The Bulk-Edge Correspondence for Abelian Quantum Hall States

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Physics is Organized by Scale

Typical Energy Scales of a Ferromagnet

- 10,000 K: Paramagnet
- 1,000 K: Curie Temperature
- 100 K: Ferromagnetic Order
- 10 K: Ferromagnetic Order
- 0 K: Ferromagnetic Order
The Mother of All Effective Hamiltonians

\[ H_{UV} = \sum_{i}^{N_{\text{electrons}}} \frac{\hat{P}_{i}^2}{2m_{\text{electron}}} + \sum_{j}^{N_{\text{protons}}} \frac{\hat{P}_{j}^2}{2m_{\text{proton}}} + V_{\text{electron-electron}} + V_{\text{electron-proton}} + \vec{E}^2 + \vec{B}^2 \]

\[ N_{\text{electrons}} \sim N_{\text{protons}} \sim 10^{23} \]
Effective Hamiltonians

\[ H_{IR} = -J \sum_{\langle i, j \rangle} S_i \cdot S_j \]

Ferromagnets

Energy

10,000 K

1,000 K

100 K

10 K

0 K
Semiconductor Heterostructures can be Surprising:

**Quantum Hall Effect**

- low temperatures < 1 K
- high magnetic fields > 1-20 T
- density \( \sim 10^{11} - 10^{12} \text{ cm}^{-2} \)
2D Electron Gas in a B Field

Von Klitzing, Dorda, & Pepper
Hall and Longitudinal Resistances

\[ R_{xy} = \frac{V_y}{I_x} \quad R_{xx} = \frac{V_x}{I_x} \]
Quantum Hall Effect

\[ R_{xy} = \nu^{-1} \frac{h}{e^2} \quad R_{xx} = R_{yy} = 0 \]
Landau levels bend up near the edge of a sample and intersect the chemical potential

Example: $v = 3$

$\frac{E}{\hbar \omega_c}$

$\mu$

Gapless excitations
1. Fermionic
2. Near the edge
3. One per filled Landau level
4. Linear dispersion
5. Chiral $y \propto k_x l_B^2$

Halperin 1982
Fractional Quantum Hall Effect

Example: $\nu = 1/3$
Topological Order and Experimental Signatures

1. Fractionalization of Charge: $e/3$ Quasiparticles

Observed in Shot Noise or current fluctuation Measurements

Shot Noise $= 2\left(\frac{e}{3}\right)I_B$

Theory: Kane-Fisher; Fendley-Ludwig-Saleur

Saminadayar, Glattli, Jin and Etienne
Experimental Signatures

1. Anyonic Statistics – generalization of Bose-Fermi statistics

\[ \psi(x_1, x_2, \ldots) \rightarrow e^{\frac{\pi i}{3}} \psi(x_2, x_1, \ldots) \]

‘Indications’/’encouragement’ of its observance in interferometry

Camino, Zhou, and Goldman
Numerical Signatures

Topology-dependent ground state degeneracy

Not observable in actual experiments as you need to fabricate a torus in the lab

Useful in numerics

\[ (\mathbb{Z}_4 \text{ top order}) \]
Edge States Provide a Window into the Bulk Physics

Tunneling into the edge from a metallic lead

\[ I \sim V^3 \]

Compared with \( \nu = 1 \)

\[ I \sim V \]

Exp: Chang, Pfeiffer, West

Theory: Kane-Fisher; Fendley-Ludwig-Saleur
When can multiple, distinct edges bound the same bulk phase?

Experimentally relevant examples bulks with distinct edge phases include (cleanest signatures):

- IQH $\nu \geq 8$
- FQH $\nu = 8/3, 16/5, 16/7, \ldots$
**Integer Quantum Hall Edge**

\[
\frac{E}{\hbar \omega_c} \quad \mu \\
\begin{array}{c}
\text{7/2} \\
\text{5/2} \\
\text{3/2} \\
\text{1/2}
\end{array}
\]

\[
S_{\nu=n} = \int dt dx \sum_{i=1}^{n} (\psi_R^{(i)})^\dagger i(\partial_t - v_i \partial_x) \psi_R^{(i)}
\]

\[
\psi_R^{(i)}(x) \psi_R^{(j)}(y) = (-1)^{\psi_R^{(j)}(y)} \psi_R^{(i)}(x)
\]
Non-Chiral Integer Edge

\[ S_{\nu=1+\nu=-1} = \int dtdx \left[ \psi_R^\dagger i(\partial_t - v_R \partial_x) \psi_R + \psi_L^\dagger i(\partial_t + v_L \partial_x) \psi_L \right] \]

Chiral edges are stable, non-chiral edges are generically unstable.

\[ \delta S_{\nu=1+\nu=-1} = M \int dtdx \left[ \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \right] \]
Fractional Quantum Hall Edges

\[ S_{\nu=1/k} = \int \frac{k}{4\pi} \int dt dx \left[ \psi_R^\dagger (\partial_t - v_R \partial_x) \psi_R + f(k) \lim_{\epsilon \to 0} (\psi_R^\dagger \psi_R)(x)(\psi_R \psi_R)(x + \epsilon) \right] \]

\[ f(1) = 0, \quad f(k \neq 1) > 0 \]

Bosonization:

\[ \psi_R \leftrightarrow e^{ik\phi} \]

\[ \psi_R^\dagger i(\partial_t - \partial_t(\partial_x \phi)) \psi_R \xrightarrow{1} \frac{1}{4\pi} \partial_x \phi(\partial_t - \partial_x \phi) \]

\[ \lim_{\epsilon \to 0} (\psi_R^\dagger \psi_R)(x)(\psi_R^\dagger \psi_R)(x + \epsilon) \xrightarrow{\theta} \frac{(k - 1)}{4\pi} \partial_x \phi(\partial_t - \partial_x \phi) \]

\[ \nu = 1/k \]
**Fractional Quantum Hall Edge**

\[ S_{\nu=1/k} = \frac{k}{4\pi} \int dt dx \left[ \partial_x \phi \left( \partial_t - v_R \partial_x \right) \phi \right] \]

\[ \phi \equiv \phi + 2\pi \]

\[ k = 1 \]

\[ \langle \psi^\dagger(x) \psi(0) \rangle \neq \frac{m}{x - vt} \quad \text{GOOD} \]

\[ \psi(x)\psi(y) = (-1)^m \psi(y)\psi(x) \]

\[ \text{general } k \]

\[ \langle e^{im\phi(x)} e^{im\phi(0)} \rangle = \frac{1}{(x - v_R t)^{m^2/2k}} \quad \text{BAD – NOT ALLOWED} \]

\[ e^{im\phi(x)} e^{in\phi(y)} = e^{\pi i \frac{mn}{k}} e^{im\phi(y)} e^{in\phi(x)} \]
Multiple Edge Modes

\[ S = \frac{1}{4\pi} \int dt dx \left[ K_{IJ} \partial_x \phi_I \partial_t \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J + 2 t_I \varepsilon_{\mu \nu} \partial_\nu \phi_I A_\mu \right] \]

# right-moving modes - # left-moving modes = signature of \( K_{IJ} \)

\[ \nu = 0 \]

\[ \nu = t_I (K^{-1})^{IJ} t_J \]

Examples of K-matrices

IQH: $\nu = N \quad K = I_N$

Laughlin: $\nu = 1/m \quad K = (m)$

Bilayer system: $\nu = 1/3 + 1/5 \quad K = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

Hierarchy: $\nu = 2/5 \quad K = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$
Bulk-Edge Correspondence

\[ S = \frac{1}{4\pi} \int dtdx \left[ K_{IJ} \partial_t \phi_I \partial_t \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J + 2t_I \epsilon_{\mu\nu} \partial_\nu \phi_I A_\mu \right] \]

\[ \text{parity}(K_{IJ}) \]

\[ \kappa_{xy} \sim \text{signature}(K_{IJ}) \]

\[ \sigma_{xy} = t_I (K^{-1})^{IJ} t_J \frac{e^2}{\hbar} \]

quasiparticles \( \leftrightarrow m_I \)

charge\((m_I) = m_I (K^{-1})^{IJ} t_J \)

statistics\((m_I, n_J) = \exp(2\pi i m_I (K^{-1})^{IJ} n_J) \)

\[ S_{CS} = \int dtdxdy \epsilon_{\mu\nu\rho} \left[ \frac{K_{IJ}}{4\pi} a_\mu \partial_\nu a_\rho + 2t_I A_\mu \partial_\nu a_\rho \right] \]
Redundancy of Edge (and Bulk) Descriptions

Are these two theories the same (ignoring the charge vector)?

\[ K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \]

Yes.

\[ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

Same operators and scaling dimensions; preserves excitation spectrum

\[ K_2 = W^{tr} K_1 W, \quad W \in GL(n, \mathbb{Z}) \]

Distinct classes of $K$-matrices can (almost always) be distinguished by their scaling dimensions

If $K$ is chiral, scaling dimensions are universal:

$$\langle e^{im_I \phi^I(0,t)} e^{-im_J \phi^J(0,0)} \rangle \sim \frac{1}{t^{\Delta_m}}$$

Scaling dimension: $\Delta_m = \frac{1}{2} m_I K_{IJ}^{-1} m_J$

When $K$ is non-chiral, $\Delta$ depends on $V$-matrix
Scaling dimensions can be used to physically distinguish edge phases

1) Tunneling across a QPC

\[ \mathcal{L}_{tun} = \sum_{m_I} v_m e^{i m_I \phi_R} e^{-i m_I \phi_L} + h.c. \]

Many terms: most relevant minimizes \( m K^{-1} m \)

Expect most relevant term to dominate backscattered current:

\[ I_b \propto |v_m|^2 V^{2m K^{-1} m - 1} \]

Chamon, Freed Wen (1994)
Kane and Fisher (1992)
Scaling dimensions can be used to physically distinguish edge phases

2) Tunneling from a metallic lead

Most relevant term minimizes $mK^{-1}m + n^2$

$I_{lead} \propto |t_m|^2 V mK^{-1}m + n^2 - 1$

Tunnel one electron:

$\mathcal{L}_{lead} = t_m \psi_{lead}^\dagger e^{imI\phi_R^I}$

$mK^{-1}t = 1$

Tunnel two electrons:

$\mathcal{L}_{lead} = t_m \psi_{lead}^\dagger \partial \psi_{lead}^\dagger e^{imI\phi_R^I}$

$mK^{-1}t = 2$

Tunnel $n$ electrons:

$\mathcal{L}_{lead} = t_m \left[ \psi_i^\dagger \partial \psi_i^\dagger \partial^2 \psi_i^\dagger \ldots \right]_{n \text{ terms}} e^{imI\phi_R^I}$

$mK^{-1}t = n$

EDGE PHASE TRANSITIONS

Plamadeala, MM, & Nayak
Cano, Cheng, MM, Nayak, Plamadeala, & Yard
The confining potential matters

Can interactions with gapped edge modes change the phase of the edge?

Chamon & Wen
Append the New Modes to the Existing K-matrix

\[ K \rightarrow K \oplus \sigma_z \]
\[ t \rightarrow (t, 1, 1) \]

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots \\
2 & 5 & 7 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & 0 & 0 \\
2 & 5 & 7 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 \\
\end{pmatrix}
\]

\(v = 1\) strip

(New quasiparticles \(e^{i\phi_{N+1}}, e^{i\phi_{N+2}}\) are electrons)
When could inter-edge tunneling open a gap?

Given an inter-edge Tunneling Operator:

\[ e^{i n_I \phi_I} + h.c. \propto \cos(n\phi) \]

Requirements on the Operator

Conserves Electrical Charge

Spin-0:

\[ n_I K_{IJ}^{-1} t_J = 0 \]
\[ n_I K_{IJ}^{-1} n_J = 0 \]

Not met for a chiral edge. Example:

\[ K = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ n_I (K^{-1})^{IJ} n_J = \frac{n_1^2}{3} + n_2^2 - n_3^2 = 0 \]

\[ n_1 = 0 \]

Haldane \textit{PRL} 74 2090 (1995)
Edge Transitions: Example 1

\[ K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_1 = (1, -1)^T \]

Enlarge: \( K_1 \rightarrow K_1 \oplus \sigma_z \quad t_1 \rightarrow (1, -1, 1, 1)^T \)

1) \[ S' = \int dx du' \cos(\phi_3 + \phi_4) \]

2) \[ S'' = \int dx du'' \cos(\phi_1 - 11\phi_2 + 2\phi_3 + 4\phi_4) \]
\[ n = (1, -11, 2, 4) \]

Strategic variable change \( \phi = W\phi' \)

\[ \begin{cases} S'' = \int dx du'' \cos(\phi'_3 + \phi'_4) \\ K \rightarrow K_2 \oplus \sigma_z \quad K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \end{cases} \]
Is the resulting theory the same or different?

Example, cont

1) Most relevant term: \( u' \cos(\phi_3 + \phi_4) \)

\[
K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}, \quad t_1 = (1, -1)^T
\]

2) Most relevant term: \( u'' \cos(\phi'_3 + \phi'_4) \)

\[
K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad t_2 = (-1, -2)^T
\]

Are these phases distinct? \( \Delta = \frac{1}{2} m_I K_{I,J}^{-1} m_J \)

\[
\Delta_{min} = 1/11, \quad \Delta_{min} = 3/11
\]
Tunneling Distinguishes the Edges

Example, cont

1) Tunneling across a QPC

\[ I_b \propto V^{2\Delta_{min} - 1} \]

\[ K_1 \rightarrow I_b^{(1)} \propto V^{-9/11} \]

\[ K_2 \rightarrow I_b^{(2)} \propto V^{-5/11} \]

2) Tunneling from a metallic lead

Both edges have a charge \( e \) operator, but different scaling dimensions

\[ I_{lead} \propto |t_m|^2 V m K^{-1} m \]

\[ K_1 \rightarrow I_{lead}^{(1)} \propto V \]

\[ K_2 \rightarrow I_{lead}^{(2)} \propto V^3 \]
Edge Transitions: Example 2

Bose-Fermi Transitions

\[ K_{\text{odd}} \oplus \sigma_z = W^T (K_{\text{even}} \oplus \sigma_z) W \]

Example

IQH \( \nu = 8 \): \( K = \mathbb{I}_8 \)

\[
\mathbb{I}_8 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
W_8^T (K_{E_8} \oplus \sigma_z) W_8 = \mathbb{I}_8 \oplus \sigma_z
\]

\[
K_{E_8} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
\end{pmatrix}
\]

Even lattice!
Can distinguish experimentally between candidate $\nu = 8$ phases

1) Tunneling across a QPC

\[ I_b \propto V^{2\Delta_{min}^{-1}} \]

\[ \mathbb{I}_8 \rightarrow I_b \propto V \quad \text{(electron tunneling)} \]

\[ K_{E_8} \rightarrow I_b \propto V^3 \quad \text{(charge 2e tunneling of composite particle)} \]
Can distinguish experimentally between candidate $\nu = 8$ phases

2) Tunneling from a metallic lead

In $I_8$ state tunnel one electron:

$$L_{\text{lead}} = t_m \psi_{\text{lead}}^{\dagger} e^{i\phi_I}$$

$$I_{\text{lead}} \propto V$$

$E_8$ state does not have a charge $e$ operator!

Tunnel two electrons from lead:

Spin-polarized: $L_{\text{lead}} = t_m \psi_{\text{lead}}^{\dagger} \partial \psi_{\text{lead}}^{\dagger} e^{im_I \phi_I^R}$ $I_{\text{lead}} \propto V^5$

Not spin-polarized: $L_{\text{lead}} = t_m \psi_{\text{lead}}^{\dagger} \psi_{\text{lead}}^{\dagger} \psi_{\text{lead}} \psi_{\text{lead}} e^{im_I \phi_I^R}$ $I_{\text{lead}} \propto V^3$
Physical Criteria for Distinct Edges

When do distinct edge phases exist for a given bulk?

Answer:

• Two distinct edges (different tunneling exponents) can border the same bulk when the braiding matrix for the bulk quasiparticles inferred from the edge modes is the same.
• The edge transition does not necessarily preserve the total number of edge modes.
• It may be necessary to add additional $\nu = 1$ modes.
MATHEMATICAL FORMULATION
Equivalence class of K-matrices = lattice

K-matrix

Lattice \( \Lambda = \{m_I e_I | m_I \in \mathbb{Z}\} \)

\( K_{IJ} = e_I \cdot e_J \)

Example: \( K = (3) \)

\[ \Lambda = \sqrt{3} \mathbb{Z} \]

Edge phase = lattice
= K-matrix equivalence class

Choose new basis:
\( \Lambda = \{m_I e'_I | m_I \in \mathbb{Z}\} \)

\( K'_{IJ} = e'_I \cdot e'_J \)

\( W^T K W = K' \)
Quasiparticles and Operators

\[ \Lambda^* = \{ m_I \mathbf{f}_I | m_I \in \mathbb{Z} \} \]

\[ f_a^I = (K^{-1})^{IL} e_{La} \]

Example: \( K = (3) \),

\[ \Lambda^* = \frac{1}{\sqrt{3}} \mathbb{Z} \]

Inner product \( \leftrightarrow \) statistics \( \leftrightarrow \) scaling dimension (on diagonal)

\[ \mathbf{f}_I \cdot \mathbf{f}_J = K_{IJ}^{-1} \]

\[ S_{m,m'} \propto e^{-2\pi i m^T K^{-1} m'} \]
Want to classify a bulk phase by its “primitive” quasiparticles

\[ \Lambda^* \] Dual lattice vectors = quasiparticles

\[ \Lambda \] Original lattice vectors = trivial particles

“Discriminant group” =

\[ \Lambda^* / \Lambda \] Group of primitive quasiparticles = quasiparticles modulo the first particle with trivial statistics with all other particles

Example: \( K = (3) \), Discriminant group = \( \mathbb{Z}_3 \)

Lattices many-to-one Discriminant groups

\[ \downarrow \] \[ \uparrow \]

Edge phases many-to-one Bulk phase
Stable Equivalence

ν₀ = 0
ν₀ = 1
Edges Are Stably Equivalent if ...

\[ \Lambda_1^*/\Lambda = \Lambda_2^*/\Lambda \]

\[ f_I^{(1)} \cdot f_J^{(1)} = f_I^{(2)} \cdot f_J^{(2)} + \text{mod } 2 \]

Chiral transitions if:

\[ \text{signature}(\Lambda_1) = \text{signature}(\Lambda_2) \]

Formalize the odd-even correspondence

Utilize another theorem from Nikulin:

For every fermionic bulk phase, there is a corresponding\(^*\) edge phase which yields the same\(^**\) bulk quasiparticles and statistics and has no gapless charge \(e\) operator

Caveats
\(^*\)might have different number of edge modes
\(^**\)mod \(e\)

Future Directions

• Generalizations – non-Abelian, higher-dimensions, interplay of symmetry.

• Could there be a physical mechanism, e.g., disorder, that initiates the flow of $V$?

• Many more edge transitions between not-fully-chiral edge states, e.g., 1/3 state.

• There exists an intriguing analogy between 4-manifold topology and the Abelian Hall states. Does this analogy run deeper?
Conclusions

- A bulk chiral quantum Hall phase generically has multiple edge phases.

- These phases can be distinguished experimentally using tunneling measurements.

- Every fermionic edge phase has a corresponding bosonic edge phase
Heterostructure Band Energy Diagram

Gate Metal

n-ALGaAs i-GaAs SI-GaAs

E_C

E_F

E_V

2DEG (Two Dimensional Electron Gas)
More topologically trivial additions

Superfluid strip

\[ K \to K \oplus \sigma_x \]

Several \( \nu = 1 \) strips

\[ K \to K \oplus \sigma_z \oplus \sigma_z \oplus \cdots \oplus \sigma_z \]

Strip of anything non-chiral with trivial quasiparticles

\[ K \to K \oplus L, \quad |\text{det}(L)| = 1 \quad \text{sig}(L) = (n, n) \]
In principle, can count the lattices in a genus by the Smith-Siegel-Minkowski mass formula

$$\sum_{\Lambda \in g} \frac{1}{|\text{Aut}(\Lambda)|} = m(K)$$

Conway and Sloane 1988

- Chiral Abelian quantum Hall states with more than 10 edge modes have multiple distinct chiral edge phases\(^1\)
- Otherwise there is a finite set of bulk states with only one edge phase; all others have multiple\(^2\)

→ Multiple edge phases are the norm, not the exception

Example: even-odd equivalence by adding edge modes

\[ \nu = 1/5 \]
\[ K = (5) \]

No non-trivial stable equivalence preserving full chirality

\[ K_1 \oplus \sigma_z = W^T(K_2 \oplus \sigma_z)W \]

\[
K_2 = \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]

\[ \nu = 3 + 1/5 \]
\[
K_1 = \begin{pmatrix}
5 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Even lattice

Candidate states at $\nu=3+1/5$ distinguishable by experiment

Example, cont

1) Tunneling across a QPC

\[ I_b \propto V^{2\Delta_{min}^{-1}} \]

\[ K_1 \rightarrow I_b^{(1)} \propto V^{-3/5} \]

\[ K_2 \rightarrow I_b^{(2)} \propto V^{3/5} \]

2) Tunneling from a metallic lead

\[ K_1 \rightarrow I_{lead}^{(1)} \propto V \] (tunnel electron)

\[ K_2 \rightarrow I_{lead}^{(2)} \propto V^5 \] (tunnel 2 electrons, spin polarized)
Candidate states at $v=3+1/5$ have same quasiparticles mod $e$

$$K_1 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad t = (1, 1, 1, 1)$$

$$K_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad t = (2, 2, 0, 0)$$

Compare quasiparticle charge:

$e^*/e \in \{0, 1/5, 2/5, 3/5, 4/5\}$ Defined mod $e$

$e^*/e \in \{0, 2/5, 4/5, 6/5, 8/5\}$ Defined mod $2e$