Helicity Evolution at Small $x$

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References: 1511.06737 1505.01176

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Introduction: Studying Proton Structure

• Deep Inelastic Scattering and the Parton Model
• Quantum evolution and the small-x limit
• The Proton Spin Crisis: Is there spin at small x?
Outline

Introduction: Studying Proton Structure
- Deep Inelastic Scattering and the Parton Model
- Quantum evolution and the small-\(x\) limit
- The Proton Spin Crisis: Is there spin at small \(x\)?

The Toolbox: Quarks and the Small-\(x\) Limit
- TMD quark distributions at large and small \(x\)
- Coherence and quasi-classical initial conditions
- Small-\(x\) evolution
Introduction: Studying Proton Structure

- Deep Inelastic Scattering and the Parton Model
- Quantum evolution and the small-x limit
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The Toolbox: Quarks and the Small-x Limit

- TMD quark distributions at large and small x
- Coherence and quasi-classical initial conditions
- Small-x evolution

The Calculation: Helicity at Small x

- Polarized initial conditions
- Evolving spin to small x
- The added complexity: Non-Ladder Diagrams
Introduction: Studying Proton Structure
An Analogy: The Proton and the Atom

The Hydrogen Atom

The Proton
An Analogy: The Proton and the Atom

The Hydrogen Atom

• Elementary bound state of a proton and electron.
• Bound by QED interactions.

The Proton

• Elementary bound state of three quarks.
• Bound by QCD interactions.
An Analogy: The Proton and the Atom

The Hydrogen Atom

- Elementary bound state of a proton and electron.
- Bound by QED interactions.
- Ground state is spherically symmetric with zero net angular momentum.

\[ J, L, F = 0 \]

\[
\begin{align*}
J &= \frac{1}{2} \\
L &= 0 \\
F &= 0
\end{align*}
\]

The Proton

- Elementary bound state of three quarks.
- Bound by QCD interactions.
- Spin \( \frac{1}{2} \) fermion can be accommodated by quark spin pairing.
The Hydrogen Atom

- Hydrogen has complex structure: fine, hyperfine...
  ... but it is well described by the Bohr model because QED is a perturbative theory.
The Importance of Proton Structure

The Hydrogen Atom

- Hydrogen has complex structure: fine, hyperfine... but it is well described by the Bohr model because QED is a perturbative theory.
- Atomic structure led to chemistry, electronics, and the nanotech revolution.
The Importance of Proton Structure

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- QCD is only perturbative at short distances...
  ... so the proton structure embodies all the non-perturbative complexity of the field theory.
The Importance of Proton Structure

The Hydrogen Atom

- Hydrogen has complex structure: fine, hyperfine... ... but it is well described by the Bohr model because QED is a perturbative theory.
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The Proton

- QCD is only perturbative at short distances... ... so the proton structure embodies all the non-perturbative complexity of the field theory.
- Proton structure will tell us about the nature of QCD and a future femtoscale revolution.
The DIS “Femto-scope”

• Deep Inelastic Scattering (DIS)

\[ e + p \rightarrow e' + X \]

Proton Rest Frame

- Proton
- Rest Frame

\[ E' \]

\[ E \]

\[ \theta \]

\[ X \]
The DIS “Femto-scope”

- Deep Inelastic Scattering (DIS)
- Kinematic variables:

\[
\begin{align*}
E & \\
E' & \\
\theta & \\
\end{align*}
\quad \rightarrow \quad 
\begin{align*}
Q^2 &= 4EE' \sin^2 \frac{\theta}{2} \\
\frac{x}{2} &= \frac{Q^2}{2m_N(E-E')} \\
S &= 2Em_N
\end{align*}
\]

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Proton Rest Frame
The DIS “Femto-scope”

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\[Q^2 = \text{“Resolution”}\]
\[x = \text{“Exposure”}\]

\[\Delta x^2_\perp < \frac{1}{Q^2}\]
\[\Delta t < \frac{1}{m_Nx}\]
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- Resolve proton substructure: \( Q^2 \gg m_N^2 \)

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- \(Q^2\) = “Resolution” \(\Delta x^2_\perp < \frac{1}{Q^2}\)
- \(x\) = “Exposure” \(\Delta t < \frac{1}{m_N x}\)

- Resolve proton substructure: \(Q^2 \gg m_N^2\)
- Bjorken scaling: asymptotic freedom!
The DIS “Femto-scope”

- **Deep Inelastic Scattering (DIS)**
- **Kinematic variables:**

\[
E' = \frac{Q^2}{2m_N(E - E')} \\
\theta = \frac{s}{2Em_N}
\]

\[
Q^2 = 4EE' \sin^2 \frac{\theta}{2}
\]

\[
x = \frac{Q^2}{2m_N(E - E')}
\]

\[
s = 2Em_N
\]

- **Resolve proton substructure:** \( Q^2 \gg m^2_N \)
- **Bjorken scaling:** asymptotic freedom!
- **Identified QCD as the fundamental theory of the strong nuclear force.**

\[e + p \rightarrow e' + X\]
Parton model: At short distances, virtual photon strikes one quark.
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• **Structure functions interpreted as parton distribution functions**

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F_2(x, Q^2) = \sum_f Z_f^2 x f(x, Q^2)
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• **Structure functions interpreted as parton distribution functions**

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• **x** has the interpretation as the quark momentum fraction

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x = \frac{k^+}{p^+} \quad 0 \leq x \leq 1
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Large \( x \): \( x \approx \frac{1}{3} \) Valence Quarks
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Small \( x \): \( x \approx 1\% \) Bremsstrahlung
• **Parton model:** At short distances, virtual photon strikes one quark.

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<thead>
<tr>
<th>Large x:</th>
<th>( x \approx \frac{1}{3} )</th>
<th>Valence Quarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small x:</td>
<td>( x \approx 1% )</td>
<td>Bremsstrahlung</td>
</tr>
<tr>
<td>Smaller x:</td>
<td>( x &lt; 1% )</td>
<td>Gluon Explosion!</td>
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</table>
Bjorken scaling occurs over a range of $Q^2$.

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• But large increases in $Q^2$ do resolve small-scale quantum fluctuations.
Bjorken scaling occurs over a range of $Q^2$

- Smaller-scale quantum fluctuations are suppressed by asymptotic freedom.

But large increases in $Q^2$ do resolve small-scale quantum fluctuations.

Short-distance fluctuations are suppressed...

...but some are enhanced by logarithms of $Q^2$

“Quantum Evolution” of the parton distributions!

$$\alpha_s(Q^2) \ln \frac{Q^2}{\Lambda^2} \sim 1$$
What’s So Special about Small $x$?

- Multiple bremsstrahlung leads to an explosion of gluon density at small $x$. 
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What’s So Special about Small $x$?

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  \[ \alpha_s \ln \frac{1}{x} \sim 1 \]
- The gluon density increases so quickly it would violate unitarity (Froissart bound)
What’s So Special about Small $x$?

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\[ \alpha_s \ln \frac{1}{x} \sim 1 \]

- The gluon density increases so quickly it would violate unitarity (Froissart bound)

- At very small $x$, nonlinear gluon fusion must lead to a **saturation** of the gluon density.
If you can do DIS with polarized protons and electrons, you can measure the spin of the quarks.
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But in 1988 the EMC Collaboration found that “only $14 \pm 9 \pm 21\%$ of the proton spin is carried by the spin of the quarks”!
• If you can do DIS with polarized protons and electrons, you can measure the spin of the quarks

• From a naive constituent quark picture, one expects the valence quarks to accommodate the proton spin.

• But in 1988 the EMC Collaboration found that “only 14 ± 9 ± 21% of the proton spin is carried by the spin of the quarks”!

• If the quark spins don’t account for the proton spin... what does?
The “Proton Spin Budget” is described by the Jaffe-Manohar Sum Rule.

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \Delta G + L_q + L_g \]
The Proton Spin Crisis

- The “Proton Spin Budget” is described by the Jaffe-Manohar Sum Rule.

\[ \frac{1}{2} = \frac{1}{2} \Delta \Sigma + \Delta G + L_q + L_g \]

- Modern measurements still cannot account for the spin of the proton!
  - Quark spins from polarized DIS
  - Gluon spins from polarized proton-proton collisions

Data from

\[ 0.001 < x < 1 \]

- \( \Delta \Sigma \approx 0.25 \) (25%)
- \( \Delta G \approx 0.2 \) (40%)
The Proton Spin Crisis

• The “Proton Spin Budget” is described by the Jaffe-Manohar Sum Rule.

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\]

• Modern measurements still cannot account for the spin of the proton!

→ Quark spins from polarized DIS
→ Gluon spins from in polarized proton-proton collisions

• Proton structure is much more complex than previously believed!

→ Orbital angular momentum?
→ Polarization at very small x?

Data from

\[
0.001 < x < 1
\]

\[
\Delta \Sigma \approx 0.25 \ (25\%)
\]

\[
\Delta G \approx 0.2 \ (40\%)
\]
The Toolbox:
Quarks and the Small-x Limit
\[ \phi_{\alpha\beta}(x, \vec{k}_\perp) = \int \frac{d^2 r}{(2\pi)^3} \frac{e^{i k \cdot r}}{4\pi^2} \langle h(p, S) | \bar{\psi}_\beta(0) \mathcal{U}[0, r] \psi_\alpha(r) | h(p, S) \rangle \]
Definition: TMD Quark Distribution

\[
\phi_{\alpha\beta}(x, \vec{k}_\perp) = \int \frac{d^2 r}{(2\pi)^3} e^{i \vec{k} \cdot \vec{r}} \langle h(p, S) | \bar{\psi}_\beta(0) U[0, r] \psi_\alpha(r) | h(p, S) \rangle
\]

Transverse Momentum Dependent Parton Distribution Functions
\[
\phi_{\alpha\beta}(x, \vec{k}_T) = \int \frac{d^2r}{(2\pi)^3} e^{i k \cdot r} \langle h(p, S) | \bar{\psi}_{\beta}(0) U[0, r] \psi_{\alpha}(r) | h(p, S) \rangle
\]

Transverse Momentum Dependent Parton Distribution Functions

\[
\sum_{\sigma\lambda} \langle h(p) | b_{\lambda \sigma}^\dagger b_{k\lambda} | h(p) \rangle \ [\bar{U}_{\sigma}(k)]_{\beta} [U_{\lambda}(k)]_{\alpha}
\]

Quark number operator + Dirac spinors
**Definition: TMD Quark Distribution**

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<td></td>
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| Quark Polarization   | $\gamma^+$       | $\gamma^+\gamma^5$           | $\gamma^+\gamma_\perp^\gamma^5$ |

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\phi_{\alpha\beta}(x, \vec{k}_\perp) = \int \frac{d^2r}{(2\pi)^3} e^{ik\cdot r} \langle h(p, S)|\bar{\psi}_\beta(0)U[0, r]|\psi_\alpha(r)|h(p, S)\rangle
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\sum_{\sigma\lambda}\langle h(p)|b_{k\sigma}^\dagger b_{k\lambda}|h(p)\rangle\ [\bar{U}_\sigma(k)]_\beta[U_\lambda(k)]_\alpha
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<tr>
<td>L</td>
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<td>Helicity</td>
<td>$h_{1L}$</td>
</tr>
<tr>
<td>T</td>
<td>$f_{1T}^\pm$</td>
<td>Sivers</td>
<td>$h_{1T}$  — Transversity</td>
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Nucleon Polarization

| $\Gamma$ | $\gamma^+$ | $\gamma^+\gamma^5$ | $\gamma^+\gamma_{\perp}^i\gamma^5$ |

**Gauge Link:** momentum redistribution due to final-state interactions

\[
\phi_{\alpha\beta}(x, \vec{k}_{\perp}) = \int \frac{d^2 - r}{(2\pi)^3} e^{i \vec{k} \cdot \vec{r}} \langle h(p, S) | \bar{\psi}_\beta(0) U[0, r] \psi_\alpha(r) | h(p, S) \rangle
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Transverse Momentum Dependent Parton Distribution Functions

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Quark number operator + Dirac spinors
Semi-Inclusive Deep Inelastic Scattering (SIDIS)

\[ e + p \rightarrow e' + h + X \]
Quark Distributions at Large $x$

**Semi-Inclusive**

**Deep Inelastic Scattering (SIDIS)**

\[ e + p \rightarrow e' + h + X \]

**Large-$x$ Kinematics:**

\[ \hat{s} \sim Q^2 \gg k_T^2 \]
\[ x = \frac{Q^2}{\hat{s} + Q^2} \sim \mathcal{O}(1) \]
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- Propagates through the gauge field before escaping.
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• Photon knocks out a quark from the proton.

• Propagates through the gauge field before escaping

• Staple-shaped gauge link encodes final-state interactions
Quark Distributions at Small $x$

**Small-x Kinematics:**

\[ \Delta t < \frac{1}{m_N x} \]
\[ \hat{s} \gg Q^2 \gg k_T^2 \]
\[ x = \frac{Q^2}{\hat{s}} \ll 1 \]
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- Photon creates a quark / antiquark pair which propagates through the proton.

⇒ Quark transport is $x$-suppressed.
Quark Distributions at Small x

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- Photon creates a quark / antiquark pair which propagates through the proton.
- Quark transport is \(x\)-suppressed.
- Proton is Lorentz-contracted to a “shockwave”.
- Gauge link covers the entire proton.
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- Photon creates a quark / antiquark pair which propagates through the proton.
- Quark transport is \( x \)-suppressed.

- Proton is Lorentz-contracted to a “shockwave”.
- Gauge link covers the entire proton.
- Infinite dipole degrees of freedom at small \( x \)

\[ S_{xy} = \frac{1}{N_c} \text{Tr} \left[ V_x V_y^\dagger \right] \]
Initial Conditions at Small $x$

- Long-lived projectile sees whole target coherently.

→ High gluon density at small $x$ enhances multiple scattering.
Initial Conditions at Small $x$

- Long-lived projectile sees whole target coherently.
- High gluon density at small $x$ enhances multiple scattering

- High density rescattering can be systematically re-summed
- Classical gluon fields!

Nucleus: $\alpha_s^2 A^{1/3} \sim 1$  Proton: $\alpha_s \rho \sim 1$

\[
\Delta t < \frac{1}{m_N x}
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**Initial Conditions at Small x**

- Long-lived projectile sees whole target coherently.
  - High gluon density at small $x$ enhances multiple scattering

- High density rescattering can be systematically re-summed
  - Classical gluon fields!

  Nucleus: $\alpha_s^2 A^{1/3} \sim 1$  
  Proton: $\alpha_s \rho \sim 1$

- Charge density defines a hard momentum scale which screens the IR gluon field.

  Both: $Q_s^2 \propto \alpha_s^2 A^{1/3} \propto \alpha_s \rho$

  $Q_s^2 \gg \Lambda^2$

\[ \Delta t < \frac{1}{m_N x} \]
High-energy radiation from a moving particle couples to $A^-$

In $A^- = 0$ gauge this radiation is suppressed.
Quantum Evolution in the Light-Cone Gauge

- High-energy radiation from a moving particle couples to $A^-$. In $A^- = 0$ gauge this radiation is suppressed.

- Quantum evolution requires long lifetimes to generate logarithms of a large phase space. Instantaneous t-channel particles do not evolve either. All evolution takes place within the moving particles.
• High-energy radiation from a moving particle couples to $A^-$ in $A^- = 0$ gauge this radiation is suppressed.

• Quantum evolution requires long lifetimes to generate logarithms of a large phase space. Instantaneous t-channel particles do not evolve either. All evolution takes place within the moving particles.

• For classical fields and leading-log evolution, $A_\perp = 0$ as well. The transverse part of the gauge link does not contribute.
The quark dipole radiates soft gluons before and after scattering.

- Evolution of the dipole scattering amplitude
- Re-sums single logarithms of $x$

\[ S_{xy} = \frac{1}{N_c} \text{Tr} [V_x V_y^+] \]

\[ \alpha_s \ln \frac{1}{x} \sim 1 \]
BK Evolution: The Small-x Gluon Cascade

\[
\frac{\partial}{\partial \ln s} \langle S_{xy} \rangle(s) = \bar{\alpha}_s \int d^2 z \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2} \left[ \langle S_{xz} S_{zy} \rangle(s) - \langle S_{xy} \rangle(s) \right]
\]

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- Some radiated gluons also rescatter in the target gauge field.
- Non-linear evolution with a hierarchy of operators

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- Evolution of the dipole scattering amplitude
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- Some radiated gluons also rescatter in the target gauge field.
- Non-linear evolution with a hierarchy of operators

- Evolution closes in the large \( N_c \) limit (BK eqn.)
  \[ Q_s^2(x) \propto \left( \frac{1}{x} \right)^{1/3} \]

\[ S_{xy} = \frac{1}{N_c} \text{Tr} [V_x V_y^\dagger] \]
• High energy (small $x$) scattering is predominantly spin independent.

BK evolution: total cross section, unpolarized quark distribution.
High energy (small x) scattering is predominantly spin independent.

**BK evolution:** total cross section, unpolarized quark distribution.

Transporting quark polarization to small x is suppressed!

Spin asymmetries, polarized quarks are suppressed at small x.
Digging for Spin Structure

High energy (small x) scattering is predominantly spin independent.

\[ 
\frac{d\Delta \sigma_q}{d^2 k} = -2 \frac{\alpha_s^2 C_F^2}{N_c} \frac{1}{s} \frac{1}{k_T^2} 
\]

BK evolution: total cross section, unpolarized quark distribution.

Transporting quark polarization to small x is suppressed!

Spin asymmetries, polarized quarks are suppressed at small x.

Sub-leading gluon exchange can also transfer spin dependence.

Gluon exchange can mix with quark exchange.
\textbf{Polarized Initial Conditions}

- \textit{"Polarized Wilson Line"} - Coherent, spin-dependent scattering.
- One spin-dependent exchange (more are suppressed)
- Dressed by multiple unpolarized scattering
“Polarized Wilson Line” - Coherent, spin-dependent scattering.

- One spin-dependent exchange (more are suppressed)
- Dressed by multiple unpolarized scattering

“Polarized Dipole Amplitude”:

- Quark (gauge link) scatters by an unpolarized Wilson line.
- Fermion (antiquark) scatters by a polarized Wilson line.

\[ G_{xy} \equiv \frac{1}{2N_c} \text{Tr} \left[ V_x V_y^\dagger (\sigma) + V_y (\sigma) V_x^\dagger \right] \]
• Kernels: Spin-dependent quark / gluon wave functions

⇒ Soft quarks and soft gluons can mix (same order)
Evolving Spin to Small $x$

- **Kernels:** Spin-dependent quark / gluon wave functions

- **Soft quarks and soft gluons can mix (same order)**

- **Requires longitudinal and transverse momentum ordering**

  $1 \gg z_1 \gg z_2 \gg \cdots \gg \frac{Q^2}{s}$

  $Q^2 \ll \frac{k_{1T}^2}{z_1} \ll \frac{k_{2T}^2}{z_2} \ll \cdots$

- **Includes “infrared” phase space:**

  $k_{1T}^2 \gg k_{2T}^2 \gg k_{1T}^2 \frac{z_2}{z_1}$
Evolving Spin to Small $x$

- **Kernels:** Spin-dependent quark / gluon wave functions
  - Soft quarks and soft gluons can mix (same order)

- Requires **longitudinal and transverse momentum ordering**
  \[
  1 \gg z_1 \gg z_2 \gg \cdots \gg \frac{Q^2}{s} \quad \quad Q^2 \ll \frac{k_{1T}^2}{z_1} \ll \frac{k_{2T}^2}{z_2} \ll \cdots
  \]
  - Includes "infrared" phase space:
    \[
    k_{1T}^2 \gg k_{2T}^2 \gg k_{1T}^2 \frac{z_2}{z_1}
    \]

- Leads to **double-log evolution.**
  - Faster evolution than unpolarized BK!
    \[
    \alpha_s \ln^2 \frac{1}{x} \sim 1
    \]
To solve, first keep only the kernels without unpolarized rescattering.

\[
\frac{\alpha_s}{2\pi} \int \frac{dz}{z} \int \frac{d\vec{k}_T^2}{k_T^2} \left( \begin{array}{cc} C_F & 2C_F \\ -N_f & 4N_c \end{array} \right)
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Solve by Mellin transform and saddle point approximation.

\[\alpha_s = 0.3\]
\[N_c = N_f = 3\]
\[G_{xy}(s) \sim \left( \frac{s}{Q^2} \right)^{1.46}\]
Solution: Ladder Evolution

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Fast growth of quark polarization at small x!

\[ S_{xy}(s) \sim \left( \frac{s}{Q^2} \right)^{0.3} \]

Large contribution to the proton spin?
• Unlike BK or DGLAP, leading-log evolution is also generated by non-ladder graphs.

→ Arises uniquely from the IR sector.

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• Quark and antiquark non-ladder graphs cancel

\[ G_{xy} \equiv \frac{1}{2N_c} \text{Tr} \left[ V_x V_y^\dagger (\sigma) + V_y (\sigma) V_x^\dagger \right] \]
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• Complication: Gluon non-ladder graphs do not cancel.
  ➡️ Ladder evolution is an unjustified truncation
• **Non-ladder gluons can stack in complex ways which still generate leading logarithms.**
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➡ Even unpolarized BK evolution in an intermediate step can contribute!
Knots of Non-Ladder Gluons

- Non-ladder gluons can stack in complex ways which still generate leading logarithms.

- Polarized gluons can "jump a rung" of evolution.

- Even unpolarized BK evolution in an intermediate step can contribute!

- Unpolarized evolution is in a color-octet state (unlike ordinary BK evolution).
Operator Evolution of the Polarized Dipole

Ladder:

\[ \partial_{Y} \]

\[ 0 \rightarrow 1 \]

Unpolarized (BK):

Ladder:

Non-Ladder:
Operator Evolution of the Polarized Dipole

Ladder:

Non-Ladder:
Operator Evolution of the Polarized Dipole

Ladder:

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• The evolution yields another infinite operator hierarchy

⇒ Closes in the large $N_c$ limit, like BK evolution.

⇒ But not physically relevant: neglects quark exchange
The evolution yields another infinite operator hierarchy

- Closes in the large $N_c$ limit, like BK evolution.

- But not physically relevant: neglects quark exchange

The transverse ordering condition is not automatically satisfied.

- Polarized dipoles can depend on their “neighbors”

- More complex than the large $N_c$ BK equation.
A Better Approximation: Large $N_c, N_f$

\[
\frac{\partial}{\partial \ln z} Q_{10}(z) = \frac{\partial}{\partial \ln z} G_{10}(z) = \frac{\partial}{\partial \ln z} A_{10}(z) = \frac{\partial}{\partial \ln z} \Gamma_{02,21}(z) + \frac{\partial}{\partial \ln z} S_{02}(z) + \frac{\partial}{\partial \ln z} G_{21}(z) - \frac{\partial}{\partial \ln z} A_{12}(z) - \frac{\partial}{\partial \ln z} Q_{10}(z) + \frac{\partial}{\partial \ln z} S_{01}(z) + \frac{\partial}{\partial \ln z} A_{21}(z)
\]

• To keep quark contributions, must also take $N_f$ large.

⇒ Must distinguish between dipoles made of actual quarks vs. large $N_c$ gluons.

⇒ Evolution equation closes, but even more complicated....
Can we solve the helicity evolution in \textit{any} systematic approximation?

- Large $N_c, N_f$?  Only large $N_c$?
- Does the growth persist at small $x$?
Outlook: The Truth is Out There!

- Can we solve the helicity evolution in **ANY systematic approximation**?
  - Large $N_c, N_f$? Only large $N_c$?
  - **Does the growth persist** at small $x$?

- What is the role of **saturation**?
  - Does multiple unpolarized scattering **reduce the intercept**?
  - Does saturation keep the **IR sector** from becoming **nonperturbative**?
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  - **Subleading evolution** of the polarized matrix element.
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- What about other polarization observables like transversity?
• Up to 35% of the proton angular momentum is unaccounted for.

Is there significant polarization at small $x$?

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
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$0.001 < x < 1$
• Up to 35% of the proton angular momentum is unaccounted for.

Is there significant polarization at small $x$?

- $0.001 < x < 1$
- $\Delta \Sigma \approx 0.25$ (25%)
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Ladder graphs: rapid growth of polarization with small $x$!

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• Quark / gluon splitting leads to double-logarithmic evolution
  ➡ Ladder graphs: rapid growth of polarization with small $x$!

• Massive complications due to non-ladder gluons and IR phase space.
  ➡ Much more to discover just around the corner!

$0.001 < x < 1$
$\Delta \Sigma \approx 0.25 \ (25\%)$
$\Delta G \approx 0.2 \ (40\%)$

$G_{xy}(s) \sim \left( \frac{s}{Q^2} \right)^{1.46}$