Extracting the Proton Radius from Electron Scattering Data

Why the proton radius is smaller in Virginia

Douglas W. Higinbotham
How many ways can YOU determine the radius of a perfect sphere?!

Image of the sphere created to test theory of relativity on the Gravity Probe B spacecraft.
Some Answers

• Diameter = 2 r
• Area = $\pi r^2$
• Volume = $\frac{4}{3} \pi r^3$ (displacement of water)
• Momentum of Inertia
  – $\frac{2}{5} m r^2$ (solid sphere)
  – $\frac{2}{3} m r^2$ (hollow sphere)
Charge Radius of the Proton

• Proton is hard as there are currently only a few ways to get the radius
  – Atomic Hydrogen Lamb Shift ( ~ 0.88 fm )
  – Muonic Hydrogen Lamb Shift ( ~ 0.84 fm)
  – And of course elastic electron scattering!

• Heavy nuclei are relatively easy (little recoil)
  – Measure charge form factor
  – Take Fourier Transform
  – Even better if you measure a diffraction minimum!
Rosenbluth Formula

From relativistic quantum mechanics one can derive the formula electron-proton scattering where one has assumed the exchange of a single virtual photon.

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \cdot \frac{E'}{E} \left[\frac{G_E^2 + \tau G_M^2}{1 + \tau} + 2\tau G_M^2 \tan^2 \frac{\theta}{2}\right]$$

where $G_e$ and $G_m$ take into account the finite size of the proton.

$$G_E = G_E(Q^2), \ G_M = G_M(Q^2); \ G_E(0)=1, \ G_M(0) = \mu_p$$

$$Q^2 = 4EE'\sin^2(\theta/2) \text{ and } \tau = Q^2/4m_p^2$$
Standard Dipole Radius: 0.81(1) fm


Due to its light mass, relativistic corrections make the proton radius more challenging.

For the proton, we extract the radius by determining the slope of $G_E$ at $q^2 = 0$.

$$G_E(q^2) = G_E(0) \left[ 1 - \frac{1}{6} \langle r_p^2 \rangle q^2 + \ldots \right]$$

$$\langle r_p^2 \rangle = -6 \frac{dG_E(q^2)}{dq^2} \bigg|_{q^2=0}$$

Took the data that was available, fit it with Taylor N=1 and 2, and got the slope at $q^2 = 0$.

**DO IT YOURSELF!!** If you make different choices you will get slightly different answers!
Proton Radius vs. Time

Mainz 2014 $G_e$ (Blue Band)


Note the spline $G_e$ fit at high $q^2$ starts to fall as soon as it is not constrained by their data.
Mainz 2014 Fitting Results ($G_e$ & $G_m$)

Did not follow a standard statistical method, such as an f-test, to determine number of parameters. Instead, the authors just state chi2 <1600 are “analyses with the good models.”

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<td>1.1436</td>
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<tr>
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PLEASE READ: Data Reduction and Error Analysis for the Physical Sciences by Bevington & Robinson and don’t just use chi2 to judge which fit is most likely. (i.e. probabilities)
Tension With World $G_m$ Results


The 1.6% shift that is suggested by their “double dipole” fit would bring Mainz to the world data.

NOTE: $G_e$ and $G_m$ do not need to be the same function!
“As can be seen, all shifts are positive, i.e., the actual cross sections as reconstructed by the fit are large then the values quoted.  Although it may look strange that all shifts are positive, the mean of the normalization falls together with the shift of the oldest measurement [57], ...” - Bernauer et al.

**Completely dismissing the fact their fit disagrees with world data?!**
Saskatoon 1974 (elastic recoil proton) missing from MANY global fits

As of 2 Feb. 2016, not even listed in the extensive Scholarpedia proton form factor articles: http://www.scholarpedia.org/article/Nucleon_Form_factors


Please add here! ->

Mainz80 and Saskatoon74

Prior to new Mainz results ~2010: Lowest, High Precision \( G_e \) Measurements

\[
f(q^2) = a0(1 + a1 \times q^2 + a2 \times q^4 \ldots + aN \times q^{2N})
\]

For a three parameter fit of the combined Mainz and Saskatoon data, we find

\[
X = \begin{pmatrix} 1.0032 \\ -0.1272 \\ 0.0110 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.367 & -1.05 & 0.684 \\ -1.05 & 3.54 & -2.55 \\ 0.684 & -2.55 & 2.03 \end{pmatrix} \times 10^{-5}
\]

with a reduced \( \chi^2 \) of 0.723, an \( a1 \) value that corresponds to a radius of 0.875 fm, and \( \sqrt{a_{11}} \) term of 0.006 fm. However, there are clearly problems as the off-diagonal terms are larger than the diagonal elements.

Continued Fraction fit of this set of data produces a similar result.

“DISCOVERED” BY MOVING THE LAST DATA POINT ONE SIGMA AND SEEING THE EFFECT ON THE RESULT

Thus, a three parameter fit of the full range of data agrees with both 0.84fm and 0.88fm radius.

Doing \( f \)-tests (see Bevington) one finds that going to second order isn’t justified by the data!
Back To The Taylor Extrapolations . . .

Hand et al.’s ideas but now with MUCH better data

D.W.H. et al., arXiv:1510.01293

K. Griffioen et. al., arXiv: 1509.0667

Classic Data from Mainz and Saskatoon data

0.84(1)fm

Monopole
Dipole
Gaussian
Taylor (N=1)

New Mainz data re-analysis

\[ G_e(Q^2) = c_1(1 + c_2 Q^2 + c_3 Q^4) \]

\[ r_{rms} = 0.850 \pm 0.019 \]

1 fm \(^{-2}\) = 0.0389 GeV\(^2\)
Of course the linear function doesn’t work to all $Q^2$, though it is amusing to note this function does have a charge radius of 0.84 fm.
Example of Precision vs. Accuracy

Shown with simply with asymptotic standard error which can be very misleading . . .

Linear Fit of Low Current Data Accurately Extrapolates The Residual Field $n_0$ (also extrapolates from 150A to 300A)
The 10th Order Polynomial Fit Precisely Describes The Data But Doesn’t Extrapolate Well
Multivariate Errors

As per the particle data handbook, one should be using a co-variance matrix and calculating the probably content of the hyper-contour of the fit. Default setting of Minuit of “up” is one.

Standard Errors often underestimate true uncertainties. (manual of gnuplot fitting has an Explicate warning about this)

<table>
<thead>
<tr>
<th>Number of Parameters</th>
<th>Confidence level (probability contents desired inside hypercontour of $\chi^2 = \chi^2_{\min} + \text{up}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50%</td>
</tr>
<tr>
<td>1</td>
<td>0.46</td>
</tr>
<tr>
<td>2</td>
<td>1.39</td>
</tr>
<tr>
<td>3</td>
<td>2.37</td>
</tr>
<tr>
<td>4</td>
<td>3.36</td>
</tr>
<tr>
<td>5</td>
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</tr>
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</tr>
<tr>
<td>7</td>
<td>6.35</td>
</tr>
<tr>
<td>8</td>
<td>7.34</td>
</tr>
<tr>
<td>9</td>
<td>8.34</td>
</tr>
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<td>10</td>
<td>9.34</td>
</tr>
<tr>
<td>11</td>
<td>10.34</td>
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If FCN is $-\log(\text{likelihood})$ instead of $\chi^2$, all values of up should be divided by 2.
Fits of the Mainz 2014 $G_E$ Rosenbluth Data

Rational Function & Dipole give radius of $\sim 0.84$ fm with Maclaurin ($j=5$ & 6)
Charge Form Factor with Dipole (0.84 fm)

Using the classic data along with the Mainz 2014 “Rosenbluth” $G_E$ Results

Within the range of the Mainz data, this result is very similar to Griffioen’s CF (N=4) fit.
The Problem with fitting the intercept

FAUX DATA: Real function $y(x) = 0.985*(1 + 0.1176*x)^{**(-2)}$ randomized point-to-point & systematically shifted (i.e. the normalization isn’t perfect)

\[ \chi^2 = 2627.63 \]
\[ N_{\text{pobs}} = 981 \]
\[ N_{\text{per}} = 2 \]
\[ P1 = 0.998983 \pm 9.66248\times10^{-5} \]
\[ P2 = 0.120615 \pm 4.50064\times10^{-5} \]

i.e. if you fit this data without letting the end point float . . . reduced $\chi^2 = 2.7$ !
Same Faux Data Now Fit with Double Dipole

Note: Now you get the same result with or without the intercept point.

FAUX DATA: Real function \( y(x) = 0.985*(1 + 0.1176*x)^{(-2)} \) randomized point-to-point & systematically shifted (i.e. the normalization isn’t perfect by 1.5%)

Three parameter with MINUIT called from PAW

\[ \chi^2 = 1078.79 \]
\[ N_{\text{par}} = 980 \]
\[ N_{\text{obs}} = 3 \]
\[ P1 = 0.0156536 \pm 0.000452187 \]
\[ P2 = 12.5115 \pm 4.76912 \]
\[ P3 = 0.117335 \pm 0.000105962 \]

Reduced chi2 very close to one, BUT this wasn’t the original function!!
# Mainz 2014 Fitting Results ($G_e$ & $G_m$)

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The Mainz 2014 supplemental material has all the data and an example Python fitting script.
The Mainz PRC ONLY Report High Order Fits !?
Proton Radius vs. Order of Polynomial Fits

$R^2$ (goodness of fit measure which runs 0 to 1) gets to 0.97 by 4th order and 0.98 by 10th....
But how well do these high order fits do against data not included in the fit?! Using the full Mainz data and a Python fitting code based on the Mainz fitting routine.

The lower order fits to the Mainz data also give agreement with world $G_M$ & a smaller radius...
So what is going on?!

Cross Section/Dipole Function (Ge=0.84fm, Gm=0.80fm)

\[
\frac{d\sigma/d\Omega}{(d\sigma/d\Omega)_{\text{Mott}}} = \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2\tau G_M^2(Q^2) \tan^2 \frac{\theta}{2}
\]

\[
\epsilon = \frac{G_E^2 + \tau G_M^2}{\epsilon (1 + \tau)}
\]

\[
\epsilon = \left[ 1 + 2(1 + \tau) \tan^2 \frac{\theta}{2} \right]^{-1}
\]

\[
\tau = \frac{Q^2}{4M_p^2}
\]
NOTE: $Q^2 = Q^2(E, \theta)$ has a kinematic max., $Q^2_{\text{max}}(E, 180^\circ)$, which these fits nicely reproduce
Summary

• One can find different proton radii depending on the function used for the fit (model dependence).
• Linear Extrapolations of the lowest $q^2$ data (Maclaurin series $N=1$) give results consistent with muonic hydrogen ($\sim 0.84$ fm)

$\bullet$ Advantages of low $Q^2$: floating normalization and tiny $G_m$ contribution. (model independence)

$\bullet$ Do NOT just shift the low $Q^2$ data without redoing the normalization (i.e. if you add a correction to the cross sections, you need to start over with the normalization procedure otherwise you are biasing the result by shifting the points without moving the intercept)

$\bullet$ Currently working with better extrapolation functions (i.e. Rational Fractions & Chebyshev polynomials) to do our own fit to the full set Mainz 2014 published data.

$\bullet$ Our preliminary model independent fits results agree with Griffioen, Carlson, Maddox’s fit and world $G_m$ data... but what about all the other global fits!?
Very Low $q^2 G_e$ dominates the cross section & very high $q^2 G_m$ dominates the cross section.

(i.e. for this fit it depends on which why you look at the plot)
IV. FIXED RADIUS FITS

We also tried fixing the radius to 0.84 fm and 0.88 fm (i.e. \( a_1 = 0.1176 \) fm\(^2\) and 0.1292 fm\(^2\) respectively) and performing a five parameter fit of the Mainz14 Rosenbluth \( G_E \) data. The \( \chi^2 \) is significantly better for the smaller radius.

TABLE III: Repeating the above \( j = 6 \) fit, but with the \( a_1 \) term fixed to the atomic hydrogen and muonic hydrogen values of the proton radius, 0.84 fm and 0.88 fm.

<table>
<thead>
<tr>
<th>Fixed Radius</th>
<th>( \chi^2 )</th>
<th>( \chi^2 / \nu )</th>
<th>( n_0 )</th>
<th>( a_2 ) ( \cdot 10^{-2} )</th>
<th>( a_3 ) ( \cdot 10^{-3} )</th>
<th>( a_4 ) ( \cdot 10^{-5} )</th>
<th>( a_5 ) ( \cdot 10^{-6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.84 fm</td>
<td>56.34</td>
<td>0.783</td>
<td>0.994(1)</td>
<td>1.12(1)</td>
<td>-0.93(2) \cdot 10^{-3}</td>
<td>5.0(1) \cdot 10^{-5}</td>
<td>1.20(5) \cdot 10^{-6}</td>
</tr>
<tr>
<td>0.88 fm</td>
<td>142.1</td>
<td>1.97</td>
<td>1.003(1)</td>
<td>1.62(1) \cdot 10^{-2}</td>
<td>-1.78(1) \cdot 10^{-3}</td>
<td>1.14(1) \cdot 10^{-4}</td>
<td>-2.90(7) \cdot 10^{-6}</td>
</tr>
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