bar{t}' t' \rangle = -\frac{1}{4\pi^4} \int d^4 q \frac{\Sigma(q)}{q^2 + \Sigma^2(q)}$$

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Miransky fixed-point? In our case the exponential factors cannot suppress them simultaneously. To avoid fine-tuning, one has to choose a cutoff at TeV scale.
Cutoff vs. Yukawa couplings

<table>
<thead>
<tr>
<th>$\frac{\alpha}{\alpha_c}$</th>
<th>1.0</th>
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<th>1.4</th>
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<th>2.2</th>
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<tr>
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<td>52.8</td>
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<td>6.47</td>
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<td>2.80</td>
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<td>$\Lambda$ (GeV)</td>
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<td>$10^5$</td>
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Table 1: The relation between the cutoff scale and the Yukawa coupling.
Cutoff vs. Yukawa couplings

Table 2: The relation between the cutoff scale and the Yukawa coupling.

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Cutoff scale $\Lambda_{\text{cutoff}}$ vs Yukawa coupling $\alpha$

$\alpha = g^2 / 4\pi$
two figures from RGE and SDE respectively
Multiple Higgs doublets

- We might have three Higgs doublets: One fundamental, two composite

\[ H_1 = (\pi^+, \pi^-, \pi^0, \sigma) \]

\[ H_2 = (\bar{b}'t', \bar{t}'b', \bar{t}'t' - \bar{b}'b', \bar{t}'t' + \bar{b}'b') \]

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- The existence of the Nambu-Goldstone bosons leads to \( \det \mathcal{M} = 0 \) – modified gap equation
Summary

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