The Toric-Boson model and quantum memory at finite temperature

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Overview

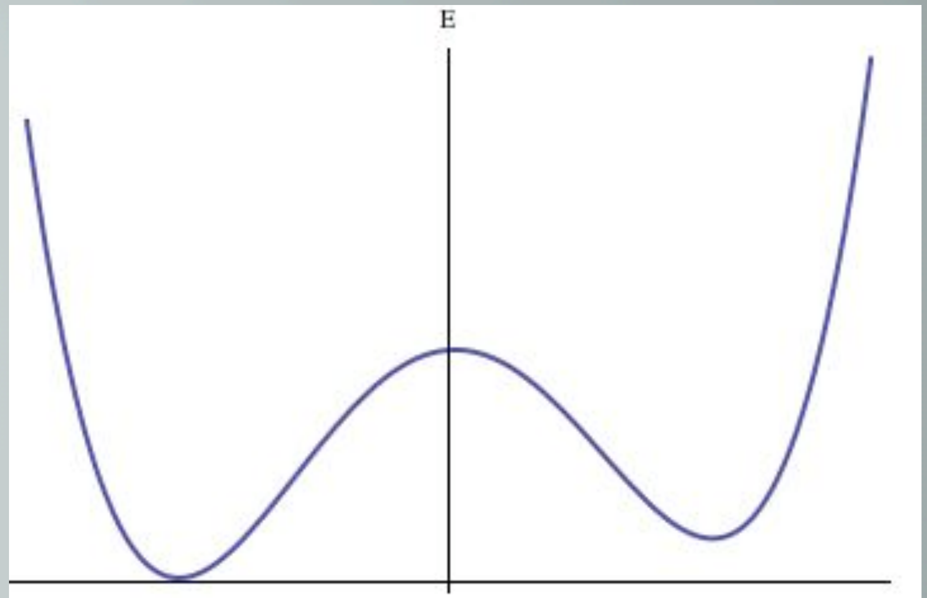
- Classical information can be stored for arbitrarily long times because of phase transitions
- Can quantum information be stored for arbitrarily long times in a passive way?
- Topological Order and robust memory for virtual processes: the Toric Code
- Thermal Fragility of the Toric Code
- The Toric-Boson model and finite T quantum memory

Memory and symmetry breaking: Weiss-Curie model

$$H = -\frac{J}{N}\left(\sum_j \sigma_j^z\right)^2$$

$$p(x \to x\prime) \simeq p(x\prime \to x)e^{-\frac{\Delta F}{k_B T}}$$

$$h(x) = -x\ln x - (1-x)\ln(1-x)$$



$$E(x) = -JNx^2$$

$$F(x,T) = E - TS = N\left[-kTh\left(\frac{x+1}{2}\right) - Jx^2\right]$$

We can see a critical temperature T_c

What is Quantum information?

A qubit is a generic superposition $\alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$

Of course $|0\rangle \rightarrow |1\rangle$ is still an error

$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \rightarrow \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ is an error! it will destroy coherence

we would end up with a state of the form $|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$

this is a classical state!

The quantum is fragile

Since we have two minima, we encode two quantum states $|0\rangle$ and $|1\rangle$ in them and we saw that the process $|0\rangle \rightarrow |1\rangle$ takes exponentially long time below the critical temperature

$\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ are only separated by a finite energy barrier

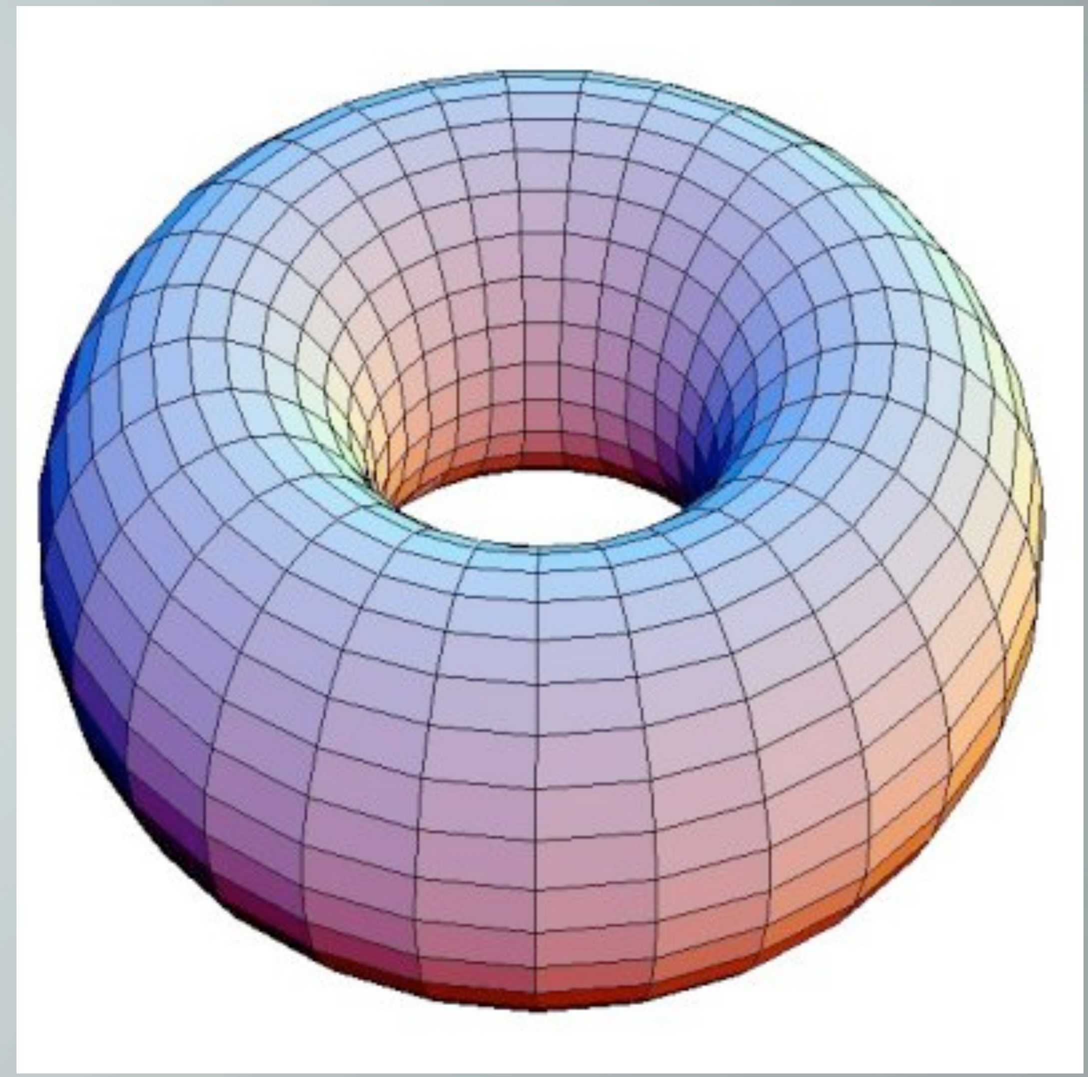
they are connected by an operator of the form $\sigma_i^z \otimes \prod_{j\neq i} I_j$

the entropy always prevails!

- In the classical case, one can have long lived metastable states whose life time scales (exponentially) with the size of the system
- The thermodynamic limit yields a breaking of ergodicity, a phase transition
- In the quantum case, the superpositions are not long lived (scaling with the size of the system) even if there is a phase transition
- Does it have to be so? Is it a law of nature and we do need error correction?

Topological Order

- The environment always acts locally
- So perhaps one can put the information in some global degrees of freedom
- For example, the



The Z2 field

$$
\begin{aligned}
W_1^{(z)} &= \prod_{j \in \gamma_1^z} \sigma_j^z \\
W_1^{(x)} &= \prod_{j \in \gamma_1^x} \sigma_j^x
\end{aligned}
$$

$$
\begin{aligned}
W_1^{(z)}|0\rangle_1 &= |0\rangle_1 \\
W_1^{(z)}|1\rangle_1 &= -|1\rangle_1 \\
W_2^{(x)}|0\rangle_1 &= |1\rangle_1 \\
W_2^{(x)}|1\rangle_1 &= |0\rangle_1
\end{aligned}
$$

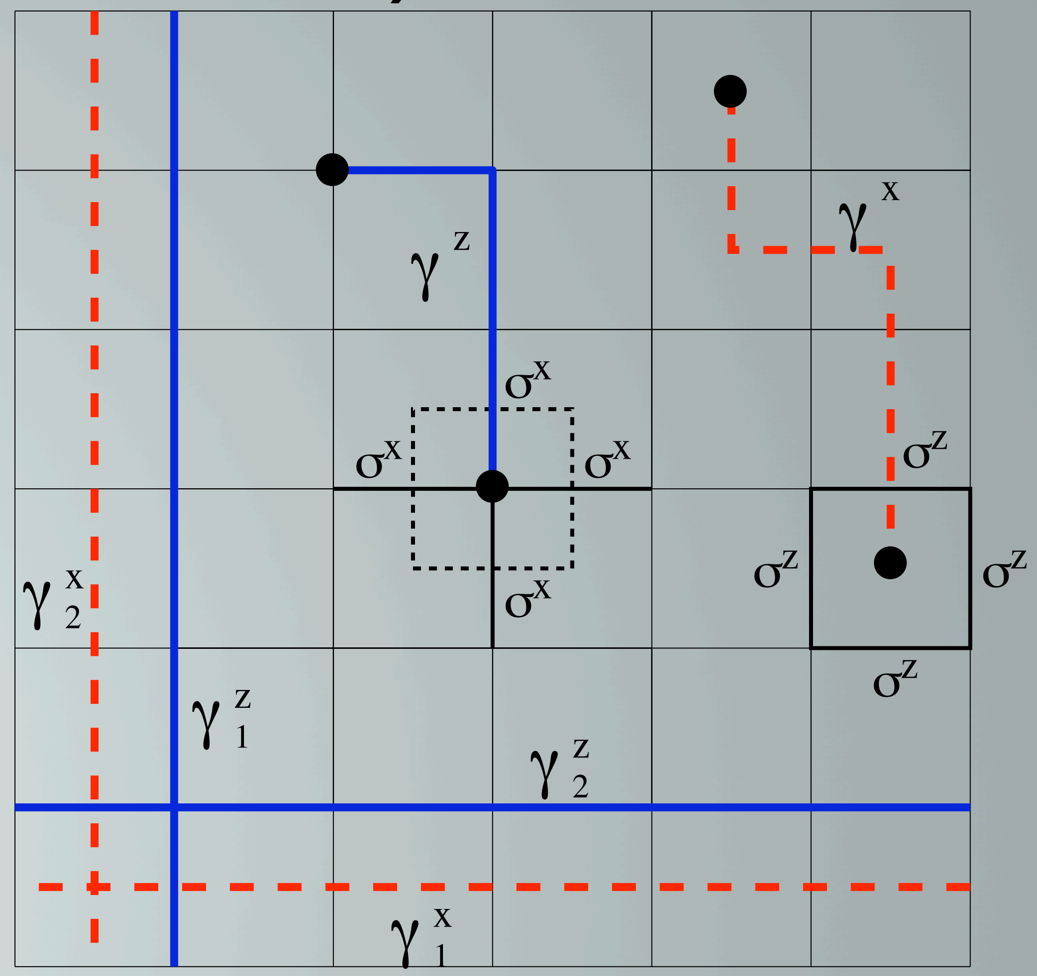
this is the algebra of a spin $\frac{1}{2}$

If we exchange $1 \leftrightarrow 2$ we make another qubit

Now what we need is that these states span the ground state

and we would like to have a gap

The Toric Code (Kitaev 1997)



$$H = -J\sum_{\Box}\Box - U\sum_{+}+$$

$$[\Box, +] = 0$$

$$\prod + = \prod \Box = 1$$

$$|+_1 +_2 \cdots +_{N-1}; \Box_1 \Box_2 \ldots \Box_{N-1}; W_1^{(z)} W_2^{(z)}\rangle$$

The low energy of this theory is like a Z_2 gauge theory

Robust ground state degeneracy

$$\mathcal{L} = \mathrm{span}\{|W_1^{(z)} W_2^{(z)}\rangle\}$$

with g holes we would have g copies of it and $\dim \mathcal{L} = 2^{2g}$

Introduce a perturbation $t \sum_j \sigma_j^x$

the splitting is $\Delta E \sim (\frac{t}{J})^L$

Thermal fragility

- The environment creates a finite density of defects and moves them around in a brownian motion
- When the defects wind around the torus and annihilate they have made an error
- Both defects are string-like objects, they move freely
- The time of recombination is microscopic:

$$t_{rec} \sim \exp(U/k_B T)$$

- Tension to the strings: if we confine the strings, we kill the topological order, the system has a quantum phase transition and the qubits are lost
- 3D model. One error is a membrane its energetics gives a phase transition, but the other error is a string. Just classical memory
- 4D model: now both errors are membranes and all works. But the real world is in 3D...
- So maybe in the real world, quantum mechanics is really microscopic! Or is it?

- What if there was a long range force between the defects that makes them attract
- then their energetics could imply a critical temperature under which they are confined
- the topological order would still be preserved because the force only sees the defects, not the spins

The Toric-Boson Model

- The recipe is to have a system made of the toric code immersed in a sea of phonons
- the phonons interact with the defects, and they experience an attractive force
- then the whole system is subject to a generic (but local) environment. Is the system robust?

we have a_k modes out of plane and b_k in plane modes

$$H_{\text{boson}} = \sum_{k\neq 0} \omega_k \; a_k^\dagger a_k + \sum_{k\neq 0, i=x,y} \Omega_k \; b_k^{(i)\dagger} b_k^{(i)}$$

The interaction Hamiltonian reads

$$H_{int} = \sum_{\ell = s,\, p} n_\ell \left[\sum g_\omega \, \varphi(x_\ell) + g_\Omega \left(\partial_x \phi^x(x_\ell) + \partial_y \phi^y(x_\ell) \right) \right]$$

The defects are placed at positions x_l: $\rho(x) = \sum_l n_l \, \delta(x - x_l)$

The Fourier transform is $\tilde{\rho}(k) = \sum_l n_l e^{ik \cdot x_l}$

Now define the shifted operators:

$$\begin{aligned} \alpha_k &\equiv a^k + \frac{g_\omega \, \tilde{\rho}^\dagger(k)}{\sqrt{2V} \, {\omega_k}^{3/2}} \\ \beta_k^{(i)} &\equiv b_k^{(i)} + \frac{i g_\Omega k_i \, \tilde{\rho}^\dagger(k)}{\sqrt{2V} \, {\Omega_k}^{3/2}} \qquad (i = x, y) \end{aligned}$$

the total Hamiltonian is

$$
\begin{aligned}
H &= H_{\text{toric}} + \sum_{k\neq 0} \omega_k \, \alpha_k^\dagger \alpha_k + \sum_{k\neq 0, i=x,y} \Omega_k \, \beta_k^{(i)\dagger} \beta_k^{(i)} \\
&- \frac{1}{V} \sum_{k\neq 0} |\tilde{\rho}(k)|^2 \left(\frac{g_\omega^2}{2\,\omega_k{}^2} + \frac{g_\Omega^2 (k_x^2 + k_y^2)}{2\Omega_k^2} \right)
\end{aligned}
$$

The last term generates a potential. If we use acousting phonons with dispersion $\omega_k = v_\omega |k|$ and $\Omega_k = v_\Omega |k|$, we obtain a gravitational potential

The gravitational potential

$$
\begin{aligned}
V_\Omega(r) + V_\omega(r) &= \frac{g_\omega^2}{v_\omega^2}\frac{1}{2\pi}\ln\frac{|r|}{a} && (1)\\
&+ \left[\frac{g_\Omega^2}{v_\Omega^2}\frac{1}{V} - \frac{g_\omega^2}{v_\omega^2}\frac{1}{2\pi}\ln\frac{\xi_L}{a}\right] && (2)\\
&- \frac{g_\Omega^2}{v_\Omega^2}\delta^{(2)}(r) && (3)
\end{aligned}
$$

ξ_L scales with the size of the system, so canceling it in (2) is paid in (3) with an infinite chemical potential

$$
\frac{g_\Omega^2}{v_\Omega^2} = \frac{g_\omega^2}{v_\omega^2}\frac{V}{2\pi}\ln\frac{\xi_L}{a} \sim L^2 \ln L
$$

The role of the in-plane phonons is to prevent the arising of defects that scale with N^2

Attention! infinite couplings! But the theory is finite!

The final expression for the effective potential is

$$V_d\left(x_1, \ldots, x_{2N}\right) = \frac{g_\omega^2}{v_\omega^2} \frac{1}{4\pi} \sum_{\ell,\ell\prime=1}^{2N} n_\ell n_{\ell\prime} \ln \frac{|x_\ell - x_{\ell\prime}|}{a}$$

- Now we have a force. But is this force robust?
- The environment could create generic terms that open a gap in the dispersion relation for the phonons
- But phonons exist in nature!
- The dispersion relation is protected by symmetry: acoustic waves to propagate in mediums, even effective mediums with some broken translational symmetry

And now, is the quantum information preserved?

Beyond a critical temperature $T_c = NGm^2/4$

the partition function of N particles of mass m interacting gravitationally becomes divergent.

Gravitational forces overcome entropy and lead to a collapse, where all the particles coalesce in a single point. Gravitational forces increase with N, so N=2 gives the critical temperature

$$t_{\mathrm{rec}} \sim \exp[\alpha \ln(L)/T] \sim L^{\alpha/T}$$

The quantum memory lasts for a time that is polynomial in the system size

Conclusions

- Classical information is stable in a passive way because there are classical metastable states with long (scaling with L) lifetime
- The question is: are there metastable quantum states? In the sense that both the states and their coherence is long lived with a quantity scaling like L?
- The toric-boson model shows that quantum states can be long lived (but we also see why usually they have microscopic lifetimes)