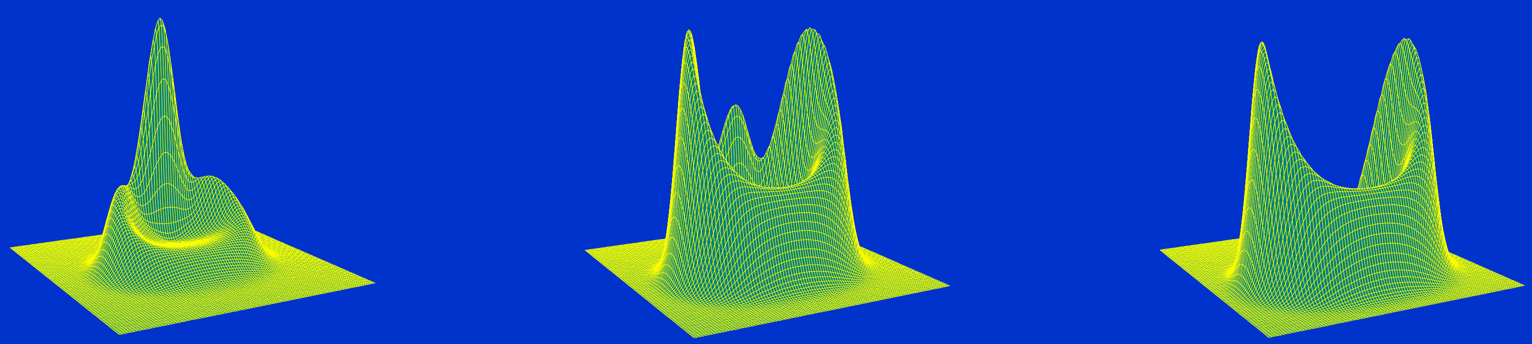
Non-classical Light and Glauber's Theory of Optical Coherence

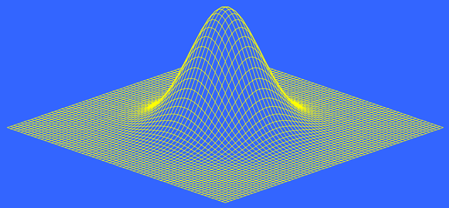
Howard Carmichael

University of Auckland

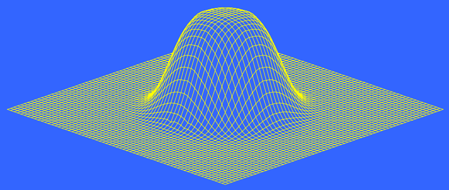


Support by the Marsden Fund of RSNZ

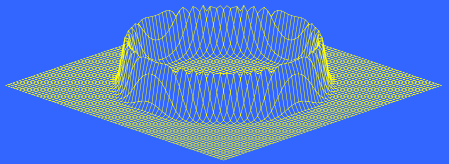
- Glauber and optical coherence:
  - background
  - quantum noise in the laser
  - classical and non-classical fields



- Historical perspective:
  - Einstein: the photoelectric effect
  - Einstein: particles and waves
  - BKS

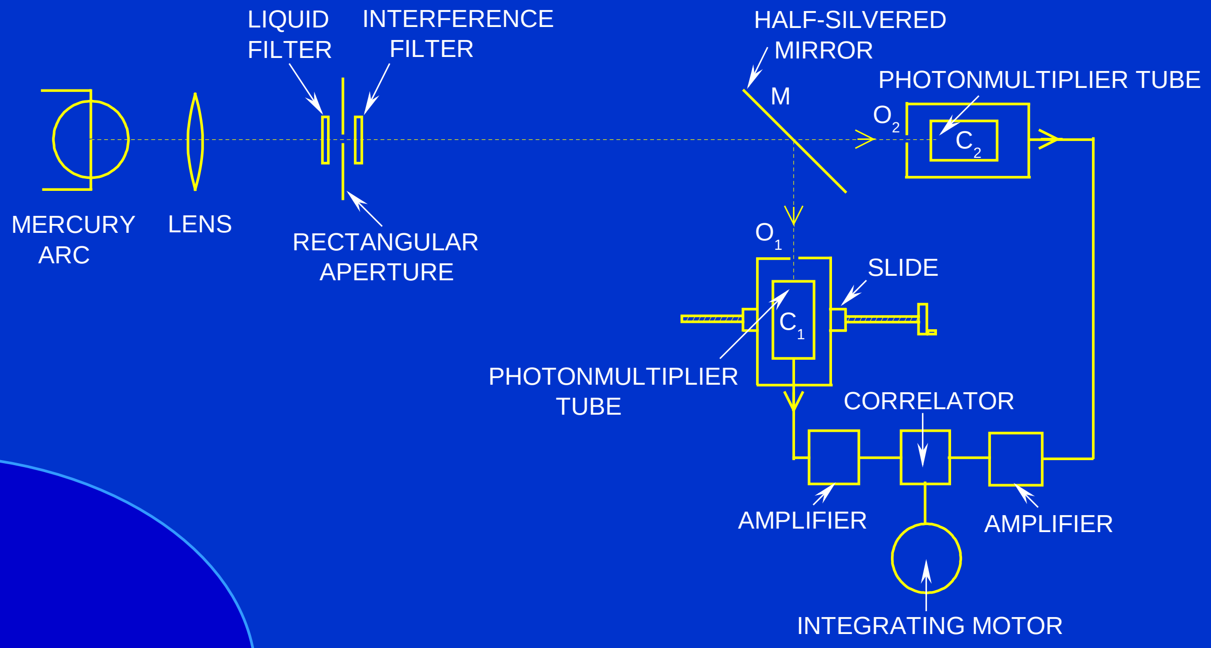


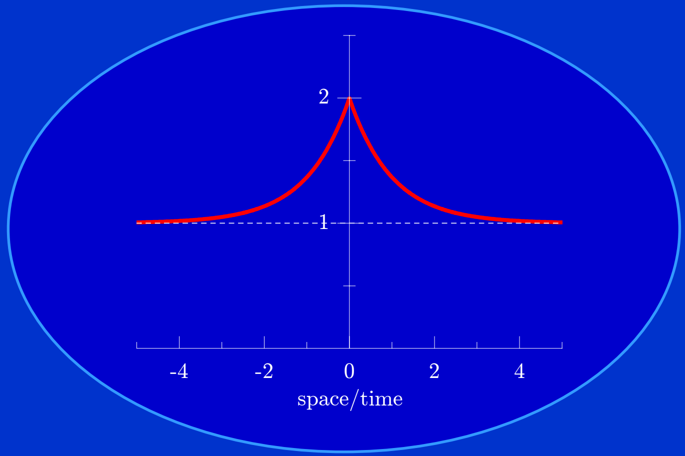
- Non-classical light:
  - “non-classical” quantum noise
  - cavity QED
  - some recent results



Hanbury Brown and Twiss light intensity interferometer

R. Hanbury Brown and R.Q. Twiss, Nature 177, 27 (1956)
R.Q. Twiss, A.G. Little, and R. Hanbury Brown, Nature 180, 324 (1957)

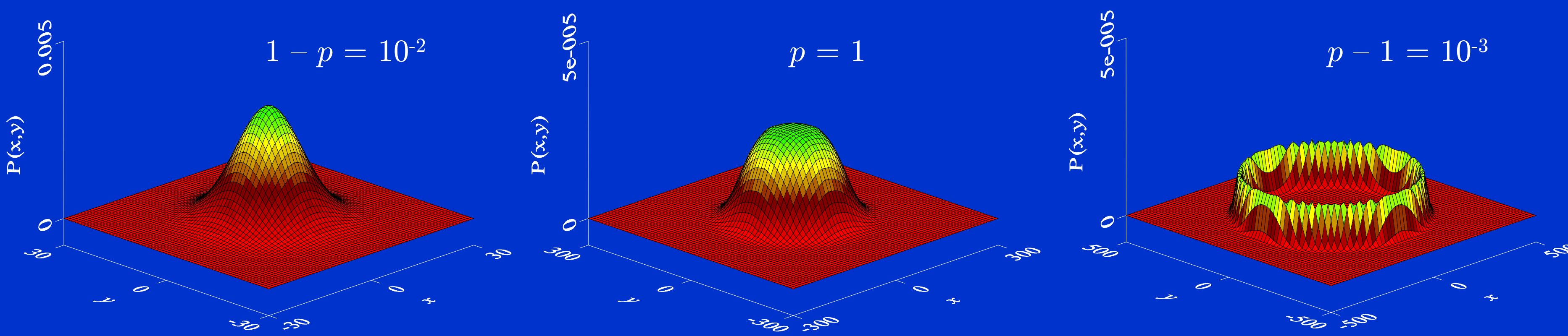




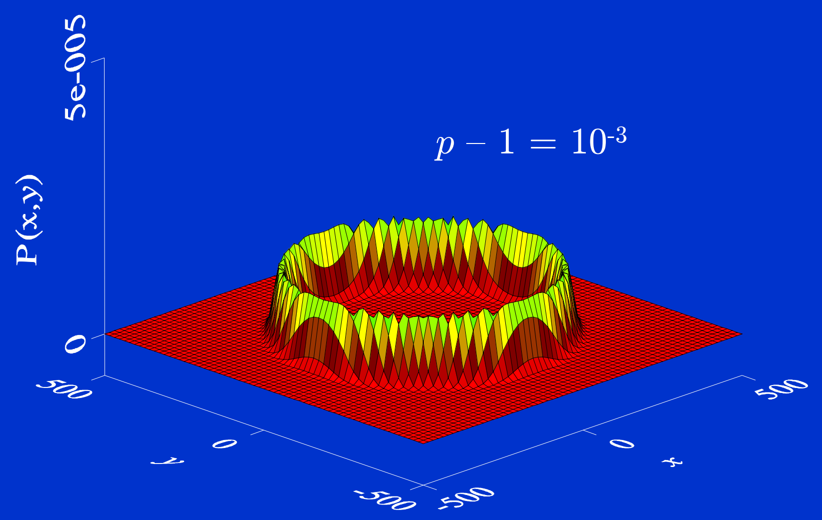
photon bunching

quantum noise in the laser

$$d\alpha = -\alpha\,(1-\overset{\text{pump parameter}}{\downarrow\atop p} + p\,|\alpha|^2)\,dt + \underbrace{\sqrt{\frac{n_{\text{spon}}}{n_{\text{sat}}}}\,(dW_1+idW_2)}_{\text{quantum noise}}$$

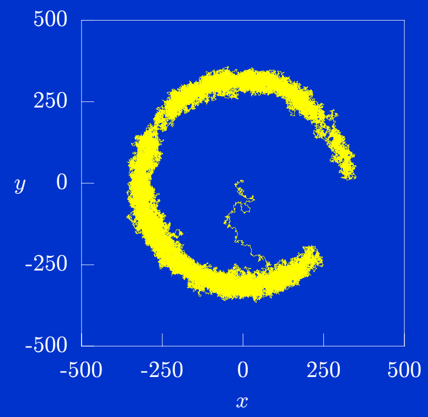


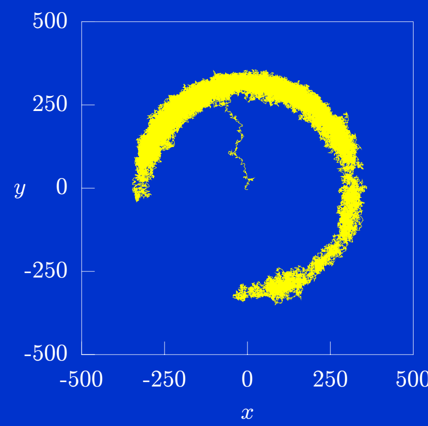
first-order coherence - laser linewidth:

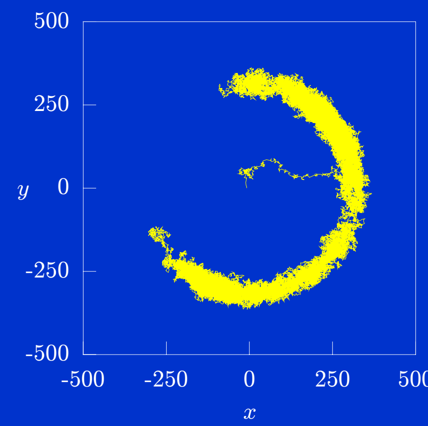


$time = 10^6\kappa^{-1}$:

$$\frac{\Delta\omega}{\kappa} = \frac{n_{\mathrm{spon}}}{2n_{\mathrm{sat}}(p-1)}$$

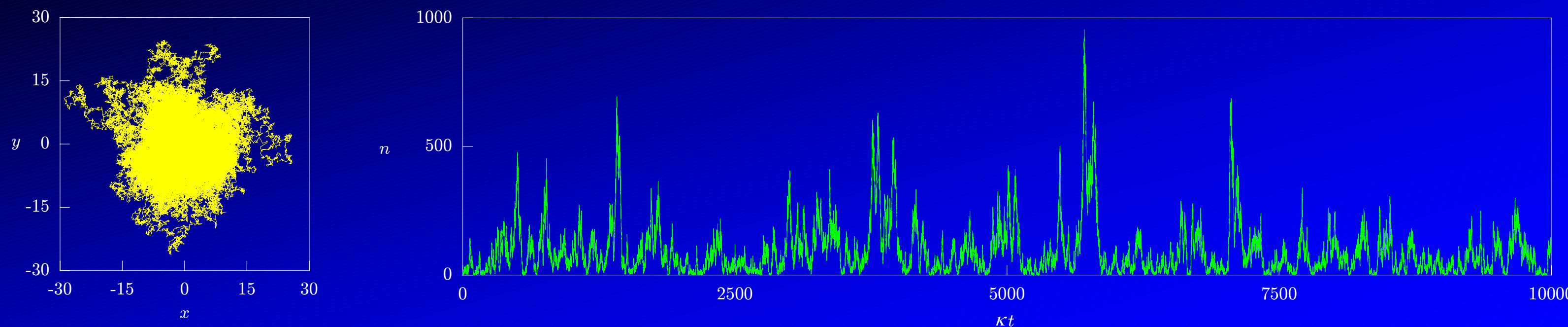




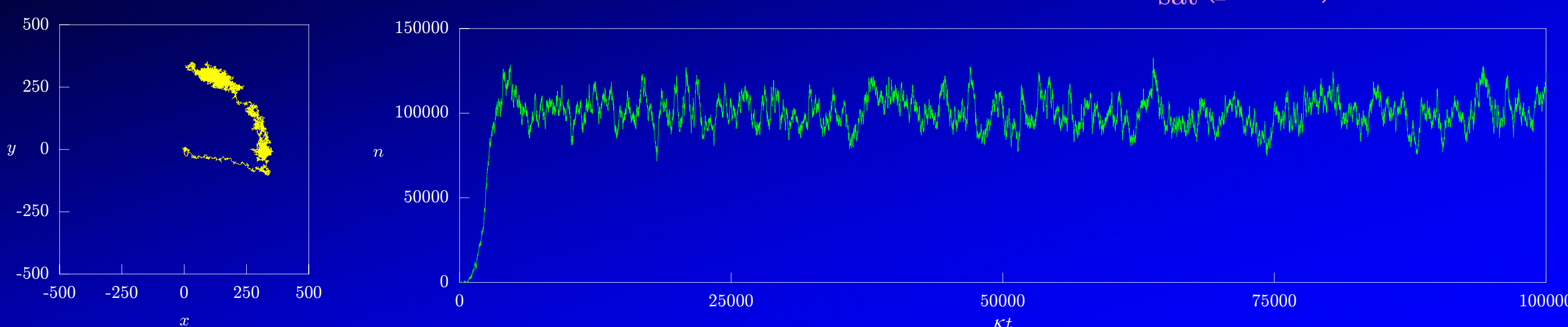


second-order coherence:

$$g^{(2)}(\tau) = 1.0 + e^{-2\kappa(1-p)|\tau|}$$



$$g^{(2)}(\tau\ ) = 1.0 + \frac{n_{\mathrm{spon}}}{n_{\mathrm{sat}}(p-1)^2}\, e^{-2\kappa(p-1)|\tau|}$$



Glauber coherence theory:

— based upon an analysis of photoelectric detection

— considers multi-photon coincidence rates
and the associated optical field correlation functions

e.g. two-photon coincidence rate is proportional to

$$G^{(2)}(r_1,t_1;r_2,t_2) = \langle \hat{E}^{\dagger}(r_1,t_1)\hat{E}^{\dagger}(r_2,t_2)\hat{E}(r_2,t_2)\hat{E}(r_1,t_1)\rangle$$

— coherence defined through the factorization
of the correlation functions

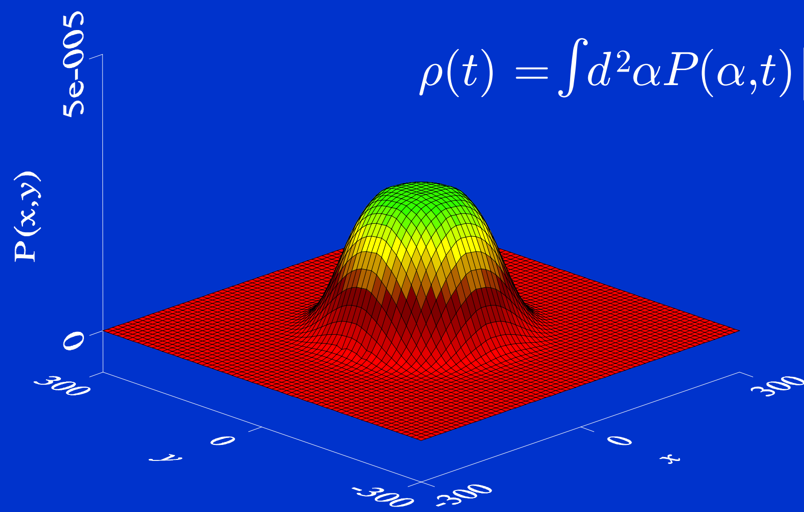
e.g. first-order coherence

$$\langle \hat{E}^{\dagger}(r_1,t_1)\hat{E}(r_2,t_2)\rangle = \mathcal{E}^*(r_1,t_1)\mathcal{E}(r_2,t_2)$$

coherent state: $\hat{E}(r,t)|\mathrm{state}\rangle = \mathcal{E}(r,t)|\mathrm{state}\rangle$

classical and non-classical fields

$$\rho(t) = \int d^2\alpha P(\alpha,t) |\alpha(t)\rangle\langle\alpha(t)|$$

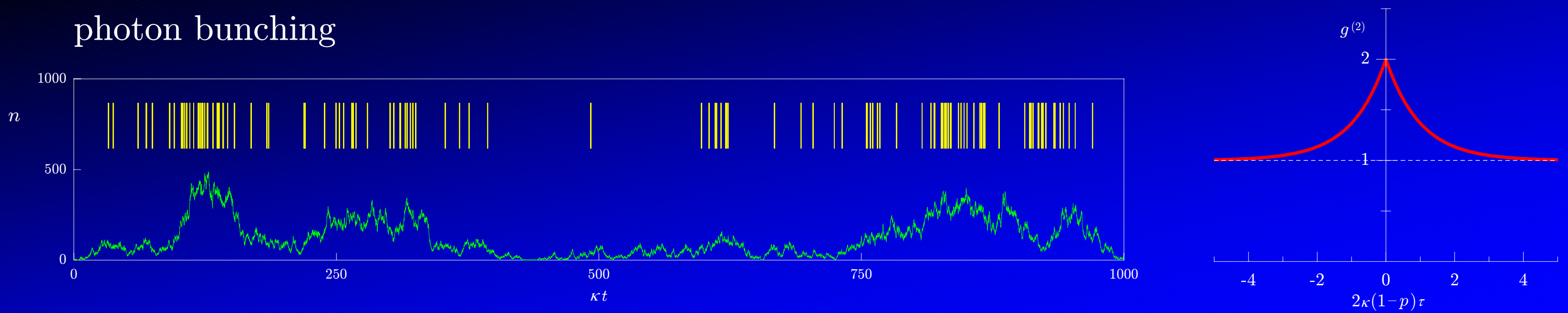


quantum average

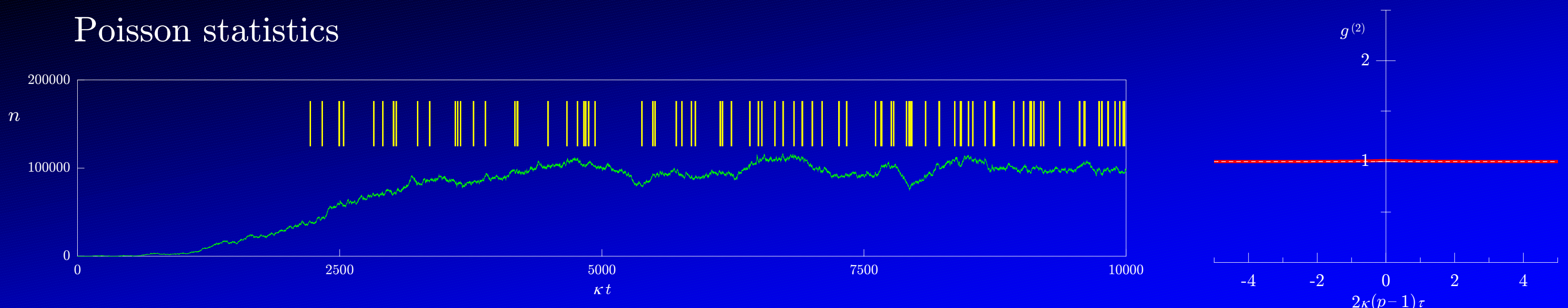
time average

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle}{\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle^2} = \frac{\langle\, |\alpha(t)|^2 |\alpha(t+\tau)|^2\rangle}{\langle\, |\alpha(t)|^2\rangle^2}$$

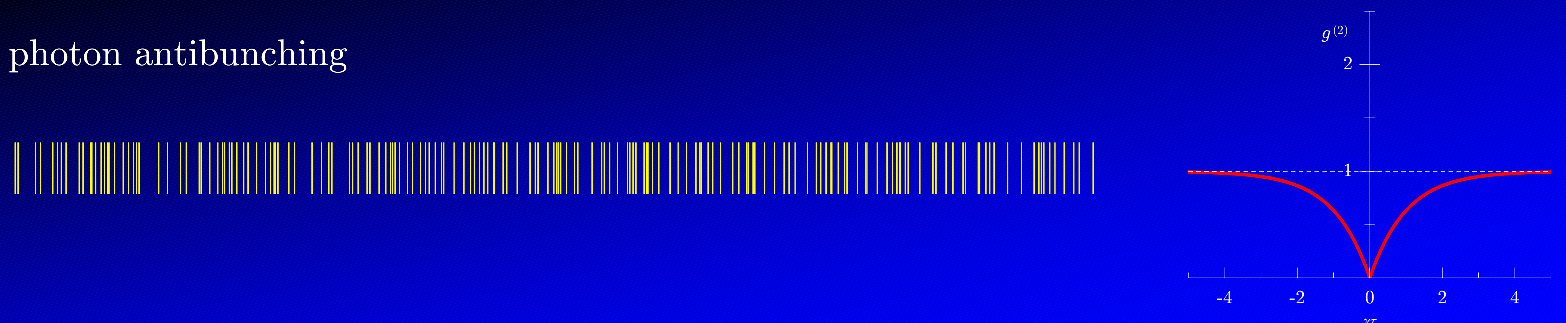
photon bunching



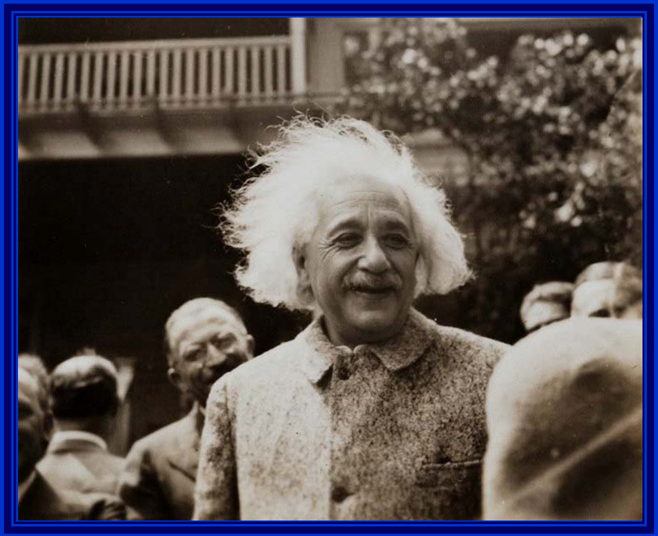
Poisson statistics



photon antibunching



Einstein: the photoelectric effect



"I have thought a hundred times as much about the quantum problems as I have about general relativity."

– recollection of Otto Stern
Jost, R., 1977, letter to A. Pais, August 17

On a Heuristic Point of View about the Creation and Conversion of Light†

A. Einstein

The wave theory of light which operates with continuous functions in space has been excellently justified for the representation of purely optical phenomena and it is unlikely ever to be replace by another theory. One should, however, bear in mind that optical observations refer to time averages and not to instantaneous values and notwithstanding the complete experimental verification of the theory of diffraction, reflexion, refraction, dispersion, and so on, it is quite conceivable that a theory of light involving the use of continuous functions in space will lead to contradictions with experience, if it is applied to the phenomena of the creation and conversion of light.

† *Ann. Physik* **17**, 132 (1905).

In fact, it seems to me that the observations on the "black-body radiation", photoluminescence, the production of cathode rays by ultraviolet light and other phenomena involving the emission or conversion of light can be better understood on the assumption that the energy of light is distributed discontinuously in space. According to the assumption considered here, when a light ray starting from a point is propagated, the energy is not continuously distributed over an ever increasing volume, but it consists of a finite number of energy quanta, localised in space, which move without being divided and which can be absorbed or emitted only as a whole.

In the following, I shall communicate the train of thought and the facts which lead me to this conclusion, in the hope that the point of view to be given may turn out to be useful for some research workers in their investigations.

1. On a Difficulty in the Theory of "Black-body Radiation"

To begin with, we take the point of view of Maxwell's theory and electron theory and consider the following case……

1. On a Difficulty in the Theory of "Black-body Radiation"
2. On Planck's Determination of Elementary Quanta
3. On the Entropy of the Radiation
4. Limiting Law for the Entropy of Monochromatic Radiation for Low Radiation Density
5. Molecular–Theoretical Investigation of the Volume-dependence of the Entropy of Gases and Dilute Solutions
6. Interpretation of the Volume-dependence of the Entropy of Monochromatic Radiation according to Boltzmann's Principle
7. On Stokes' Rule
8. On the Production of Cathode Rays by the Illumination of Solids
9. On the Ionisation of Gases by Ultraviolet Light

the electron is then $h\nu - \phi$ and this must be the kinetic energy of the emerging electron,[4]

$$\boxed{K = h\nu - \phi} \qquad (12)$$

Einstein's photoelectric equation

This is Einstein's photoelectric equation. It shows that the kinetic energy is indeed a linearly increasing function of the frequency, in agreement with the data of Figure 40.8. According to Eq. (12), the slope of the straight line in Figure 40.8 should equal Planck's constant.

Einstein's photoelectric equation was verified in detail by a long series of meticulous experiments by R. A. Millikan (the data in Figure 40.8 are due to him). In order to obtain reliable results, Millikan found it necessary to take extreme precautions to avoid contamination of the surface of the photosensitive electrode. Since the surfaces of metals exposed to air quickly accumulate a layer of oxide, he developed a technique for shaving the surfaces of his metals in a vacuum by means of a magnetically operated knife.

The results of these experiments gave strong support to the quantum theory of light. This success of Einstein's theory was all the more striking in view of the failure of the classical wave theory of light to account for the features of the photoelectric effect. According to the wave theory, the crucial parameter that determines the ejection of a photoelectron should be the intensity of light. If an intense electromagnetic wave strikes an electron, it should be able to jolt it loose from the metal, regardless of the frequency of the wave. Furthermore, the kinetic energy of the ejected electron should be a function of the intensity of the wave. The observational evidence contradicts these predictions of the wave theory: A wave with a frequency below the threshold frequency never ejects an electron, regardless of its intensity. And, furthermore, the kinetic energy depends on the frequency, and not on the intensity. High-intensity light ejects more photoelectrons, but does not give the individual electrons more kinetic energy.

Today, the photoelectric effect finds many practical applications in sensitive electronic devices for the detection of light. For instance, in a photomultiplier tube, an incident photon ejects an electron from an electrode; this electron is accelerated toward a second electrode (called a dynode; see Figure 40.9) where its impact ejects several secondary electrons; these, in turn, are accelerated toward a third electrode where their impact ejects tertiary electrons, etc. Thus, one electron from the first electrode generates an avalanche of electrons. In a high-gain photomultiplier tube, a pulse of 10^9 electrons emerges from the last electrode, delivering a measurable pulse of current to an external circuit. In this way, the photomultiplier tube can detect the arrival of individual photons. Some sensitive television cameras, such as the image orthicon, rely on the same multiplier principle to convert the arrival of a photon at a photosensitive faceplate into a measurable pulse of current.

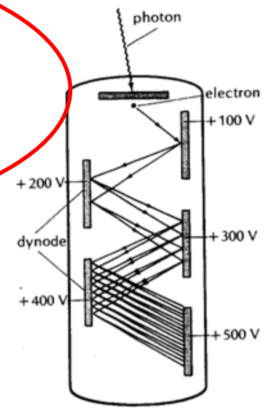


Fig. 40.9 Schematic diagram of a photomultiplier tube. The secondary electrodes are called dynodes. For the purpose of this diagram it has been assumed that each electron impact on a dynode releases two electrons. The arrows show an avalanche of electrons.

EXAMPLE 4. The work function for platinum is 9.9×10^{-19} J. What is the threshold frequency for the ejection of photoelectrons from platinum?

This success
of Einstein's theory
was all the more striking
in view of the failure
of the classical wave theory of light
to account for the features
of the photoelectric effect.
According to the wave theory,
the crucial parameter
that determines the
ejection of a photoelectron
should be the intensity of light.
If an intense electromagnetic wave
strikes an electron,
it should be able to jolt it
loose from the metal,
regardless of the
frequency of the wave.
Furthermore, the kinetic energy
of the ejected electron
should be a function of
the intensity of the wave.

encountered a time-dependent potential of this kind in Section 5.5 in discussing t-dependent two-level problems.

Again assume that only one of the eigenstates of H_0 is populated initially. Perturbation (5.6.39) is assumed to be turned on at $t = 0$, so

$$c_n^{(1)} = \frac{-i}{\hbar}\int_0^t \left(\mathcal{V}_{ni} e^{i\omega t'} + \mathcal{V}_{ni}^{\dagger} e^{-i\omega t'}\right) e^{i\omega_{ni} t'}\,dt'$$

$$= \frac{1}{\hbar}\left[\frac{1-e^{i(\omega+\omega_{ni})t}}{\omega+\omega_{ni}}\mathcal{V}_{ni} + \frac{1-e^{i(\omega_{ni}-\omega)t}}{-\omega+\omega_{ni}}\mathcal{V}_{ni}^{\dagger}\right] \tag{5.6.40}$$

where $\mathcal{V}_{ni}^{\dagger}$ actually stands for $(\mathcal{V}^{\dagger})_{ni}$. We see that this formula is similar to the constant perturbation case. The only change needed is

$$\omega_{ni} = \frac{E_n - E_i}{\hbar} \to \omega_{ni} \pm \omega. \tag{5.6.41}$$

So as $t \to \infty$, $|c_n^{(1)}|^2$ is appreciable only if

$$\omega_{ni} + \omega \simeq 0 \quad \text{or} \quad E_n \simeq E_i - \hbar\omega \tag{5.6.42a}$$

$$\omega_{ni} - \omega \simeq 0 \quad \text{or} \quad E_n \simeq E_i + \hbar\omega. \tag{5.6.42b}$$

Clearly, whenever the first term is important because of (5.6.42a), the second term is unimportant, and vice versa. We see that we have no energy-conservation condition satisfied by the quantum-mechanical system alone; rather the apparent lack of energy conservation is compensated by the energy given out to—or energy taken away from—the "external" potential $V(t)$. Pictorially, we have Figure 5.9. In the first case (*stimulated emission*), the quantum-mechanical system gives up energy $\hbar\omega$ to V; this is clearly possible only if the initial state is excited. In the second case (*absorption*), the quantum-mechanical system receives energy $\hbar\omega$ from V and ends up as an excited state. Thus a time-dependent perturbation can be regarded as an inexhaustible source or sink of energy.

In complete analogy with (5.6.34), we have

$$\begin{aligned} w_{i\to[n]} &= \frac{2\pi}{\hbar}\overline{|\mathcal{V}_{ni}|^2}\rho(E_n)\Big|_{E_n \cong E_i - \hbar\omega} \\ w_{i\to[n]} &= \frac{2\pi}{\hbar}\overline{|\mathcal{V}_{ni}^{\dagger}|^2}\rho(E_n)\Big|_{E_n \cong E_i + \hbar\omega} \end{aligned} \tag{5.6.43}$$

or, more commonly,

$$w_{i\to n} = \frac{2\pi}{\hbar}\left\{\begin{matrix}|\mathcal{V}_{ni}|^2 \\ |\mathcal{V}_{ni}^{\dagger}|^2\end{matrix}\right\}\delta(E_n - E_i \pm \hbar\omega). \tag{5.6.44}$$

Note also that

$$|\mathcal{V}_{ni}|^2 = |\mathcal{V}_{in}^{\dagger}|^2, \tag{5.6.45}$$

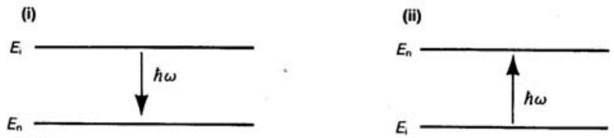


FIGURE 5.9. (i) Stimulated emission: Quantum-mechanical system gives up $\hbar\omega$ to V (possible only if initial state is excited). (ii) Absorption: Quantum mechanical system receives $\hbar\omega$ from V and ends up as an excited state.

which is a consequence of

$$\langle i|\mathcal{V}^{\dagger}|n\rangle = \langle n|\mathcal{V}|i\rangle^* \tag{5.6.46}$$

(remember $\mathcal{V}^{\dagger}|n\rangle \overset{\text{DC}}{\leftrightarrow} \langle n|\mathcal{V}$). Combining (5.6.43) and (5.6.45), we have

$$\frac{\text{emission rate for } i \to [n]}{\text{density of final states for } [n]} = \frac{\text{absorption rate for } n \to [i]}{\text{density of final states for } [i]}, \tag{5.6.47}$$

where in the absorption case we let i stand for final states. Equation (5.6.47), which expresses symmetry between emission and absorption, is known as **detailed balancing.**

To summarize, for constant perturbation, we obtain appreciable transition probability for $|i\rangle \to |n\rangle$ only if $E_n \simeq E_i$. In contrast, for harmonic perturbation we have appreciable transition probability only if $E_n \simeq E_i - \hbar\omega$ (stimulated emission) or $E_n \simeq E_i + \hbar\omega$ (absorption).

5.7. APPLICATIONS TO INTERACTIONS WITH THE CLASSICAL RADIATION FIELD

Absorption and Stimulated Emission

We apply the formalism of time-dependent perturbation theory to the interactions of atomic electron with the classical radiation field. By a **classical radiation field** we mean the electric or magnetic field derivable from a classical (as opposed to quantized) radiation field.

The basic Hamiltonian, with $|\mathbf{A}|^2$ omitted, is

$$H = \frac{\mathbf{p}^2}{2m_e} + e\phi(\mathbf{x}) - \frac{e}{m_e c}\mathbf{A}\cdot\mathbf{p}, \tag{5.7.1}$$

which is justified if

$$\nabla\cdot\mathbf{A} = 0; \tag{5.7.2}$$

that of x^2, of order R^2_{atom}, and so on, we see that the approximation of replacing (5.7.15) by its leading term is an excellent one.

Now we have

$$\langle n|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle \to \hat{\boldsymbol{\varepsilon}}\cdot\langle n|\mathbf{p}|i\rangle. \tag{5.7.19}$$

In particular, we take $\hat{\boldsymbol{\varepsilon}}$ along the x-axis (and $\hat{\mathbf{n}}$ along the z-axis). We must calculate $\langle n|p_x|i\rangle$. Using

$$[x, H_0] = \frac{i\hbar p_x}{m}, \tag{5.7.20}$$

we have

$$\begin{aligned}\langle n|p_x|i\rangle &= \frac{m}{i\hbar}\langle n|[x, H_0]|i\rangle \\ &= im\omega_{ni}\langle n|x|i\rangle.\end{aligned} \tag{5.7.21}$$

Because of the approximation of the dipole operator, this approximation scheme is called the **electric dipole approximation**. We may here recall [see (3.10.39)] the selection rule for the dipole matrix element. Since $\mathbf{x}$ is a spherical tensor of rank 1 with $q = \pm 1$, we must have $m' - m = \pm 1$, $|j' - j| = 0,1$ (no $0 \to 0$ transition). If $\hat{\boldsymbol{\varepsilon}}$ is along the y-axis, the same selection rule applies. On the other hand, if $\hat{\boldsymbol{\varepsilon}}$ is in the z-direction, $q = 0$; hence, $m' = m$.

With the electric dipole approximation, the absorption cross section (5.7.14) now takes a simpler form upon using (5.7.19) and (5.7.21) as

$$\sigma_{abs} = 4\pi^2\alpha\omega_{ni}|\langle n|x|i\rangle|^2\delta(\omega - \omega_{ni}). \tag{5.7.22}$$

In other words, σ_{abs} treated as a function of ω exhibits a sharp δ-function-like peak whenever $\hbar\omega$ corresponds to the energy-level spacing at $\omega \simeq (E_n - E_i)/\hbar$. Suppose $|i\rangle$ is the ground state, then ω_{ni} is necessarily positive; integrating (5.7.22), we get

$$\int \sigma_{abs}(\omega)\, d\omega = \sum_n 4\pi^2\alpha\omega_{ni}|\langle n|x|i\rangle|^2. \tag{5.7.23}$$

In atomic physics we define **oscillator strength**, f_{ni}, as

$$f_{ni} \equiv \frac{2m\omega_{ni}}{\hbar}|\langle n|x|i\rangle|^2. \tag{5.7.24}$$

It is then straightforward (consider $[x,[x, H_0]]$) to establish the **Thomas-Reiche-Kuhn sum rule,**

$$\sum_n f_{ni} = 1. \tag{5.7.25}$$

In terms of the integration over the absorption cross section, we have

$$\int \sigma_{abs}(\omega)\, d\omega = \frac{4\pi^2\alpha\hbar}{2m_e} = 2\pi^2 c\left(\frac{e^2}{m_e c^2}\right). \tag{5.7.26}$$

Notice how $\hbar$ has disappeared. Indeed, this is just the oscillation sum rule already known in classical electrodynamics (Jackson 1975, for instance). Historically, this was one of the first examples of how "new quantum mechanics" led to the correct classical result. This sum rule is quite remarkable because we did not specify in detail the form of the Hamiltonian.

Photoelectric Effect

We now consider the **photoelectric effect**—that is, the ejection of an electron when an atom is placed in the radiation field. The basic process is considered to be the transition from an atomic (bound) state to a continuum state $E > 0$. Therefore, $|i\rangle$ is the ket for an atomic state, while $|n\rangle$ is the ket for a continuum state, which can be taken to be a plane-wave state $|\mathbf{k}_f\rangle$, an approximation that is valid if the final electron is not too slow. Our earlier formula for $\sigma_{abs}(\omega)$ can still be used, except that we must now integrate $\delta(\omega_{ni} - \omega)$ together with the density of final states $\rho(E_n)$.

Our basic task is to calculate the number of final states per unit energy interval. As we will see in a moment, this is an example where the matrix element depends not only on the final state energy but also on the momentum *direction*. We must therefore consider a group of final states with both similar momentum directions and similar energies.

To count the number of states it is convenient to use the box normalization convention for plane-wave states. We consider a plane-wave state normalized if when we integrate the square modulus of its wave function for a cubic box of side L, we obtain unity. Furthermore, the state is assumed to satisfy the periodic boundary condition with periodicity of the side of the box. The wave function must then be of form

$$\langle \mathbf{x}|\mathbf{k}_f\rangle = \frac{e^{i\mathbf{k}_f\cdot\mathbf{x}}}{L^{3/2}}, \tag{5.7.27}$$

where the allowed values of k_x must satisfy

$$k_x = \frac{2\pi n_x}{L}, \ldots, \tag{5.7.28}$$

with n_x a positive or negative integer. Similar restrictions hold for k_y and k_z. Notice that as $L \to \infty$, k_x, k_y, and k_z become continuous variables.

The problem of counting the number of states is reduced to that of counting the number of dots in three-dimensional lattice space. We define n such that

$$n^2 = n_x^2 + n_y^2 + n_z^2. \tag{5.7.29}$$

specifically, we work with a monochromatic field of the plane wave for

$$\mathbf{A} = 2A_0 \hat{\boldsymbol{\varepsilon}} \cos\left(\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t\right) \tag{5.7.3}$$

where $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\mathbf{n}}$ are the (linear) polarization and propagation direction. Equation (5.7.3) obviously satisfies (5.7.2) because $\hat{\boldsymbol{\varepsilon}}$ is perpendicular to the propagation direction $\hat{\mathbf{n}}$. We write

$$\cos\left(\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t\right) = \frac{1}{2}\left[e^{i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} + i\omega t}\right] \tag{5.7.4}$$

and treat $-(e/m_e c)\mathbf{A}\cdot\mathbf{p}$ as time-dependent potential, where we express $\mathbf{A}$ in (5.7.3) as

$$\mathbf{A} = A_0 \hat{\boldsymbol{\varepsilon}}\left[e^{i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} + i\omega t}\right]. \tag{5.7.5}$$

Comparing this result with (5.6.39), we see that the $e^{-i\omega t}$-term in

$$-\left(\frac{e}{m_e c}\right)\mathbf{A}\cdot\mathbf{p} = -\left(\frac{e}{m_e c}\right) A_0 \hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\left[e^{i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} + i\omega t}\right] \tag{5.7.6}$$

is responsible for absorption, while the $e^{+i\omega t}$-term is responsible for stimulated emission.

Let us now treat the absorption case in detail. We have

$$\mathscr{V}_{ni}^{\dagger} = -\frac{eA_0}{m_e c}\left(e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right)_{ni} \tag{5.7.7}$$

and

$$w_{i\to n} = \frac{2\pi}{\hbar}\frac{e^2}{m_e^2 c^2}|A_0|^2 |\langle n|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle|^2 \delta(E_n - E_i - \hbar\omega). \tag{5.7.8}$$

The meaning of the δ-function is clear. If $|n\rangle$ forms a continuum, we simply integrate with $\rho(E_n)$. But even if $|n\rangle$ is discrete, because $|n\rangle$ cannot be a ground state (albeit a bound-state energy level), its energy is not infinitely sharp; there may be a natural broadening due to a finite lifetime (see Section 5.8); there can also be a mechanism for broadening due to collisions. In such cases, we regard $\delta(\omega - \omega_{ni})$ as

$$\delta(\omega - \omega_{ni}) = \lim_{\gamma\to 0}\left(\frac{\gamma}{2\pi}\right)\frac{1}{\left[(\omega-\omega_{ni})^2 + \gamma^2/4\right]}. \tag{5.7.9}$$

Finally, the incident electromagnetic wave itself is not perfectly monochromatic; in fact, there is always a finite frequency width.

We derive an absorption cross section as

$$\frac{(\text{Energy/unit time) absorbed by the atom } (i\to n)}{\text{Energy flux of the radiation field}}. \tag{5.7.10}$$

For the energy flux (energy per area per unit time), classical electromagnetic

$$\mathbf{A} = 2A_0 \hat{\boldsymbol{\varepsilon}} \cos\left(\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t\right) \tag{5.7.3}$$

where $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\mathbf{n}}$ are the (linear) polarization and propagation direction. Equation (5.7.3) obviously satisfies (5.7.2) because $\hat{\boldsymbol{\varepsilon}}$ is perpendicular to the propagation direction $\hat{\mathbf{n}}$. We write

$$\cos\left(\frac{\omega}{c}\hat{\mathbf{n}}\cdot\mathbf{x} - \omega t\right) = \frac{1}{2}\left[e^{i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} + i\omega t}\right] \tag{5.7.4}$$

and treat $-(e/m_e c)\mathbf{A}\cdot\mathbf{p}$ as time-dependent potential, where we express $\mathbf{A}$ in (5.7.3) as

$$\mathbf{A} = A_0 \hat{\boldsymbol{\varepsilon}}\left[e^{i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} + i\omega t}\right]. \tag{5.7.5}$$

Comparing this result with (5.6.39), we see that the $e^{-i\omega t}$-term in

$$-\left(\frac{e}{m_e c}\right)\mathbf{A}\cdot\mathbf{p} = -\left(\frac{e}{m_e c}\right) A_0 \hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\left[e^{i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} - i\omega t} + e^{-i(\omega/c)\hat{\mathbf{n}}\cdot\mathbf{x} + i\omega t}\right] \tag{5.7.6}$$

is responsible for absorption, while the $e^{+i\omega t}$-term is responsible for stimulated emission.

Let us now treat the absorption case in detail. We have

$$\mathscr{V}_{ni}^{\dagger} = -\frac{eA_0}{m_e c}\left(e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right)_{ni} \tag{5.7.7}$$

and

$$w_{i\to n} = \frac{2\pi}{\hbar}\frac{e^2}{m_e^2 c^2}|A_0|^2 |\langle n|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{x})}\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}|i\rangle|^2 \delta(E_n - E_i - \hbar\omega). \tag{5.7.8}$$

The meaning of the δ-function is clear. If $|n\rangle$ forms a continuum, we simply integrate with $\rho(E_n)$. But even if $|n\rangle$ is discrete, because $|n\rangle$ cannot be a ground state (albeit a bound-state energy level), its energy is not infinitely sharp; there may be a natural broadening due to a finite lifetime (see Section 5.8); there can also be a mechanism for broadening due to collisions. In such cases, we regard $\delta(\omega - \omega_{ni})$ as

$$\delta(\omega - \omega_{ni}) = \lim_{\gamma\to 0}\left(\frac{\gamma}{2\pi}\right)\frac{1}{\left[(\omega-\omega_{ni})^2 + \gamma^2/4\right]}. \tag{5.7.9}$$

Finally, the incident electromagnetic wave itself is not perfectly monochromatic; in fact, there is always a finite frequency width.

We derive an absorption cross section as

$$\frac{(\text{Energy/unit time) absorbed by the atom } (i\to n)}{\text{Energy flux of the radiation field}}. \tag{5.7.10}$$

For the energy flux (energy per area per unit time), classical electromagnetic

Einstein: particles and waves

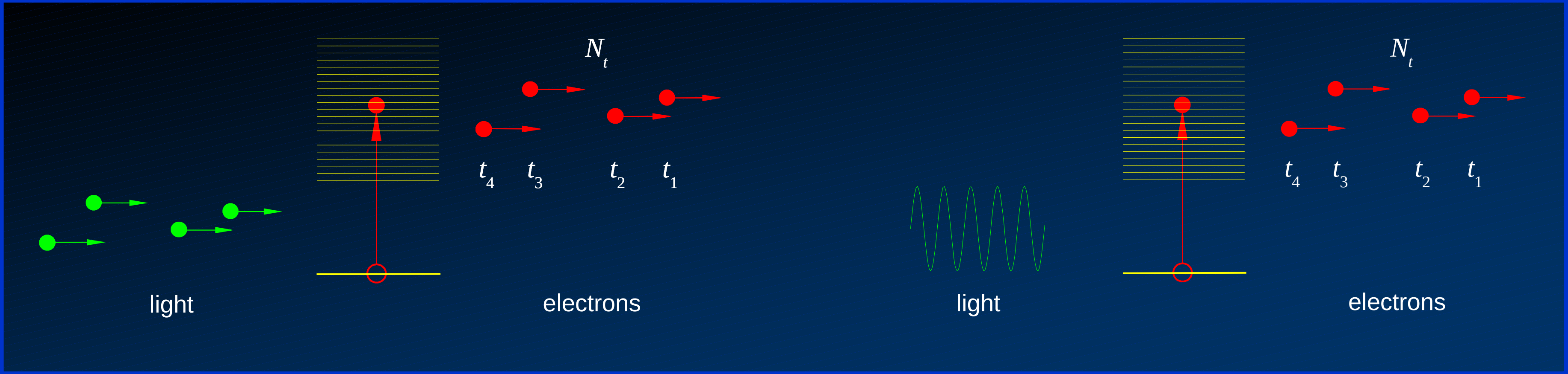
I already attempted earlier to show that our current foundations
of the radiation theory have to be abandoned …
it is my opinion that the development of theoretical physics will bring us to
a theory of light which can be interpreted as a kind of fusion
of the wave and the emission theory …
[the] wave structure and [the] quantum structure …
are not to be considered mutually incompatible …
it seems to follow from the Jeans law that we will have to modify our current theories,
not to abandon them completely.

$$\langle \mathcal{E}^2(\nu,T)\rangle = \Big(h\nu\rho + \frac{c^3}{8\pi\nu^2}\rho^2\Big)V d\nu$$

particle
term

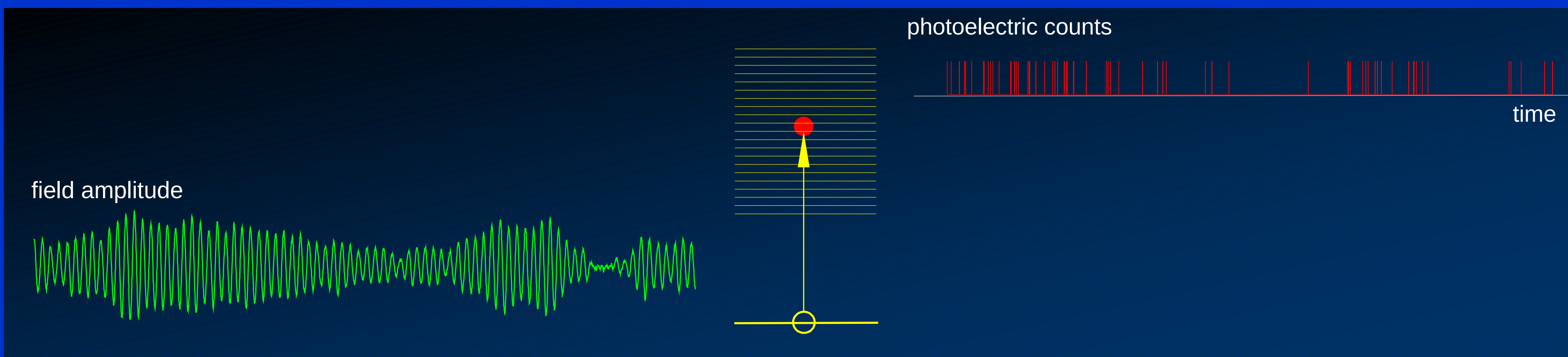
wave
term

Bohr Kramers Slater



Bohr, N., H.A. Kramers, and J.C. Slater,
1924, Philos. Mag. 47, 785

$$\langle \mathcal{E}^2(\nu,T)\rangle = \left(h\nu\rho + \frac{c^3}{8\pi\nu^2}\rho^2\right)V d\nu$$

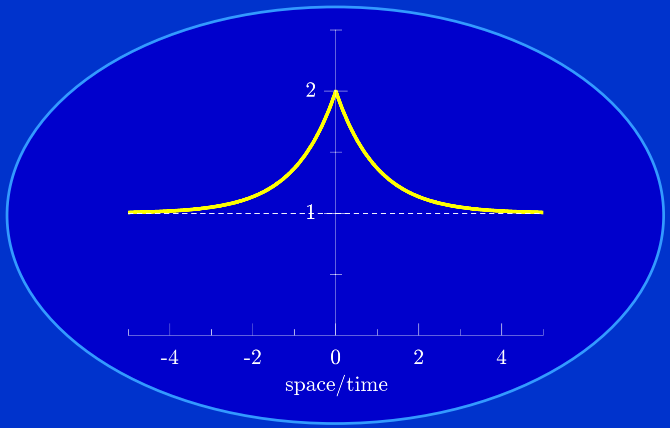


$$(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle + \langle n\rangle^2$$

particle term

wave term

Hanbury Brown and Twiss

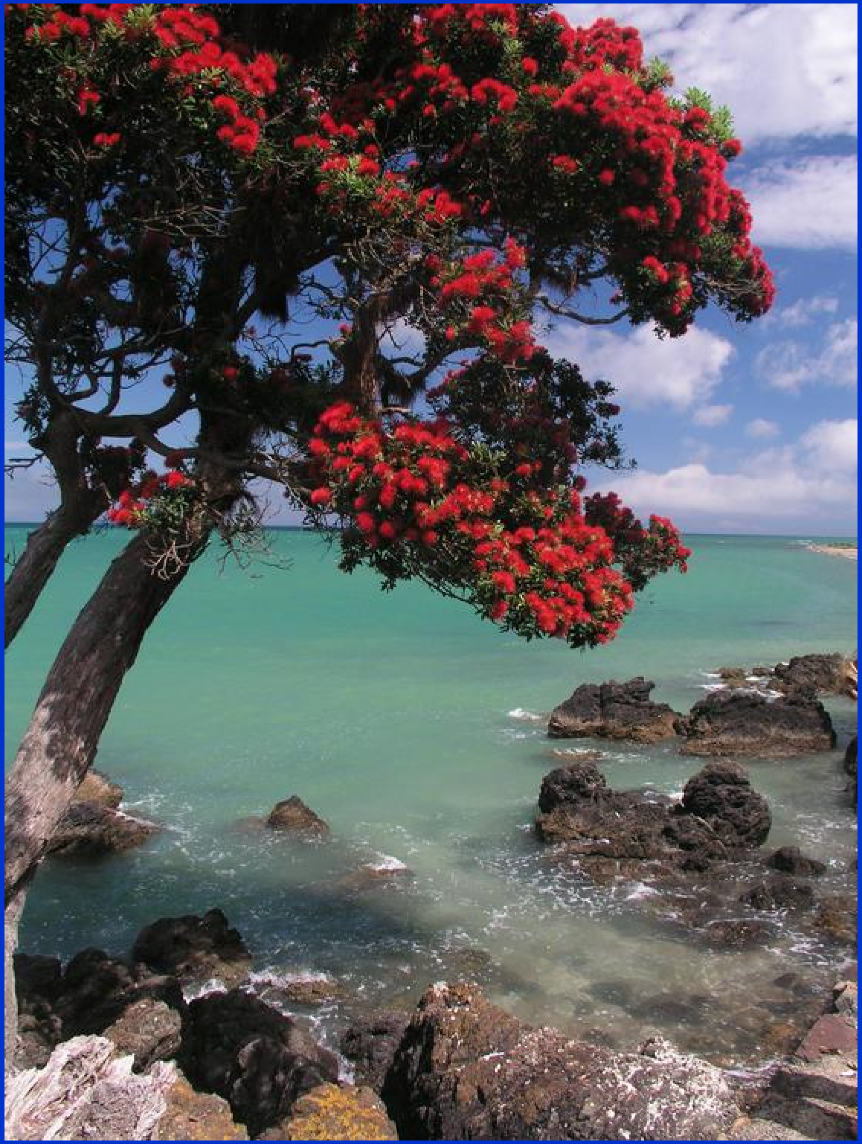


photon bunching

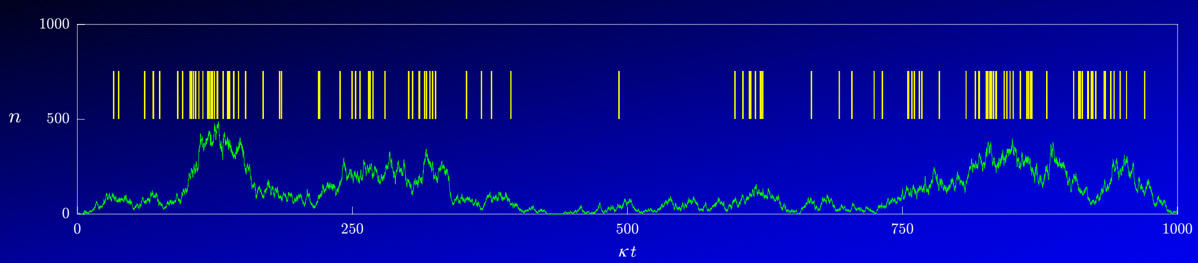
$$(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle + \langle n\rangle^2$$

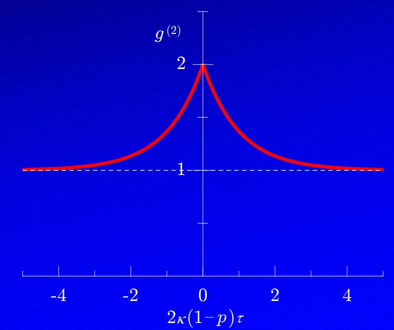
$$\langle n^2\rangle - \langle n\rangle = \langle n\rangle^2 + \langle n\rangle^2 = 2\langle n\rangle^2$$

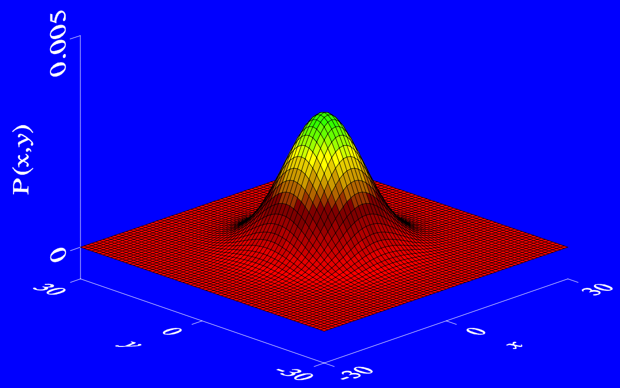
$$\frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2} = 2 \qquad \text{(for thermal light)}$$



nonclassical light

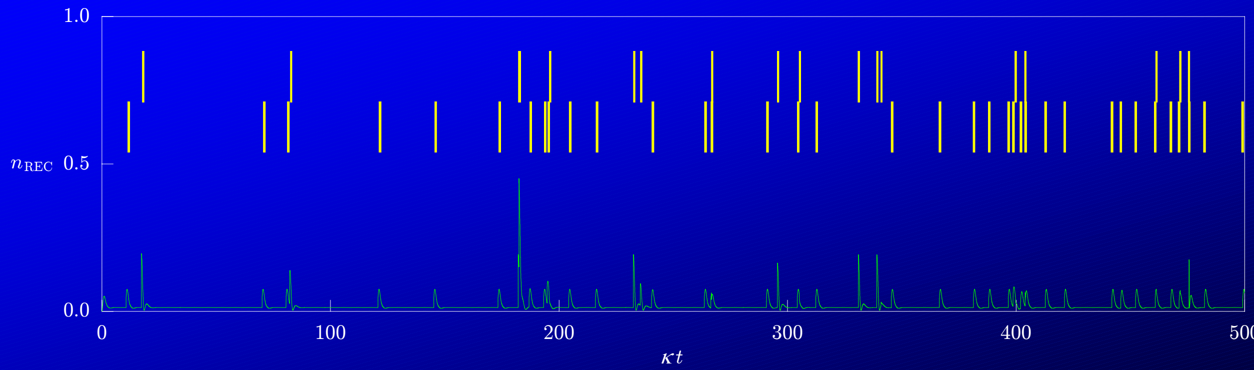


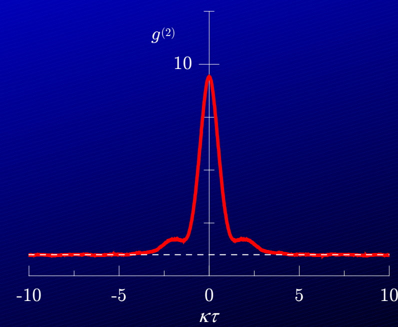




$$\rho(t) = \int d^2\alpha P(\alpha,t)\,|\alpha(t)\rangle\langle\alpha(t)|$$

photon number expectation

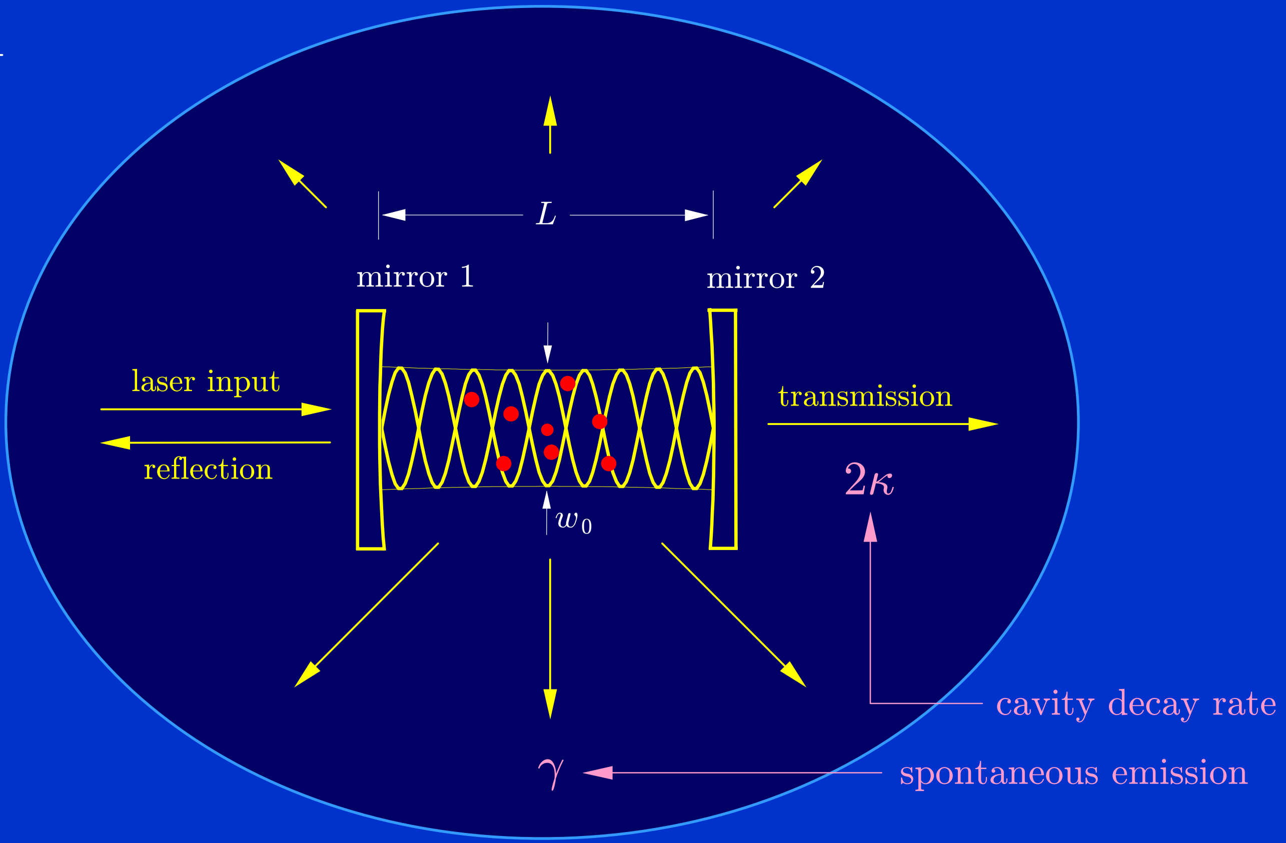




cavity QED

dipole coupling strength:

$$g = \sqrt{\frac{\omega_C d^2}{2\hbar\varepsilon_0 V}}$$



quantum trajectories:

non-unitary Schrödinger evolution:

$$\frac{d|\bar\psi_{\mathrm{REC}}\rangle}{dt} = \frac{1}{i\hbar}\hat H_B(t)|\bar\psi_{\mathrm{REC}}\rangle$$

$$\frac{1}{i\hbar}\hat H_B(t) = \underbrace{\mathcal{E}(\hat a^\dagger - \hat a)}_{\text{laser input}} + \underbrace{\sum_j g(r_j(t))(\hat a^\dagger \hat\sigma_{j-} - \hat a\hat\sigma_{j+})}_{\text{dipole interaction}} - \kappa\hat a^\dagger\hat a - \frac{\gamma}{2}\sum_j \hat\sigma_{j+}\hat\sigma_{j-}$$

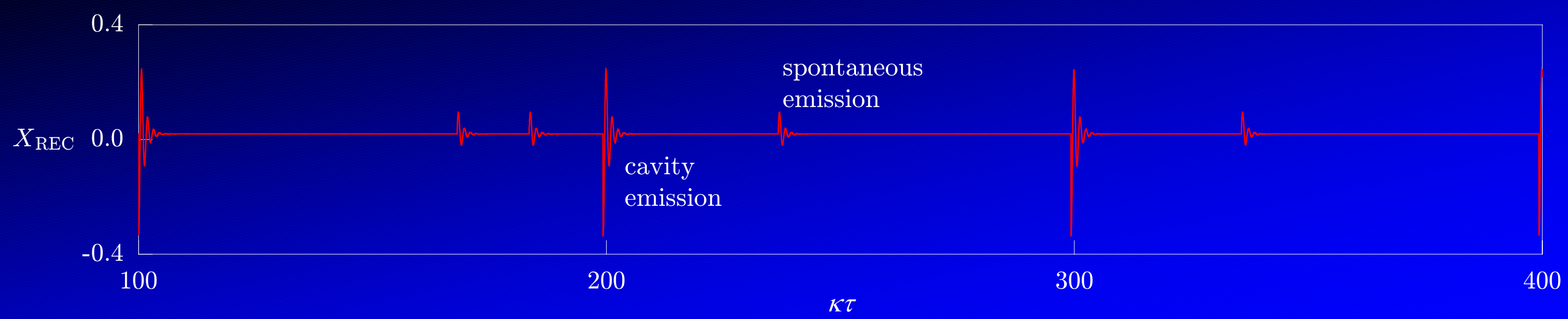
spontaneous emission

cavity loss

quantum jumps:

$$|\bar\psi_{\mathrm{REC}}\rangle \longrightarrow \begin{cases} \hat a|\bar\psi_{\mathrm{REC}}\rangle & \text{at rate} \quad 2\kappa\langle(\hat a^\dagger\hat a)(t)\rangle_{\mathrm{REC}} \\ \hat\sigma_{j-}|\bar\psi_{\mathrm{REC}}\rangle & \text{at rate} \quad \gamma\langle(\hat\sigma_{j+}\hat\sigma_{j-})(t)\rangle_{\mathrm{REC}} \end{cases}$$

weak excitation:



$$g^{(2)}_{\{r_j\}}(\tau) = \left[h_{\{r_j\}}(\tau) \right]^2$$

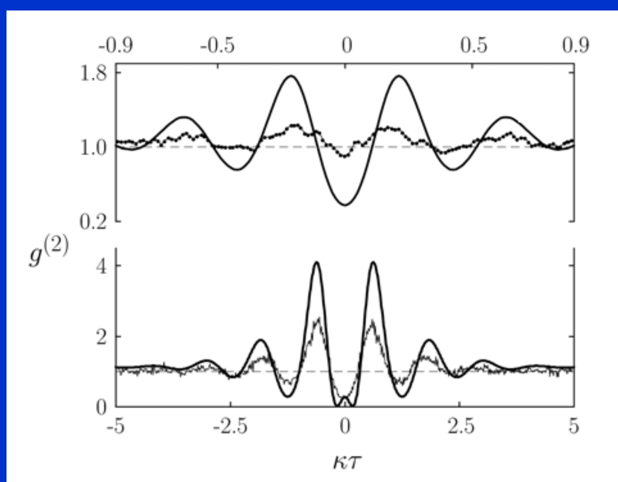
$$h_{\{r_j\}}(\tau) = 1.0 - A_{\{r_j\}} e^{-\frac{1}{2}(\kappa+\gamma/2)|\tau|} \left[\cos(\Omega_{\{r_j\}}\tau) + \frac{\frac{1}{2}(\kappa+\gamma/2)}{\Omega_{\{r_j\}}} \sin(\Omega_{\{r_j\}}\tau) \right]$$

$$\Omega_{\{r_j\}} = \sqrt{\sum_j g^2(r_j) - \frac{1}{4}(\kappa-\gamma/2)^2}$$

photon antibunching in cavity QED:

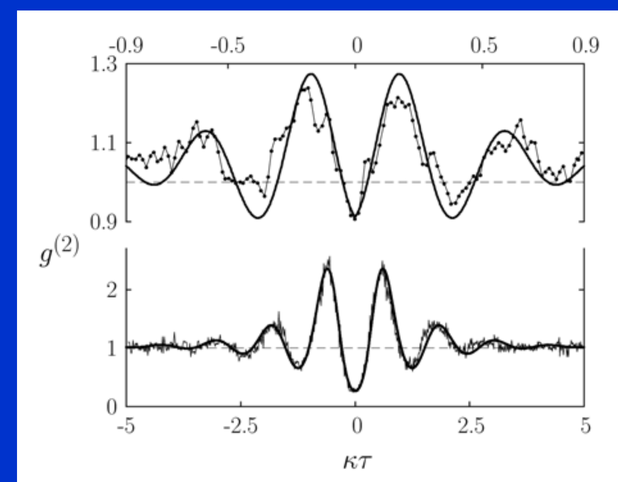
G. Rempe, R.J. Thompson, R.J. Brecha, W.D. Lee, and H.J. Kimble, Phys. Rev. Lett. **67**, 1727 (1991)

G.T. Foster, S.L. Mielke, and L.A. Orozco, Phys. Rev. A **61**, 053821 (2000)



Rempe *et al.*

Foster *et al.*



end