Uncovering the Fibonacci Phase
in $\mathbb{Z}_3$ Parafermion Systems

E. Miles Stoudenmire
Perimeter Institute

University of Virginia 2015
Collaborators:

David Clarke - Caltech / Maryland

Roger Mong - Caltech / Pittsburgh

Jason Alicea - Caltech
In this talk:

- Two-dimensional lattice model containing square lattice, triangular lattice, and decoupled chain limits

- Site degrees of freedom are “parafermions”

- Strong evidence for emergent Fibonacci anyon quasiparticle on isotropic triangular lattice (and likely “$t_1$-$t_2$” model as well)
Technique used in this talk is the **density matrix renormalization group (DMRG)**

Works by “compressing” many-body wavefunction

\[
|\Psi\rangle = \begin{bmatrix}
\vdots
\end{bmatrix}
\]

\(d^N\) components

DMRG:

\[
|\tilde{\Psi}\rangle = \begin{bmatrix}
\vdots
\end{bmatrix}
\sim Nd\) components

Mean field:

\[
|\tilde{\Psi}_{MF}\rangle = \begin{bmatrix}
\vdots
\end{bmatrix}
\]
DMRG can address wide variety of systems

Frustrated magnets
Infinite, finite T
triangular lattice
Heisenberg model:

![Graph showing susceptibility vs temperature with data points for different orders of DMRG+NLC.](image)

- Prokofiev et al. BDMC
- Order 5   DMRG+NLC
- Order 4.5 DMRG+NLC
- Order 4   DMRG+NLC

Stoudenmire, unpublished

Fermions
e.g. continuum 1d systems

Lattice models of ‘anyons’ in two dimensions...
A major goal of 21st century physics: build a scalable quantum computer

**Ingredients:**
- Qubits (e.g. a spin 1/2)
- Unitary operations on these qubits

**Challenges:**
- Stability (decoherence)
- Usefulness (universal computation?)
Promising approach for dealing with decoherence is topological quantum computing.

In certain topological phases qubit space can be ‘hidden’

Information stored non-locally: decoherence protection

*for Majorana fermion case
More quasiparticles $\rightarrow$ additional qubits
More quasiparticles $\rightarrow$ additional qubits
More quasiparticles $\rightarrow$ additional qubits

Qubits can be manipulated by ‘braiding’ quasiparticles

$\sim \hat{U}$
Encouraging progress in *engineering* such “non-Abelian anyons”

**Microscopic platforms for Majorana zero modes**

Fu, Kane PRB 79, 161408(R) (2009)

Lutchyn et al. PRL 105, 077001 (2010)


Stoudenmire, Alicea, Starykh, Fisher PRB 84, 014503 (2011)

**Experimental realization?**


A. Das et al., Nat. Phys. 8, 887 (2012).
New platforms under way for parafermions, simplest generalization of Majorana fermions

Clarke, Alicea, and Shtengel, Nat. Commun. 4, 1348 (2013)
Barkeshli and Qi, PRX 2, 031013 (2012)

Majorana: 2-level system

Z3 Parafermion: 3-level system
New platforms under way for \textit{parafermions}, simplest generalization of Majorana fermions

![Diagram of 2/3 FQHE and 1/3 FQHE bilayer]

\[\text{Clarke, Alicea, and Shtengel, Nat. Commun. 4, 1348 (2013)}\]
\[\text{Barkeshli and Qi, PRX 2, 031013 (2012)}\]

\text{Majorana:} \quad 2\text{-level system}

\text{Z}_3 \text{ Parafermion:}^* \quad 3\text{-level system}

*also different commutation relations from Majorana
New platforms under way for *parafermions*, simplest generalization of Majorana fermions

2/3 FQHE

1/3 FQHE bilayer

crossed tunneling region

Clarke, Alicea, and Shtengel, Nat. Commun. 4, 1348 (2013)

Barkeshli and Qi, PRX 2, 031013 (2012)

Schemes will continue to improve....

But why pursue them?
Because Majorana fermions \& parafermions insufficient for \textit{universal} quantum computation

\[ \sim \hat{U} \]

Not enough operations

\[ \sim \hat{U} \]
Yet parafermions may hold the key...

This talk:
parafermions could hybridize to yield
Fibonacci anyons
Yet parafermions may hold the key...

This talk:
parafermions could hybridize to yield
Fibonacci anyons
Unlike parafermion

Fibonacci anyons

\sim 3\text{-level system}

\sim 1\text{ level system}
Unlike parafermion

3-level system

Fibonacci anyons

1, 1 level system
Unlike parafermion

\[ \sim 3 \text{-level system} \]

Fibonacci anyons

\[ \sim 1, 1, 2 \text{ level system} \]
Unlike parafermion

3-level system

Fibonacci anyons

$\sim 1, 1, 2, 3$ level system
Unlike parafermion

3-level system

Fibonacci anyons

$\sim 1, 1, 2, 3, 5$ level system
More importantly, Fibonacci quasiparticles have universal braiding.

More importantly, Fibonacci quasiparticles have universal braiding. 

\[ \sim U \] 

enough operations for quantum computing
Finally, parafermion lattice model just a “crutch”

“Smeared” limit could be sufficient for Fibonacci*

*Barkeshli, Vaezi PRL 113, 236804 (2014). Also see: Liu et al. arxiv:1502.05391; Geraedts et al. arxiv:1502.01340 for negative result
Hybridizing Parafermions
Warmup #1: parafermion dimer

\[ H = -\frac{t}{2}(\omega \alpha_i^\dagger \alpha_j + \bar{\omega} \alpha_j^\dagger \alpha_i) \quad [\omega = e^{i2\pi/3}] \]

Simplest parafermion Hamiltonian

Strongly interacting, despite appearance
Warmup #1: parafermion dimer

\[ H = -\frac{t}{2} (\omega \alpha_i^\dagger \alpha_j + \bar{\omega} \alpha_j^\dagger \alpha_i) \quad [ \omega = e^{i2\pi/3} ] \]

Hamiltonian (by mapping to ‘clock’ variables):

\[
\begin{bmatrix}
  t/2 & -t \\
  -t & t/2
\end{bmatrix}
\]

Positive \( t > 0 \), unique ground state

Negative \( t < 0 \), two ground states

Sign of \( t \) important!
Warmup #2: two-leg ladder

\[ i \longrightarrow j = -t (\omega \alpha_i^\dagger \alpha_j + \text{H.c}) \]

'Squeezed' system will 'point' us toward 2d Fibonacci phase

Can understand in two limits:

- \( t_1 \gg t_2, t_3 \)
- \( t_2 \gg t_1, t_3 \)
Warmup #2: two-leg ladder

\[ \begin{align*}
\alpha_i & \leftrightarrow \alpha_j \\
& = -t (\omega \alpha_i^\dagger \alpha_j + H.c)
\end{align*} \]

\[ t_1 \gg t_2, t_3 \]

Parafermions “pair” along rungs

Remain in trivial gapped phase for small \( t_2, t_3 \)
Warmup #2: two-leg ladder

\[ \begin{align*}
  t_2 & >> t_1, t_3 \\
  \end{align*} \]

Parafermions “pair” diagonally
Fractionalized 3-fold degenerate edge state

\[ i \quad j = -t (\omega \alpha_i^\dagger \alpha_j + H.c) \]

Remain in topological phase for small \( t_1, t_3 \)
Warmup #2: two-leg ladder

Phases compete for $t_1 \approx t_2$

DMRG results for phase boundary:

![Diagram of phase boundary with trivial and topological regions]

Continuous transition along $t_1 = t_2$ line
Warmup #2: two-leg ladder

Duality argument shows transition exactly at $t_1 = t_2$!

Suggestive field theory picture

For $t_1 = t_2 = 0$, $t_3 > 0$, each chain described by ‘$Z_3$ parafermion’ conformal field theory (CFT)$^{1,2}$

---

Warmup #2: two-leg ladder

Duality argument shows transition exactly at $t_1 = t_2$!

Suggestive field theory picture

For $t_1 = t_2 = 0$, $t_3 > 0$, each chain described by ‘$Z_3$ parafermion’ conformal field theory (CFT)$^{1,2}$
Warmup #2: two-leg ladder

Fine tuning $0 < t_1 = t_2 \ll 1$ couples only left mover of bottom chain to right mover of top chain
Warmup #2: two-leg ladder

Fine tuning $0 < t_1 = t_2 \ll 1$ couples only left mover of bottom chain to right mover of top chain.

Other fields remain gapless, critical ladder described by single Z$_3$ pfn. field theory.
$Z_3$ parafermion CFT has central charge $c=4/5$ (=0.8) Confirmed by DMRG on critical ladder

\[ S_L(x) = 0.80006 x + 1.06871 \]

$t_1 = t_2 = t_3$

$\frac{1}{3} \log \left[ \frac{L}{\pi} \sin \left( \frac{\pi x}{L} \right) \right]$  

L=60, PBC
Critical $t_1 = t_2$ line will serve as a precursor of Fibonacci phase in 2d
Towards Two Dimensions
Upon adding more legs, critical line could become stable 2d phase
Upon adding more legs, critical line could become stable 2d phase
Upon adding more legs, critical line could become stable 2d phase
Why expect this?

Iterating field theory argument for weak edge modes get separated by *macroscopic* distance $t_1 = t_2 > 0$.
Why expect this?

Iterating field theory argument for weak edge modes get separated by *macroscopic* distance $t_1 = t_2 > 0$
Why expect this?

Iterating field theory argument for weak edge modes get separated by *macroscopic* distance

\[ t_1 = t_2 > 0 \]
Why expect this?

Iterating field theory argument for weak edge modes get separated by *macroscopic* distance $t_1 = t_2 > 0$. 
Coupled chain picture thus ‘points’ in interesting direction in parameter space to explore

\[ t_1 = t_2 > 0 \]
Presence of chiral gapless edge modes suggests we will reach a *topological phase*

Which one?
Edge theory has six primary fields \( \{1, \psi, \psi^\dagger, \sigma, \sigma^\dagger, \epsilon\} \)

\( \psi \) and \( \psi^\dagger \) are continuum limit of lattice parafermions.

Treating \( \psi \) and \( \psi^\dagger \) as local leaves two sectors:

\[
\{1, \psi, \psi^\dagger\} \quad \{\epsilon, \sigma, \sigma^\dagger\} \quad (= \{1, \psi, \psi^\dagger\} \times \epsilon)
\]

This implies

\[\Rightarrow \] two degenerate ground states

\[\Rightarrow \] one non-trivial quasiparticle (Fibonacci anyon)

\[\Rightarrow \] counting of low ‘energy’ entanglement spectra

This phase called the \textit{Fibonacci phase}
Prior reasoning based on weakly-coupled chains

Do subleading interactions eventually couple edge modes?

Stable to finite $t_1, t_2$ ?

Does Fibonacci phase persist to isotropic point $t_1 = t_2 = t_3$ ?

Stability for $t_1 \neq t_2$ ?

Approach isotropic $t_1 = t_2 = t_3$ limit non-perturbatively with DMRG on cylinders
Two-dimensional results: **Fibonacci**
Line of attack

- Gradually increase $t_1 = t_2 \overset{\text{def}}{=} t_\perp$ and number of legs $N_y = 4, 6, 8, 10$

- Apply DMRG to infinitely long cylinders (iDMRG)
Line of attack

- Gradually increase $t_1 = t_2 = t_\perp$ and number of legs $N_y = 4, 6, 8, 10$

- Apply DMRG to infinitely long cylinders (iDMRG)
Immediately see two quasi-degenerate ground states

Energy splitting of ground states versus $t_\perp$ for $N_y = 4$: 

\[
\frac{(E_\epsilon - E_1)}{|E_1|}
\]
For small $t_\perp = 0.2$, y- correlation length apparently less than circumference of $N_y = 4$ cylinder

Seeing two-dimensional topological states?
Q:
How to observe physics with no local order parameter?
How to distinguish degenerate ground states?

A:
Entanglement entropy and entanglement ‘spectrum’ by ‘cutting’ the wavefunction

\[ |1\rangle \quad \text{and} \quad |\varepsilon\rangle \]
“Entanglement spectrum” is set of probabilities for system to be in different states near the cut

\[ |\Psi\rangle = \sum_n |n\rangle \times p_n \]

\[ \text{S} = - \sum_n p_n \log p_n \]

“Entanglement entropy” measures log(# states) system fluctuates through

\[ p_n \overset{\text{def}}{=} e^{-\tilde{E}_n} \]
Entanglement spectra of ground states show sharp degeneracies

\[ |1, \varepsilon\rangle = \sum_n \langle n | e^{-\tilde{E}_n} \]

Spectrum of “virtual edge” has precise agreement with field theory (Z3 parafermion CFT) of edge spectrum

\[ t_{\perp} = 0.2 \]
\[ N_y = 4 \]

\[ \tilde{E}_n = -\log p_n \]
(shifted and rescaled)
From finite-size scaling, can measure topological entanglement entropy

Prediction for these topological states$^{1,2,3}$

\[ S_1 = aN_y - \gamma_1 \]
\[ S_\varepsilon = aN_y - \gamma_\varepsilon \]

Constrained quantum fluctuations

\[ D = \sqrt{1 + \phi^2} \]
\[ \phi = (1 + \sqrt{5})/2 \]

2) Kitaev, Preskill PRL 96, 110404 (2006)
3) Zhang, Grover, Turner, Oshikawa, Vishwanath, PRB 85, 235151 (2012)
Topological entanglement entropy, state $|1\rangle$
(two strengths of $t_\perp$)*

$$S_n = \log(1 \ll D)$$

$\gamma_1 = \log(1/D)$

* Up to $-\log \sqrt{3}$ shift
Topological entanglement entropy, state $|\varepsilon\rangle$
(two strengths of $t_\perp$)*

![Graph showing $S_n$ vs. $N_y$ with two lines for $t_\perp = 0.4$ and $t_\perp = 0.6$]

$-\gamma_\varepsilon = \log(\varphi/D)$

* Up to $-\log \sqrt{3}$ shift
Topological entanglement entropy shows completeness of ground states

<table>
<thead>
<tr>
<th>$t_\perp$</th>
<th>$\gamma_1$</th>
<th>$\gamma_\varepsilon$</th>
<th>$e^{-2\gamma_1} + e^{-2\gamma_\varepsilon}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>$\log \mathcal{D} \approx 0.6430$</td>
<td>$\log(\mathcal{D}/\varphi) \approx 0.1617$</td>
<td>1</td>
</tr>
<tr>
<td>$t_\perp = 0.4^a$</td>
<td>0.6235</td>
<td>0.1393</td>
<td>1.0442</td>
</tr>
<tr>
<td>$t_\perp = 0.4^b$</td>
<td>0.6306</td>
<td>0.1538</td>
<td>1.0186</td>
</tr>
<tr>
<td>$t_\perp = 0.6$</td>
<td>0.6498</td>
<td>0.1562</td>
<td>1.0043</td>
</tr>
</tbody>
</table>

All ground states accounted for

$^a$ Ny=4,6,8 fitted

$^b$ Only Ny=4,6, fitted
Approach isotropic limit on larger cylinders, energy splitting:

Fibonacci phase at isotropic triangular lattice and beyond
Strong evidence that isotropic triangular lattice of \( \mathbb{Z}_3 \) parafermions lies deep within Fibonacci phase

Weakly-coupled wires approach safely guided us deep into gapped, topological phase
Initial results for anisotropic square lattice yield no evidence of Fibonacci phase

Different phase?

\[
t_1 \quad t_3
\]
Adiabatically move toward square lattice

- fix value of $t_1$
- gradually reduce $t_2$ to zero

Measure entanglement entropy along these lines
Observe peaks in entanglement entropy

Results for $N_y = 8$ (largest)

Empirically fit peaks to quadratic to estimate location
Combining results for $N_y = 4, 6, 8$

Peak locations:

---

![Graph showing peak locations for $N_y = 4, 6, 8$.]
RG argument predicts critical line along

\[ t_{2c} - t_{1c} = C(t_{2c} + t_{1c})^{8/5} \]
Transforming $N_y = 8$ fit under all permutations of $t_1, t_2, t_3$ gives estimate for phase boundary

\[ \mathbf{t} = (1, 0, 0) \]

\[ \mathbf{t} = (t_1, t_2, t_3) \]

\[ \mathbf{t} = (0, 1, 0) \]

\[ \mathbf{t} = (0, 0, 1) \]

\[ \mathbf{t} \]

\[ \mathbf{t} \]

Isotropic triangular point

Isotropic square point
Square lattice in different phase, but direct attack not useful

Two degen. ground states, but large finite-size effects

\( \vec{t} = (1, 0, 0) \)
\( \vec{t} = (0, 1, 0) \)
\( \vec{t} = (0, 0, 1) \)

\( \vec{t} = (t_1, t_2, t_3) \)

△ Isotropic triangular point
□ Isotropic square point

Energy splitting:
So we attacked from limit of decoupled chains with negative-sign of interactions

\[ \vec{t} = (0, -1, 0) \]
\[ \vec{t} = (1, 0, 0) \]
\[ \vec{t} = (0, 1, 0) \]
\[ \vec{t} = (0, 0, 1) \]
\[ \vec{t} = (t_1, t_2, t_3) \]

Similar coupled chain argument + DMRG numerics finds topological phase but no Fibonacci anyon
Wrap up
In this talk, showed that an isotropic, next-neighbor model of coupled parafermions realizes a highly non-trivial 2D phase (Fibonacci phase)

Could guide search for ‘smeared out’ limit of such a model, for example
- uniform superconductor coupled to 2/3 fractional QHE
- coupled fractional QHE bilayers
More generally, weakly-coupled chain analytics + DMRG style numerics = fruitful approach for discovering simple lattice models deep in interesting phases

Other short-range lattice models for topological phases?

Useful for finding 2D phases without gapless edges?
“Beyond DMRG” methods are coming

Efficient schemes for contracting / optimizing infinite 2D variational wavefunctions (so called Tensor Product States / PEPS)$^{1,2}$

Known how to write topological states as simple tensor product states...

Study proximate phases by adding small number of variational parameters

---

1) Evenbly, Vidal 1412.0732 (2014)
2) Lubasch, Cirac, Banuls, PRB 90, 064425 (2014)
Summary

- Isotropic triangular lattice of \((Z_3)\) parafermions lies deep within Fibonacci phase

- Isotropic square lattice likely hosts a different (Abelian) topological phase

- Powerful combination of coupled-chain analytics + DMRG numerics