Nonextremal Black Holes, Subtracted Geometry and Holography

Mirjam Cvetič
Einstein’s theory of gravity predicts **Black Holes**

Due to its high mass density, the space-time is curved so much that objects traveling toward it reach a point of no return \( \rightarrow \) **Horizon** (& eventually reaches **space-time singularity**)

Black holes `behave’ as thermodynamic objects

\[ S = \frac{1}{4} A_{\text{horizon}} \]

\( A_{\text{horizon}} = \) area of the black hole horizon \((w/ \hbar=c=G_N=1)\)
Key Issue in Black Hole Physics:

How to relate Bekenstein-Hawking - thermodynamic entropy: \( S_{\text{thermo}} = \frac{1}{4} A_{\text{hor}} \) (\( A_{\text{hor}} = \) area of the black hole horizon; \( c=\hbar=1;G_N=1 \)) to Statistical entropy: \( S_{\text{stat}} = \log N_i \)?

Where do black hole microscopic degrees \( N_i \) come from?
Black Holes in String Theory

The role of D-branes
D(iriichlet)-branes
boundaries of open strings with charges at their ends

I. Implications for particle physics (charged excitations)-no time
II. Implications for Black Holes

Dual D-brane interpretation:

extended massive gravitational objects

D-branes in four-dimensions:

part of their world-volume on compactified space & part in internal compactified space
D-branes as gravitational objects wrap cycles in internal space: intersecting D-branes in compact dimensions & charged black holes in four dim. space-time (w/ each D-brane sourcing charge $Q_i$)

Prototype: four-charge black hole w/ $S = \pi \sqrt{Q_1 Q_2 P_3 P_4}$

M.C. & Youm 9507090

Cartoon of (toroidal) compactification

Thermodynamic BH Entropy & Statistical field theory interpretation

D-branes as a boundary of strings: microscopic degrees $N_i$ are string excitations on intersecting D-branes w/ $S = \log N_i$

Strominger & Vafa ’96

the same!
Microscopic origin of entropy for extremal (BPS), multi-charged black holes with \( M = \sum_i |Q_i| + \sum_i |P_i| \) (schematic)

M-mass, \( Q_i \)- el., \( P_i \)- magn. charges

Systematic study of microscopic degrees quantified via: AdS/CFT (Gravity/Field Theory) correspondence

[A string theory on a specific Curved Space-Time (in D-dimensions) related to specific Field Theory (in (D-1)- dimensions) on its boundary → Holographic Approach]

Maldacena’97

For multicharged (near)-BPS black holes: AdS\(_3\)/CFT\(_2\) correspondence
The rest of the talk:

Highlight recent progress on studies of Internal Structure of Non-Extremal Black Holes
Outline:

I. General asymptotically flat black holes in string theory
   [in four (&five) dimensions – prototype STU black holes]
   thermodynamics, suggestive of conformal symmetry

II. Subtracted Geometry: non-extremal black holes in
    asymptotically conical box
    manifest conformal symmetry

III. Variational Principle and Subtracted Geometry
    conserved charges and thermodynamics

IV. Holography via 2D Einstein-Maxwell-Dilaton gravity
    full holographic dictionary

V. Outlook
Background:

Initial work on subtracted geometry
M.C., Finn Larsen 1106.3341, 1112.4846, 1406.4536
M.C., Gary Gibbons 1201.0601
M.C., Monica Guica, Zain Saleem 1301.7032
...

Recent: variational principle, conserved charges and thermodynamics of subtracted geometry
Ok Song An, M.C., Ioannis Papadimitriou, 1602.0150

Most recent: subtracted geometry and AdS$_2$ holography
M.C., Ioannis Papadimitriou, 1608.07018
I. 4D general non-extremal black holes in string theory, asymptotically flat (zero cosmological constant $\Lambda=0$)

\[ M - \text{mass}, \; Q_i, \; P_i - \text{multi-charges}, \; J - \text{angular momentum} \]

\[ w/ \quad M > \Sigma_i |Q_i| + \Sigma_i |P_i| \]

Prototype solutions of a sector of \textit{maximally supersymmetric} \textit{D=4 Supergravity}
\textit{[sector of toroidally compactified effective string theory]} $\rightarrow$ so-called \textit{STU model}
STU Model Lagrangian

[A sector of toroidally compactified effective string theory]

\[ 2\kappa_4^2 \mathcal{L}_4 = R \star 1 - \frac{1}{2} \star d\eta_a \wedge d\eta_a - \frac{1}{2} e^{2\eta_a} \star d\chi^a \wedge d\chi^a \]

\[ - \frac{1}{2} e^{-\eta_0} \star F^0 \wedge F^0 - \frac{1}{2} e^{2\eta_a-\eta_0} \star (F^a + \chi^a F^0) \wedge (F^a + \chi^a F^0) \]

\[ + \frac{1}{2} C_{abc} \chi^a F^b \wedge F^c + \frac{1}{2} C_{abc} \chi^a \chi^b F^0 \wedge F^c + \frac{1}{6} C_{abc} \chi^a \chi^b \chi^c F^0 \wedge F^0 \]

(\(a=1,2,3;\) \(C_{abc}\)-anti-symmetric tensor)

w/ \(A^0\) & three gauge fields \(A^a\), the three dilatons \(\eta^a\) and the three axions \(\chi^a\).

Black holes: explicit solutions of equations of motion for the above Lagrangian w/ metric, four gauge potentials and three axio-dilatons

Prototype, four-charge rotating black hole, originally obtained via solution generating techniques

M.C., Youm 9603147
Chong, M.C., Lü, Pope 0411045

Four- SO(1,1) transfs. time-reduced Kerr BH

\[ H = \begin{pmatrix} \cosh \delta_i & \sinh \delta_i \\ \sinh \delta_i & \cosh \delta_i \end{pmatrix} \]

Full four-electric and four-magnetic charge solution only recently obtained

Chow, Compère 1310.1295;1404.2602
Compact form of the metric for rotating four-charge black holes

\[ ds^2_4 = -\Delta_0^{-1/2}G(dt + A)^2 + \Delta_0^{1/2}\left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G}\sin^2 \theta d\phi^2\right) \]

\[ X = r^2 - 2mr + a^2 = 0 \text{ outer & inner horizon} \]
\[ G = r^2 - 2mr + a^2 \cos^2 \theta , \]
\[ A = \frac{2ma \sin^2 \theta}{G} [(\Pi_c - \Pi_s)r + 2m\Pi_s] d\phi , \]
\[ \Delta_0 = \prod_{I=0}^{3}(r + 2m \sinh^2 \delta_I) + 2a^2 \cos^2 \theta [r^2 + mr \sum_{I=0}^{3} \sinh^2 \delta_I + 4m^2(\Pi_c - \Pi_s)\Pi_s] \]
\[ - 2m^2 \sum_{I<J<K} \sinh^2 \delta_I \sinh^2 \delta_J \sinh^2 \delta_K] + a^4 \cos^4 \theta . \]
\[ G_4M = \frac{1}{4}m \sum_{I=0}^{3I<J<K} \cosh 2\delta_I , \text{ Mass} \]
\[ G_4Q_I = \frac{1}{4}m \sinh 2\delta_I , (I = 0, 1, 2, 3) \text{ Four charges} \]
\[ G_4J = ma(\Pi_c - \Pi_s) , \text{ Angular momentum} \]

Special cases:
- \( \delta_I = \delta \) Kerr-Newman
- \& a = 0 \ Reisner-Nordström;
- \( \delta_I = 0 \) Kerr
- \& a = 0 \ Schwarzschild;
- \( \delta_I \rightarrow \infty \) m\rightarrow 0 \ w/m \exp(2 \delta_I)-finite extremal (BPS) black hole

Or equivalently: m, a, \( \delta_I \) (I=0,1,2,3)
Thermodynamics of outer & inner horizons suggestive of weakly interacting 2-dim. CFT w/ "left-" & "right-moving" excitations

Area of outer horizon $S_+ = S_L + S_R$

Area of inner horizon $S_- = S_L - S_R$

Surface gravity (inverse temperature) of

outer horizon $\beta_H = \frac{1}{2} (\beta_L + \beta_R)$

inner horizon $\beta_\Sigma = \frac{1}{2} (\beta_L - \beta_R)$

Similar structure for angular velocities $\Omega_+, \Omega_-$ and momenta $J_+, J_-.$

Depend only on four parameters: $m, a, \Pi_c \equiv \prod_{I=0}^{3} \cosh \delta_I, \quad \Pi_s \equiv \prod_{I=0}^{3} \sinh \delta_I$

Shown more recently, all independent of the warp factor $\Delta_0$!

M.C., Youm '96
M.C., Larsen '97

M.C., Larsen '11
II. Subtracted Geometry - Motivation

Quantify this ``conventional wisdom’’ M.C., Larsen ‘97-’99 that also non-extremal black holes might have microscopic explanation in terms of dual 2D CFT

Focus on the black hole “by itself” →
enclose the black hole in a box  (à la Gibbons Hawking) →
an equilibrium system w/ conformal symmetry manifest

The box leads to a ``mildly” modified geometry changing only the warp factor $\Delta_0 \rightarrow \Delta$
(same horizon thermodynamic quantities)

Subtracted Geometry M.C., Larsen ’11
Determination of new warp factor $\Delta_0 \rightarrow \Delta$

via massless scalar field wave eq.: wave eq. separable & the radial part is solved by hypergeometric functions w/ SL(2,R)$^2$

(manifest conformal symmetry)

The general Laplacian (with warp factor $\Delta_0$ – implicit):

$$\Delta_0^{-\frac{1}{2}} [\partial_r X \partial_r - \frac{1}{X} (A_{\text{red}} \partial_t - \partial_\phi)^2 + \frac{A_{\text{red}}^2 - \Delta_0}{G} \partial_t^2 + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2]$$

w/ $A_{\text{red}} = \frac{G}{a \sin^2 \theta} A = 2m [(\Pi_c - \Pi_s) r + 2m \Pi_s]$

$G = r^2 - 2mr + a^2 \cos^2 \theta$

$\Delta_0 \rightarrow \Delta$ such that wave eq. is separable: $f(r) + g(\theta)$ (true for $\Delta_0$ and $\Delta$)

& the radial part is solved by hypergeometric functions:

$f(r) + g(\theta)$-const. $\rightarrow$ uniquely fixes $\Delta$
Subtracted geometry for rotating four-charge black holes

\[ ds^2_4 = -\Delta_0^{-1/2} G(dt + A)^2 + \Delta_0^{1/2} \left( \frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2 \theta d\phi^2 \right) \]

\[ X = r^2 - 2mr + a^2, \]
\[ G = r^2 - 2mr + a^2 \cos^2 \theta, \]
\[ A = \frac{2ma \sin^2 \theta}{G} [(\Pi_c - \Pi_s)r + 2m\Pi_s] d\phi, \]
\[ \Delta_0 = \prod_{I=0}^{3} (r + 2m \sinh^2 \delta_I) + 2a^2 \cos^2 \theta [r^2 + mr \sum_{I=0}^{3} \sinh^2 \delta_I + 4m^2 (\Pi_c - \Pi_s) \Pi_s \]
\[ - 2m^2 \sum_{I<J<K} \sinh^2 \delta_I \sinh^2 \delta_J \sinh^2 \delta_K] + a^4 \cos^4 \theta. \]

\[ \Delta_0 \rightarrow \Delta = (2m)^3 r (\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2 \theta \]

Comments: while \( \Delta_0 \sim r^4, \Delta \sim r \) (not asymptotically flat!)
subtracted geometry depends only on four parameters:

\[ m, \ a, \ \Pi_c \equiv \prod_{I=0}^{3} \cosh \delta_I, \ \Pi_s \equiv \prod_{I=0}^{3} \sinh \delta_I \]
Matter fields (gauge potentials and scalars)

Scalars: \( \eta_1 = \eta_2 = \eta_3 \equiv \eta, \chi_1 = \chi_2 = \chi_3 \equiv \chi, \)

Running dilaton: \( e^\eta = \frac{(2m)^2}{\sqrt{\Delta}}, \quad \chi = \frac{a(\Pi_c - \Pi_s)}{2m} \cos \theta. \)

Gauge potentials: \( A^1 = A^2 = A^3 \equiv A. \)

\[
A^0 = \frac{(2m)^4 a (\Pi_c - \Pi_s)}{\Delta} \sin^2 \theta d\phi + \frac{(2ma)^2 \cos^2 \theta (\Pi_c - \Pi_s)^2 + (2m)^4 \Pi_c \Pi_s}{(\Pi_c^2 - \Pi_s^2) \Delta} dt,
\]

\[
A = \frac{2m \cos \theta}{\Delta} \left( [\Delta - (2ma)^2 (\Pi_c - \Pi_s)^2 \sin^2 \theta] d\phi - 2ma (2m \Pi_s + r(\Pi_c - \Pi_s)) dt \right),
\]

Non-extremal black hole immersed in constant magnetic field

\( w/ \ \Delta = (2m)^3 (\Pi_c^2 - \Pi_s^2) r + (2m)^4 \Pi_s^2 - (2ma)^2 (\Pi_c - \Pi_s)^2 \cos^2 \theta \)
Remarks:

Asymptotic geometry of subtracted geometry is of Lifshitz-type w/ a deficit angle:

\[ ds^2 = -\left(\frac{R}{R_0}\right)^{2p} dt^2 + B^2 dR^2 + R^2 (d\theta^2 + \sin^2 \theta^2 d\phi^2) \]

\[ p=3, \ B=4 \]

→ black hole in an "asymptotically conical box"

M.C., Gibbons 1201.0601

→ the box conformal to AdS\(_2 \times S^2\)

→ confining, but "softer" than AdS
Origin of subtracted geometry

i. Subtracted geometry – as a scaling limit of near-horizon black hole w/ three-large charges $Q$, (mapped on $m, a, \Pi_c, \Pi_s$)

$$\tilde{r} = r\epsilon, \quad \tilde{t} = t\epsilon^{-1}, \quad \tilde{m} = m\epsilon, \quad \tilde{a} = a\epsilon.$$  
$$\epsilon \to 0.$$ 

$$2\tilde{m}\sinh^2\tilde{\delta} \equiv Q = 2m\epsilon^{-1/3}(\Pi_c^2 - \Pi_s^2)^{1/3}, \quad \sinh^2\tilde{\delta}_0 = \frac{\Pi_s^2}{\Pi_c^2 - \Pi_s^2}.$$  

ii. Subtracted geometry - as an infinite boost Harrison transformations on the original BH

$$\text{SO}(1,1): \quad H \sim \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \quad \beta \to 1.$$  

iii. Subtracted geometry – as turning off certain integration constants in harmonic functions of asymptotically flat black holes

Baggio, de Boer, Jottar, Mayerson 1210.7695  
An, M.C., Papardimitriou 1602.0150

→ non-extremal black hole microscopic properties associated with its horizon are captured by a dual field theory of subtracted geometry
Lift of subtracted geometry on a circle $S^1$ to five-dimension turns out to locally factorize $AdS_3 \times S^2$ ([SL(2,R)$^2 \times SO(3)]/Z_2$ symmetry)

[globally $S^2$ fibered over Bañados-Teitelboim-Zanelli (BTZ) black hole w/ mass $M_3$, angular momentum $J_3$ & 3d cosmol. const. $\Lambda=\ell^3$]

\[
\begin{align*}
    ds_5^2 &= (ds_{S^2}^2 + ds_{BTZ}^2) \\
    ds_{S^2}^2 &= \frac{1}{4} \ell^2 \left( d\theta^2 + \sin^2 \theta d\bar{\phi}^2 \right) \\
    ds_{BTZ}^2 &= -(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2) dt_3^2 + \frac{\ell^2 r_3^2}{(r_3^2 - r_{3+}^2)(r_3^2 - r_{3-}^2)} dr_3^2 + r_3^2 (d\phi_3 + \frac{r_3 + r_{3-}}{r_3} dt_3)^2
\end{align*}
\]

\[
\begin{align*}
    \bar{\phi} &= \phi + \frac{16ma(\Pi_c - \Pi_s)}{\ell^3}(z + t) \\
    \phi_3 &= \frac{z}{R}, \\
    t_3 &= \frac{\ell}{R} t, \\
    r_3^2 &= \frac{16(2mR)^2}{\ell^4} \left[ 2m(\Pi_c^2 - \Pi_s^2)r + (2m)^2\Pi_s^2 - a^2(\Pi_c - \Pi_s)^2 \right]
\end{align*}
\]

Conformal symmetry of $AdS_3$ can be promoted to Virasoro algebra of dual two-dimensional CFT

Standard statistical entropy (via $AdS_3/CFT_2$)

$\rightarrow$ Reproduces entropy of 4D black holes à la Brown-Hennaux

à la Cardy
Subtracted geometry \[ \Delta_0 \rightarrow \Delta = A r + B \cos^2 \theta + C; \ A,B,C\text{-horrendous} \] also works for most general black holes of the STU Model (specified by mass, four electric and four magnetics charges and angular momentum)

Chow, Compère 1310.1295;1404.2602

All also works in parallel for subtracted geometry of most general five-dimensional black holes (specified by mass, three charges and two angular momenta)

M.C., Youm 9603100
Further developments

Quantum aspects of subtracted geometries:

i) **Quasi-normal modes** - exact results for scalar fields
two damped branches → no black hole bomb
M.C., Gibbons 1312.2250, M.C., Gibbons, Saleem 1401.0544

ii) **Entanglement entropy** – minimally coupled scalar
M.C., Satz, Saleem 1407.0310

iii) **Vacuum polarization** $\langle \varphi^2 \rangle$ analytic expressions
at the horizon: static M.C., Gibbons, Saleem, Satz 1411.4658
   rotating M.C., Satz, Saleem 1506.07189
outside & inside horizon: rotating M.C., Satz 1609....

iv) **Thermodynamics of subtracted geometry**
   via Komar integral: M.C., Gibbons, Saleem 1412.5996 (PRL)
   → Systematic approach via variational principle
III. Thermodynamics via variational principle
An, M.C., Papadimitriou 1602.0150

Following lessons from AdS geometries
Heningson, Skenderis’98; Balasubramanian, Kraus’99; deBoer, Verlinde’99,…
achieved through an algorithmic procedure for subtracted geometry:

• integration constants, parameterizing solutions of the eqs. of motion, separated into `normalizable` - free to vary & `non-normalizable` modes – fixed

• non-normalizable modes – fixed only up to transformations induced by local symmetries of the bulk theory (radial diffeomorphisms & gauge transf.)

• covariant boundary term, $S_{ct}$, to the bulk action - determined by solving asymptotically the radial Hamilton-Jacobi eqn. $\Rightarrow$

Skenderis, Papadimitriou’04, Papadimitriou’05

total action $S + S_{ct}$ independent of the radial coordinate

• first class constraints of Hamiltonian formal. lead to conserved charges associated with Killing vectors

• conserved charge satisfy the first law of thermodynamics
• Identify normalizable and non-renormalizable modes

Introduce new coordinates:

Rescaled radial coord.: \( \ell^4 r \leftrightarrow (2m)^3 (\Pi^2_c - \Pi^2_s) r + (2m)^4 \Pi^2_s - (2ma)^2 (\Pi_c - \Pi_s)^2, \)

Rescaled time: \( \frac{k}{\ell^3 t} \leftrightarrow \frac{1}{(2m)^3 (\Pi^2_c - \Pi^2_s)} t, \)

Trade four parameters \( m, a, \Pi_c, \Pi_s \) for:

\[
\ell^4 r_\pm = (2m)^3 m (\Pi^2_c + \Pi^2_s) - (2ma)^2 (\Pi_c - \Pi_s)^2 \pm \sqrt{m^2 - a^2 (2m)^3 (\Pi^2_c - \Pi^2_s)}
\]

\[
\ell^3 \omega = 2ma (\Pi_c - \Pi_s), \quad B = 2m,
\]

\( r_+, r_-, \omega \) - normalizable modes

B - non-renormalizable mode

(fixed up to bulk diffeomorphisms \& global gauge symmetries)
`Vacuum’ solution

obtained by turning off $r_+, r_-, \omega$ – three normalizable modes:

Asymptotically conical box – conformal to $\text{AdS}_2 \times \text{S}^2$

\[
\begin{align*}
    ds^2 &= \sqrt{r} \left( \ell^2 \frac{dr^2}{r^2} - r k^2 dt^2 + \ell^2 d\theta^2 + \ell^2 \sin^2 \theta d\phi^2 \right) \\
    e^\eta &= \frac{B^2/\ell^2}{\sqrt{r}}, \quad \chi = 0, \quad A^0 = 0, \quad A = B \cos \theta d\phi
\end{align*}
\]

Non-normalizable (fourth) mode $B$, along with $\ell$ and $k$, fixed up to radial diffeomorphism:

\[
    r \rightarrow \lambda^{-4} r \quad k \rightarrow \lambda^3 k, \quad \ell \rightarrow \lambda \ell, \quad B \rightarrow B
\]

and global $U(1)$ symmetry:

\[
    e^\eta \rightarrow \mu^2 e^\eta, \quad \chi \rightarrow \mu^{-2} \chi, \quad A^0 \rightarrow \mu^3 A^0, \quad A \rightarrow \mu A, \quad ds^2 \rightarrow ds^2
\]

which keep $kB^3/\ell^3$ - fixed
• **Radial Hamiltonian formalism**

to determine $S_{ct}$, to the bulk action $S$

Suitable radial coordinate $u$, such that constant-$u$ slices $\Sigma_u$

$$\Sigma_u \to \partial \mathcal{M} \quad \text{as} \quad u \to \infty.$$ 

Decomposition of the metric and gauge fields:

$$ds^2 = (N^2 + N_i N^i) du^2 + 2 N_i du dx^i + \gamma_{ij} dx^i dx^j$$

$$A^L = a^\Lambda du + A^\Lambda_i dx^i,$$

Decomposition leads to the radial Lagrangian $L$ w/ canonical momenta:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}}$$

$$\pi^I = \frac{\delta L}{\delta \dot{\phi}^I}$$

$$\pi^i = \frac{\delta L}{\delta \dot{A}^\Lambda_i}$$

w/ momenta conjugate to $N$, $N_i$, and $a^\Lambda$ vanish.
Hamiltonian:

\[ H = \int d^3x \left( \pi^{ij} \dot{\gamma}_{ij} + \pi_I \dot{\varphi}^I + \pi^i_\Lambda \dot{A}_i^\Lambda \right) - L = \int d^3x \left( N \mathcal{H} + N_i \mathcal{H}^i + a^\Lambda \mathcal{F}_{\Lambda} \right) \]

First class constraints \( \mathcal{H} = \mathcal{H}^i = \mathcal{F}_{\Lambda} = 0 \), - Hamilton Jacobi eqs.

& momenta as gradients of Hamilton’s principal function \( S(\gamma, A^\Lambda, \varphi^I) \):

\[
\begin{align*}
\pi^{ij} &= \frac{\delta L}{\delta \dot{\gamma}_{ij}} \\
\pi_I &= \frac{\delta L}{\delta \dot{\varphi}^I} \\
\pi_i^\Lambda &= \frac{\delta L}{\delta \dot{A}_i^\Lambda} \\
\end{align*}
\]

w/ original

\[
\begin{align*}
\pi^{ij} &= \frac{\delta S}{\delta \gamma_{ij}} \\
\pi_i^\Lambda &= \frac{\delta S}{\delta A_i^\Lambda} \\
\pi_I &= \frac{\delta S}{\delta \varphi^I} \\
\end{align*}
\]

deBoer, Verlinde’99,... Skenderis, Papadimitriou ’04,...

Solve asymptotically (for `vacuum’ asymptotic solutions) for

\[ S(\gamma, A^\Lambda, \varphi^I) = - S_{\text{ct}} \]

\( S(\gamma, A^\Lambda, \varphi^I) \) coincides with the on-shell action, up to terms that remain finite as \( \Sigma_u \rightarrow \partial \mathcal{M} \). In particular, divergent part of \( S[\gamma, A^\Lambda, \varphi^I] \) coincides with that of the on-shell action.
• Hamiltonian Formalism with ``Renormalized" Action

\[ S_{\text{reg}} = S_4 + S_{\text{ct}} \quad S_{\text{ren}} = \lim_{r \to \infty} S_{\text{reg}} \quad \text{Finite-independent of } r \]

**Covariant** \( S_{\text{ct}} \) **calculated** for vacuum asymptotic solutions (for non-flat, conformal to \( \text{AdS}_2 \times \text{S}^2 \) geometry)

\[ S_{\text{ct}} = - \frac{1}{\kappa_4^2} \int d^3x \sqrt{-\gamma} \frac{B}{4} e^{\eta/2} \left( \frac{4 - \alpha}{B^2} + (\alpha - 1)e^{-\eta}R[\gamma] - \frac{\alpha}{2}e^{-2\eta}F_{ij}F^{ij} + \frac{1}{4}e^{-4\eta}F_{ij}^0F^{0ij} \right) \]

**Renormalized canonical momenta:**

\[ \Pi^{ij} = \pi^{ij} + \frac{\delta S_{\text{ct}}}{\delta \gamma^{ij}}, \quad \Pi^i_\Lambda = \pi^i_\Lambda + \frac{\delta S_{\text{ct}}}{\delta A^\Lambda_i}, \quad \Pi_I = \pi_I + \frac{\delta S_{\text{ct}}}{\delta \varphi^I} \]
• **Conserved Charges**

Conserved currents, a consequence of the first class constraints

\[ F_\Lambda = 0 \] Conserved currents for gauge potentials: \[ D_i \Pi^i = 0, \quad D_i \Pi^0 = 0. \]

Conserved charges:

\[ Q_{4}^{(m)} = - \int_{\partial M \cap C} d^2 x \, \Pi^t, \quad Q_{4}^{0(e)} = - \int_{\partial M \cap C} d^2 x \, \Pi^{0t} \]

\[ = \frac{3B}{4G_4} \]

\[ = \frac{\ell^4}{4G_4 B^3} (\sqrt{r_+ r_-} + \omega^2 \ell^2) \]

\[ H_i = 0 \] Conserved currents:

\[ - 2D_j \Pi^j_i + \Pi \eta \partial_i \eta + \Pi \chi \partial_i \chi + F_{ij}^0 \Pi^{0j} + F_{ij} \Pi^j \approx 0 \]

Conserved ``charges``:

\[ Q[\zeta] = \int_{\partial M \cap C} d^2 x \, (2\Pi^j_j + \Pi^{0j} A^j_j + \Pi^t A^j_j) \zeta^j \]

Asymptotic Killing vector \( \xi^i \)

**Mass:** \( M_4 = - \int_{\partial M \cap C} d^2 x \, (2\Pi^t_t + \Pi^0_0 A^0_t + \Pi^t A_t) = \frac{\ell k}{8G_4} (r_+ + r_-) \)

**Angular Momentum:** \( J_4 = \int_{\partial M \cap C} d^2 x \, (2\Pi^t_\phi + \Pi^0_\phi A^0_\phi + \Pi^t A_\phi) = - \frac{\omega \ell^3}{2G_4} \)
• Thermodynamic relations and the first law

Free Energy: \( I_4 = S_{\text{ren}}^E = -S_{\text{ren}} = \beta_4 \mathcal{G}_4 = \frac{\beta_4 \ell k}{8G_4} \left( (r_- - r_+) + 2\omega^2 \ell^2 \sqrt{\frac{r_-}{r_+}} \right) \)

Quantum statistical relation: \( \mathcal{G}_4 = M_4 - T_4 S_4 - \Omega_4 J_4 - \Phi^0(e) Q^0(e) \)

First law: \( \text{d}M_4 - T_4 \text{d}S_4 - \Omega_4 \text{d}J_4 - \Phi^0(e) \text{d}Q^0(e) - \Phi^m(m) \text{d}Q^m(m) = 0. \)

Smarr’s Formula: \( M_4 = 2S_4 T_4 + 2\Omega_4 J_4 + Q^0(e) \Phi^0(e) + Q^m(m) \Phi^m(m) \)

Varying parameters: \( r_+, r_-, \omega, \) and \( B, k, \ell \) subject to \( kB^3/\ell^3 \) –fixed

original parameters \( m, a, \Pi_c, \Pi_s \) & a scaling parameter
IV. Holography via 2D Einstein-Maxwell-Dilaton

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4D STU fields can be consistently Kaluza-Klein reduced on $S^2$ by one-parameter family of Ansätze:

$$e^{-2\eta} = e^{-2\psi} + \lambda^2 B^2 \sin^2 \theta, \quad \chi = \lambda B \cos \theta$$

$$e^{-2\eta} A^0 = e^{-2\psi} A^{(2)} + \lambda B^2 \sin^2 \theta d\phi, \quad A + \chi A^0 = B \cos \theta d\phi$$

$$e^\eta ds_4^2 = ds_2^2 + B^2 \left( d\theta^2 + \frac{\sin^2 \theta}{1 + \lambda^2 B^2 \sin^2 \theta} (d\phi - \lambda A^{(2)})^2 \right)$$

$ds_2^2, \Psi, A^{(2)}$ -fields of 2D Einstein-Maxwell-Dilaton Gravity:

$$S_{2D} = \frac{1}{2\kappa_2^2} \left( \int d^2 x \sqrt{-g} e^{-\psi} \left( R[g] + \frac{2}{L^2} - \frac{1}{4} e^{-2\psi} F_{ab} F^{ab} \right) + \int dt \sqrt{-\gamma} e^{-\psi} 2K \right)$$

$B = 2L; \lambda$-independent

$$\lambda = \omega \ell^3 / B^3$$ rotational parameter of subtracted geometry
Web of Theories

Subtracted geometry

4D STU model

$S^1$ uplift

$k_5^2 = R_z k_4^2$

$R_z = 2\pi L k \left(\frac{B}{\ell}\right)^3$

$k\omega L \in \mathbb{Z}$

Locally: $\text{AdS}_3 \times S^2$

5D Einstein-Maxwell-Chern-Simons

$\omega$-twisted

$S^2$ reduction

$\kappa_3^2 = \frac{\kappa_5^2}{\pi L^2}$

KK Ansatz

2D Einstein-Maxwell-Dilaton

NCFT$_1$

$S^1$ reduction

$\kappa_2^2 = \frac{\kappa_3^2}{R_z}$

3D Einstein-Hilbert w/ specific BCs

RG

projected CFT$_2$
General solution of 2D EMD Gravity – running dilaton

Fefferman-Graham gauge: \( ds^2 = du^2 + \gamma_{tt}(u, t) dt^2, \quad A_u = 0 \)

Analytic general solution:

\[
e^{-\psi} = \beta(t)e^{u/L} \sqrt{\left(1 + \frac{m - \beta''(t)/\alpha^2(t)}{4\beta^2(t)} \right) L^2 e^{-2u/L} \right)^2 - \frac{Q^2 L^2}{4\beta^4(t)} e^{-4u/L}
\]

\[
\sqrt{-\gamma} = \frac{\alpha(t)}{\beta'(t)} \partial_t e^{-\psi}
\]

\[
A_t = \mu(t) + \frac{\alpha(t)}{2\beta'(t)} \partial_t \log \left( \frac{4L^{-2}e^{2u/L} \beta^2(t) + m - \beta''(t)/\alpha^2(t) - 2Q/L}{4L^{-2}e^{2u/L} \beta^2(t) + m - \beta''(t)/\alpha^2(t) + 2Q/L} \right)
\]

Leading asymptotic behavior:

\[
\gamma_{tt} = -\alpha^2(t)e^{2u/L} + O(1), \quad e^{-\psi} \sim \beta(t)e^{u/L} + O(e^{-u/L}), \quad A_t = \mu(t) + O(e^{-2u/L})
\]

running dilaton

- Arbitrary functions \( \alpha(t), \beta(t) \) and \( \mu(t) \) identified with the sources of the corresponding dual operators
- 4D uplift results in asymptotically conformally AdS\(_2\)xS\(_2\) subtracted geometries, generalized to include arbitrary time-dependent sources
Repeat Radial Hamiltonian Formalism in 2D

Radial ADM decomposition:
\[ ds^2 = (N^2 + N_t N^t) du^2 + 2 N_t du dt + \gamma_{tt} dt^2 \]

Counterterm Action:
\[ S_{ct} = -\frac{1}{\kappa_2^2} \int dt \sqrt{-\gamma} \ L^{-1} (1 - u_o L \Box_t) e^{-\psi} \]

Renormalized one-point functions:
\[ \mathcal{T} = 2 \hat{\pi}_t^t, \quad \mathcal{O}_\psi = -\hat{\pi}_\psi, \quad \mathcal{J}^t = -\hat{\pi}^t \]

\[ \hat{\pi}_t^t = \frac{1}{2\kappa_2^2} \lim_{u \to \infty} e^{u/L} \left( \partial_u e^{-\psi} - e^{-\psi} L^{-1} \right) \]

\[ \hat{\pi}^t = \lim_{u \to \infty} \frac{e^{u/L}}{\sqrt{-\gamma}} \pi^t \]

\[ \hat{\pi}_\psi = -\frac{1}{\kappa_2^2} \lim_{u \to \infty} e^{u/L} e^{-\psi} (K - L^{-1}) \]
Explicit one-point functions:

$$\mathcal{T} = -\frac{L}{2\kappa^2} \left( \frac{m}{\beta} - \frac{\beta''}{\beta\alpha^2} \right), \quad \mathcal{J}^t = \frac{1}{\kappa^2} \frac{Q}{\alpha}, \quad O_\psi = \frac{L}{2\kappa^2} \left( \frac{m}{\beta} - \frac{\beta''}{\beta\alpha^2} - 2\frac{\beta'\alpha'}{\alpha^3} + 2\frac{\beta''}{\alpha^2} \right)$$

Ward Identities:

$$\partial_t \mathcal{T} - O_\psi \partial_t \log \beta = 0, \quad \mathcal{D}_t \mathcal{J}^t = 0$$

Conformal anomaly:

$$\mathcal{T} + O_\psi = \frac{L}{\kappa^2} \left( \frac{\beta''}{\alpha^2} - \frac{\beta'\alpha'}{\alpha^3} \right) = \frac{L}{\kappa^2} \partial_t \left( \frac{\beta'}{\alpha} \right) \equiv A$$

Exact generating function ( $$\mathcal{T} = \frac{\delta S_{\text{ren}}}{\delta \alpha}$$, $$O_\psi = \frac{\beta}{\alpha} \frac{\delta S_{\text{ren}}}{\delta \beta}$$, $$\mathcal{J}^t = -\frac{1}{\alpha} \frac{\delta S_{\text{ren}}}{\delta \mu}$$):

$$S_{\text{ren}}[\alpha, \beta, \mu] = -\frac{L}{2\kappa^2} \int dt \left( \frac{m\alpha}{\beta} + \frac{\beta'^2}{\beta\alpha} + \frac{2\mu Q}{L} \right)$$

Legandre transformed generating function ( $$\text{w/ } \alpha(t) = \beta(t)$$):

$$\Gamma_{\text{eff}} = S_{\text{ren}} + \int dt \alpha \left( \mathcal{T} + O_\psi \right) = \frac{L}{\kappa^2} \int dt \left( \{\tau, t\} - \mu Q/L - m \right)$$

$$\{\tau, t\} = \frac{\tau'''}{\tau'} - \frac{3}{2} \frac{\tau''^2}{\tau'}$$

Schwarzian derivative

$$-\alpha^2(t) dt^2 = -\left( d\tau(t) \right)^2$$

c.f., Sadchev, Ye, Kitaev '93,… Almeheiri, Polochinski '14; Maldacena, Stanford, Yang '16; Engelsoy, Merens, Verlinde '16,…
Asymptotic symmetries and conserved charges

Asymptotic symmetries: subset of Penrose-Brown-Henneaux (PBH) transformations (diffeomorphisms and gauge transformations preserving the Fefferman-Graham gauge) that preserve boundary conditions:

\[ \delta_{\text{PBH}} \alpha = \partial_t (\varepsilon \alpha) + \alpha \sigma / L, \quad \delta_{\text{PBH}} \beta = \varepsilon \beta' + \beta \sigma / L, \quad \delta_{\text{PBH}} \mu = \partial_t (\varepsilon \mu + \varphi) \]

\[ \delta_{\text{PBH}} \text{(sources)} = 0 \rightarrow \text{constrain functions } \varepsilon(t), \sigma(t) \text{ and } \varphi(t) \text{ in term of two constants } \xi_{1,2} \]

Conserved Charges: boundary terms obtained by varying the action with respect to the asymptotic symmetries (and Ward identities) \( \rightarrow \)

\[ \text{U}(1) \times \text{U}(1): \quad Q_1 = - \left( \beta T - \frac{L}{2\kappa^2} \frac{\beta'^2}{\alpha^2} \right) = \frac{mL}{2\kappa^2}, \quad Q_2 = \alpha J^t = \frac{Q}{\kappa^2}. \]

3D perspective: two copies of the Virasoro algebra with the Brown-Henneaux central charge. Only \( L^\pm_0 \) are realized non-trivially in 2D.
Holography depends on the structure of non-extremal constant dilaton solutions and choice of boundary conditions

Provided systematic holographic dictionary for each choice

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Note: non-extremal running dilaton solution

extremal running-dilaton solution with RG flow to IR fixed point

extremal constant dilaton solution

non-extremal constant dilaton branch (`Coulomb phase’)

(Q=mL/2)

VEV of irrelevant scalar op.

does not lift into subtracted geometry)
Summary/Outlook with focus on AdS\(_2\) Holography

- Provided consistent KK Ansätze that allow us to uplift any solution of 2D EMD gravity to 4D STU solutions, which are non-extremal 4D black holes, asymptotically (conformally) AdS\(_2\times\)S\(^2\) – subtracted geometry. [Works also for 5D solutions asymptotically (conformally) AdS\(_2\times\)S\(^3\).]

- 2D EMD gravity has a well defined UV fixed point, described by a sector of 2D CFT.

- Constructed complete holographic dictionary of 2D EMD gravity theory obtained by an S\(^2\) reduction of 4D STU subtracted geometry & constant dilaton solutions.

- Many aspects of the holographic description are generic and should apply to generic 2D dilaton gravity theories.