Hydrodynamic Noise and Bjorken Expansion

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Strong interactions

- **Nuclear force** holds protons/neutrons together in a nucleus.
  This is the force that makes the Sun shine.

- Like van-der-Waals forces in ED, nuclear force is a “shadow” of a much stronger force.
  This force holds proton’s constituents together.

- Like in ED the strong force field can carry waves – gluons.

- Unlike ED the gluon-mediated force grows with separation. Confinement.
  The “charge” comes in 3 varieties – colors.
  The QFT of the gluons and quarks – **Quantum Chromodynamics**.

- QCD is the most elegant piece of the Standard Model.
  The quest to “solve” QCD produced the most advanced ideas and tools in Theoretical Physics – from Lattice to String theory.
Can the gas of color-neutral nucleons be “ionized”? Can the (confined) quarks be set free by heating the gas of hadrons to extremely high temperature, revealing the QCD constituents and the “color” forces? Quark-Gluon Plasma.

What temperature would be needed?

\[ k_B T = \frac{\hbar c}{R_{\text{proton}}} \sim 200 \text{ MeV} \] (over \(10^5\) of \(T\) in the Sun’s core)

The Universe was that hot (and hotter) at the beginning.

Today we can recreate such conditions by smashing large atomic nuclei.
Lattice

- QCD is a quantum field theory, i.e., everything measurable is, in principle, predictable: hadron masses, scattering amplitudes, equation of state, etc.

- The theory says “calculate path integral” (Feynman), i.e., a weighted sum over all space-time trajectories of the variables, i.e., the fields.

- It is an infinitely difficult problem (for a non-trivial theory), because the number of variables (and integrations) is $\infty$.

- Lattice approach allows to reach this infinity gradually, by starting on a coarse, finite lattice and refining it. QCD on the lattice (Wilson).

- The question the lattice can answer is: what happens in QCD at finite $T$ in equilibrium?
  - E.g., how does energy density, pressure, entropy, etc. depend on $T$?
  - Does hadron gas become QGP at $T \sim 200$ MeV? Yes, it does.

- But non-equilibrium (e.g., transport) properties are still a challenge for today’s lattice methods.
Heavy-ion collision

- Heavy-ion collision creates matter in not quite as static a state as what we can study on the lattice.
- The created fireball evolves (explodes): “little bang”. Transport properties are important.
- Detectors measure the particle type and momentum distributions in the final state, when the density drops so that the particles free-stream – freeze-out.

This state is thermal, with temperature about 160 MeV.

Similarity to BB and CMB.
Hydrodynamic description

The hydrodynamic description of the heavy ion collision goes back to Landau (1953).

Approach: take the equation of state, set initial conditions, and solve hydrodynamic equations to get particle yields, spectra, etc.

- Good agreement with data.
- Sensitive to viscosity. (Azimuthal asymmetry)

Recent interest is due to the remarkably small implied value of viscosity.

Expressed as the ratio $\eta/s$ it is much smaller than one would predict in a weakly-coupled QCD plasma (Arnold-Moore-Yaffe, 2003).

The ratio $\eta/s$ is a measure of the coupling strength.

For weak coupling $\frac{\eta}{s} \sim \frac{1}{(\text{coupling})^2}$ – must be large.

- AdS/CFT calculation in SYM theory at infinite coupling: $\eta/s = 1/(4\pi)$.

- QGP at RHIC is not a gas, but a very good (perfect?) liquid. sQGP.
Fluctuations and viscosity

Can viscosity be measured in a different, complementary way?

Idea: fluctuation-dissipation theorem requires fluctuations (hydrodynamic noise) and dissipation (viscosity) to be proportional to each other.

The magnitude of fluctuations (correlations) can be measured.

The hydrodynamic correlations can be determined theoretically and depend on viscosity.

Correlations over large $\Delta \eta$ are induced by local fluctuations (hydrodynamic noise) propagating with the speed of sound.

Similarity to the fluctuations in the CMB.
Relativistic Hydrodynamics

Hydrodynamics: the effective theory for slow, long scale variations of the variables characterizing local thermal equilibrium.

Variables: conserved quantities – energy, momentum, charge densities.

Equations: conservation laws – $\nabla_\mu T^{\mu\nu} = 0$, $\nabla_\mu J^\mu = 0$.

Defining variables involves choosing the local rest frame: $T^{00} = \epsilon$, $J^0 = n$.

Simplifying choice(s): $T^{0i} = 0$ (Landau) or $J^i = 0$ (Eckart).

Use 3 components of $u^\mu$ instead of momentum density.

The 4 equations involve 10 components of $T^{\mu\nu}$. Thus the remaining 6 must be expressed in terms of $\epsilon$ and $u^\mu$.

In equilibrium the medium is homogeneous and ($T^{00} = \epsilon$, $T^{11} = p$, …)

$$T^{\mu\nu}_{eq} = \epsilon u^\mu u^\nu - P(\epsilon)(g^{\mu\nu} - u^\mu u^\nu).$$

$\nabla_\mu T^{\mu\nu}_{eq} = 0$ is ideal hydrodynamics.
Viscous hydrodynamics

Deviations from equilibrium are due to (slow) spatial variations of $\epsilon$ and $u^\mu$. I.e.,

$$T^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + \left(\text{gradients of } \epsilon \text{ and } u^\mu\right)_{\Delta T^{\mu\nu}}$$

Subject to $\Delta T^{\mu\nu} u_\nu = 0$ ($T^{\mu\nu} u_\nu = \epsilon u^\mu$ by definition), the most general form is ($\Delta^\mu = h^{\mu\nu} \nabla_\nu$, $h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$)

$$\Delta T^{\mu\nu} = \eta \left[ \Delta^\mu u^\nu + \Delta^\nu u^\mu - \frac{2}{3} h^{\mu\nu} (\nabla \cdot u) \right] - \zeta h^{\mu\nu} (\nabla \cdot u)$$

Second law of thermodynamics for entropy $s = \beta(\epsilon + P)$ flow

$$\nabla_\mu (s u^\mu) = -\beta \Delta T^{\mu\nu} \nabla_\mu u_\nu = \frac{\eta}{2T} \left[ \Delta^\mu u^\nu + \Delta^\nu u^\mu - \frac{2}{3} h^{\mu\nu} (\nabla \cdot u) \right]^2 + \frac{\zeta}{T} (\nabla \cdot u)^2$$
Fluctuations and Noise

So far hydro. eqns. describe evolution of the average values of the variables. In a thermodynamic ensemble the variables fluctuate.

The origin of the noise is local, but these fluctuations propagate according to hydrodynamic equations. I.e., hydrodynamics can describe long-range correlations.

This means

$$\Delta T^{\mu\nu} = \Delta T_{\text{visc}}^{\mu\nu} + S^{\mu\nu}.$$

Locality means \( \langle S^{\mu\nu}(x) S^{\alpha\beta}(0) \rangle \sim \delta^4(x) \). The magnitude is determined by the condition that the equilibrium distribution is given by \( e^S \) (Einstein).
Fluctuations and Noise

Generically, for a system of many variables $x_i$, obeying

$$\dot{x}_i = - \sum_j \gamma_{ij} X_j + y_i$$

where $X_j = - \partial S / \partial x_i$, the required noise is $\langle y_i(t) y_j(0) \rangle = (\gamma_{ij} + \gamma_{ji}) \delta(t)$.

Applying to $x_i \sim \epsilon, u^\mu$ one finds

$$\langle S^{\mu\nu}(x) S^{\alpha\beta}(0) \rangle = 2T \left[ \eta \left( h^{\mu\alpha} h^{\nu\beta} + h^{\mu\beta} h^{\nu\alpha} \right) + \left( \zeta - \frac{2}{3} \eta \right) h^{\mu\nu} h^{\alpha\beta} \right] \delta^4(x)$$

This, with $T^{\mu\nu} = T_{eq}^{\mu\nu} + \Delta T_{visc}^{\mu\nu} + S^{\mu\nu}$ defines a system of stochastic equations

$$\nabla_\mu T^{\mu\nu} = 0.$$ 

The correlation functions of $T^{\mu\nu}$ can be now calculated by solving in terms of $S^{\mu\nu}$.

Usually, this is applied to fluctuations around a static equilibrium solution. Our goal is to apply this to determine correlations in an expanding fireball.
Bjorken expansion

Bjorken (1983) suggested that the central region of the heavy-ion collisions can be described by a solution of the hydrodynamic equations which is boost-invariant.

The Bjorken flow is conveniently viewed in Bjorken coordinates:

\[ t = \tau \cosh \xi \quad \text{and} \quad z = \tau \sinh \xi. \]

In these coordinates the fluid is at rest locally: \( u^\mu = (1, 0, 0, 0)_\text{Bj} \).

The average quantities depend only on \( \tau \). But the fluctuations depend also on \( \xi \) and \( x_\perp \), e.g., \( \epsilon = \epsilon_0(\tau) + \delta \epsilon(\tau, \xi, x_\perp) \).

We integrate (average) over \( x_\perp \) and consider, effectively, a 1+1 dimensional problem. In this case \( S^{\mu\nu} u_\nu = 0 \) means

\[ S^{\mu\nu} = w(\tau) f(\xi, \tau) h^{\mu\nu} \]

where \( f \) is random noise (\( w = \epsilon + p \))

\[ \langle f(\xi_1, \tau_1) f(\xi_2, \tau_2) \rangle = \frac{2T(\tau_1)}{A\tau_1 w^2(\tau_1)} \left[ \frac{4}{3} \eta(\tau_1) + \zeta(\tau_1) \right] \delta (\tau_1 - \tau_2) \delta (\xi_1 - \xi_2) \]
The only nontrivial function is \( \epsilon(\tau) \), and it obeys

\[
\frac{d(\tau s)}{d\tau} = \frac{\nu}{\tau T} s
\]

I.e., entropy per unit rapidity, \( \tau s A \), increases only due to viscosity.

Convenient notation: \( \nu \equiv (4\eta/3 + \zeta)/s \).

E.g., for \( s \sim T^3 \Rightarrow T \sim \tau^{-1/3} + \text{visc. corrections} \)
Hydrodynamic equations for fluctuations

Fluctuations, $\epsilon = \epsilon_0(\tau) + \delta \epsilon(\xi, \tau)$, $u^z = \sinh(\xi + \omega(\xi, \tau))$, obey

$$
\tau \frac{\partial \delta \epsilon}{\partial \tau} + \delta w + \omega f - \frac{\delta (\nu s)}{\tau} + \left[ w - 2 \frac{\nu s}{\tau} \right] \frac{\partial \omega}{\partial \xi} = 0
$$

$$
\tau \frac{\partial}{\partial \tau} \left[ \omega \left( w - \frac{\nu s}{\tau} \right) \right] + 2 \omega \left( w - \frac{\nu s}{\tau} \right) + \frac{\partial}{\partial \xi} \left[ \delta P + w f - \frac{\delta (\nu s)}{\tau} \right] - \frac{\nu s}{\tau} \frac{\partial^2 \omega}{\partial \xi^2} = 0.
$$

Easy to obtain by $P \rightarrow P + \omega f - \frac{\nu s}{\tau} \left( 1 + \frac{\partial \omega}{\partial \xi} \right)$ in ideal equations.

Convenient variable: $\rho \equiv \delta s/s = \delta \epsilon/w$. Also, Fourier transform $\xi \rightarrow k$.

Solve for $X = \rho, \omega$ in terms of $f$.

$$
\tilde{X}(k, \tau) = - \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'} \tilde{G}_X(k; \tau, \tau') \tilde{f}(k, \tau')
$$

Then calculate correlations $\langle XY \rangle$ using known $\langle ff \rangle$. 
Correlations

The equal-(proper)time correlation function at the freeze-out time $\tau_f$:

$$C_{XY}(\xi_1 - \xi_2; \tau_f) \equiv \langle X(\xi_1, \tau_f) Y(\xi_2, \tau_f) \rangle = \frac{2}{A} \int_{\tau_0}^{\tau_f} \frac{d\tau}{\tau^3} \frac{\nu(\tau)}{w(\tau)} G_{XY}(\xi_1 - \xi_2; \tau_f, \tau), \quad (1)$$

where the Fourier transform of $G_{XY}(\xi; \tau_f, \tau)$ is given by

$$\tilde{G}_{XY}(k; \tau_f, \tau) \equiv \tilde{G}_X(k; \tau_f, \tau) \tilde{G}_Y(-k; \tau_f, \tau). \quad (2)$$

Thus

$$G_{XY}(\xi_1 - \xi_2; \tau_f, \tau) = \int_{-\infty}^{\infty} d\xi G_X(\xi_1 - \xi; \tau_f, \tau) G_Y(\xi_2 - \xi; \tau_f, \tau). \quad (3)$$
Solution: inviscid case, linear EOS

$$\tau \frac{\partial \tilde{\psi}}{\partial \tau} + D \tilde{\psi} + \tilde{n} = 0,$$

with (for inviscid case)

$$\tilde{\psi} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\omega} \end{pmatrix}; \quad D = D_0 \equiv \begin{pmatrix} 0 & ik \\ ikv_s^2 & 1 - v_s^2 \end{pmatrix}, \quad \tilde{n} = \begin{pmatrix} 1 \\ ik \end{pmatrix} \tilde{f}.$$

$$\tilde{\psi}(k, \tau) = - \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'} \tilde{U}(k; \tau, \tau') \tilde{n}(k, \tau')$$

where

$$\tilde{U}(k; \tau, \tau') = T \exp \left\{ - \int_{\tau'}^{\tau} \frac{d\tau''}{\tau''} D(k, \tau'') \right\}$$

If \( v_s^2 \equiv dP/d\epsilon = \text{const} \) (linear EOS):

$$\tilde{U}(k; \tau, \tau') = \frac{(\tau'/\tau)^{\lambda_-}}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ & -ik \\ -ikv_s^2 & -\lambda_- \end{pmatrix} - \frac{(\tau'/\tau)^{\lambda_+}}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_- & -ik \\ -ikv_s^2 & -\lambda_+ \end{pmatrix}.$$
Response functions and sound horizon

\[ \tilde{G}_\rho(k; \tau, \tau') = \left( \frac{\tau'}{\tau} \right) ^ \alpha \left[ \cosh \left( \beta \ln(\tau/\tau') \right) + \left( \frac{\alpha + k^2}{\beta} \right) \sinh \left( \beta \ln(\tau/\tau') \right) \right] \]

Note: \( \beta = \sqrt{\alpha^2 - v_s^2 k^2} \) is pure imaginary if \( |k| > (1 - v_s^2)/2v_s \).

Acoustic oscillations.

\[ G_X - \text{meromorphic function of } k. \text{ Sole singularity is at } k = \infty. \]

Cauchy theorem gives:

\[ G_X(\xi; \tau, \tau') = 0 \quad \text{when} \quad |\xi| > v_s \ln(\tau/\tau') \quad \text{— sound horizon} \]

In the local rest frame the velocity of the front, \( \tau d\xi/d\tau \), equals \( v_s \).
Singularities at the sound horizon

Oscillatory behavior at large $k$ translates into sound front in $\xi$:

$$G_{\rho\rho}(\xi; \tau_f, \tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\xi} \tilde{G}_{\rho}(k; \tau_f, \tau) \tilde{G}_{\rho}(-k; \tau_f, \tau) \rightarrow$$

$$\frac{1}{4v_s^2} \left( \frac{\tau}{\tau_f} \right)^{2\alpha} \left[ \delta''(\xi - 2v_s \ln(\tau_f/\tau)) + \delta''(\xi + 2v_s \ln(\tau_f/\tau)) - 2\delta''(\xi) \right]$$
The wake

\[ G_{\rho\rho}^{\text{reg}} \equiv G_{\rho\rho} - G_{\rho\rho}^{\text{sing}} \]

\[ v_s^2 = \frac{1}{3} \text{ and } \ln(\tau_f/\tau) = 2, 4, 6 \]

If the dispersion was linear, there would only be the sound front. However,

\[ \omega = i\lambda_{\pm} = i\alpha \pm \sqrt{v_s^2 k^2 - \alpha^2}. \]

Also note that one eigenvalue \( \lambda_- \sim k^2 \), for \( k \to 0 \).

Diffusion-like, but no dissipation.

\[ G_{\rho\rho} \text{ becomes Gaussian at large } \tau_f/\tau. \text{ Width } \langle \Delta\xi^2 \rangle = 2v_s^2/\alpha \cdot \ln(\tau_f/\tau). \]

Sum rule: \( \int_{-\infty}^{\infty} d\xi G_{\rho\rho}(\xi; \tau_f, \tau) = 1. \)
Viscosity and taming of singularities

For $\nu = \text{const}$ can be solved by perturbation in $\frac{\nu}{\tau T} \ll 1$.

Not assuming $k^2 \times \frac{\nu}{\tau T}$ to be small.

$$\tilde{G}_\rho(k; \tau, \tau') = \left(\frac{\tau'}{\tau}\right)^\alpha \left[ \cosh \left( \beta \ln(\tau/\tau') \right) + \left( \alpha + k^2 + \frac{\nu k^2}{2\tau' T(\tau')} \right) \frac{\sinh \left( \beta \ln(\tau/\tau') \right)}{\beta} \right]$$

$$\times \exp \left[ -\frac{\nu k^2}{4\alpha} \left( \frac{1}{\tau' T(\tau')} - \frac{1}{\tau T(\tau)} \right) \right]$$

For $G_{XX}(\xi; \tau_f, \tau)$ – Gaussian smearing with width

$$\sigma^2 = \frac{\nu}{\alpha} \left( \frac{1}{\tau T} - \frac{1}{\tau_f T_f} \right)$$
Cooper-Frye prescription:

\[
p^0 \frac{dN_s}{d^3p} = d_s \int_{\Sigma_f} d^3\sigma \mu p^\mu f_s(x, p); \quad f_s(x, p) = (e^{p \cdot u/T} \pm 1)^{-1}
\]

Fluctuations \( \rho \) and \( \omega \) (i.e., \( T \) and \( u \)) translate into \( \delta N \):

\[
\delta \left( \frac{dN}{d\eta} \right) = \frac{d_s A \tau_f T_f^3}{(2\pi)^2} \int d\xi \frac{\rho \nu_s^2 + \omega \tanh(\eta - \xi)}{\cosh^2(\eta - \xi)} \Gamma \left( 4, \frac{m_0}{T_f} \cosh(\eta - \xi) \right)
\]

Translation from \( \xi \) to \( \eta \) leads to additional (thermal) smearing.

\[
\left\langle \delta \frac{dN}{d\eta_1} \delta \frac{dN}{d\eta_2} \right\rangle \left\langle \frac{dN}{d\eta} \right\rangle^{-1} = \frac{45d_s}{4\pi^4 N_{\text{eff}}(T_0)} \frac{\nu}{T_f \tau_f} \left( \frac{T_0^2}{T_f^2} \right)^{\nu_s^{-2} - 2} K(\Delta \eta),
\]

\[
K(\Delta \eta)
\]
Conclusions and Outlook

- Hydrodynamics predicts long range correlations induced by thermal noise.

- Wake: magnitude is proportional to $\nu$. Nontrivial consequence of expansion. Magnitude of correlations in static equilibrium is determined by the static (thermodynamic) quantities only.

- Ridge: need to determine $\phi$-dependence. Expect a narrow (thermal) peak.

- More realistic (numerical) calculations need to be compared with experiment.