Casimir effect due to a single boundary as a manifestation of the Weyl problem

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(i) The electromagnetic field has zero-point energy

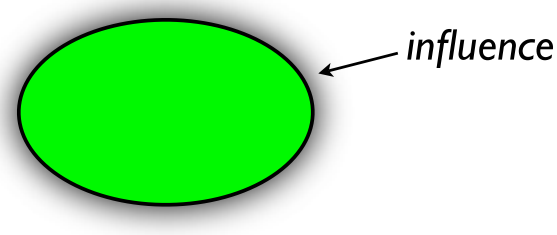
$$E = \sum_{\nu} \frac{1}{2}\hbar\omega_{\nu}$$

modes

Not really: field modes of sufficiently high energy should not enter the count since they are unaffected by the geometry; a physical cutoff is inevitable

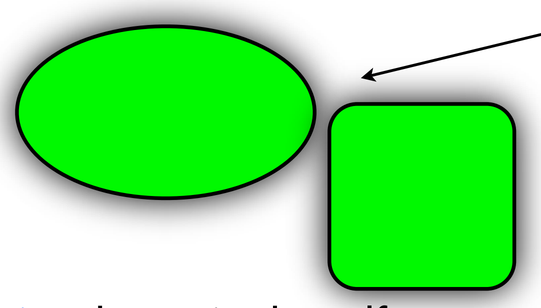
whose density is infinite.

(ii) An object



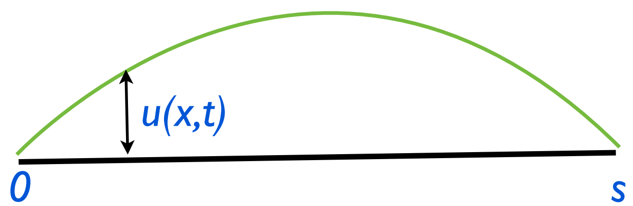
modifies the spectrum. This gives a self-energy relative to the vacuum. This is also infinite.

(iii) Objects that are close to each other have overlapping influence:



This gives a finite change in the self-energy. The outcome is the Casimir force measured in modern experiments.

Example: a scalar field $u(x,t)$ *on a one-dimensional Dirichlet interval*



$$\omega_n = c q_n = \pi c n / s$$

speed of "light"/sound

$$E = \frac{\pi \hbar c}{2s} \sum_{n=1}^{\infty} n$$

diverges. But real materials become transparent at short wavelengths. So we can cut off the sum at some *n=N>>1*. Then

$$E = \frac{\pi \hbar c}{4s} (N^2 + N + \underset{\text{sharp}}{0})$$

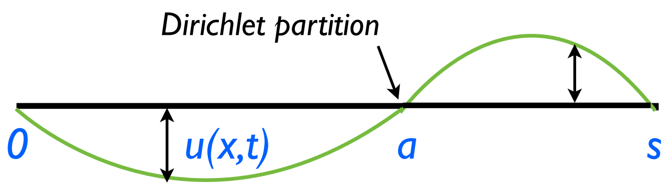
The upper limit is of the order $N = \omega_0 s / c$ where ω_0 is the cutoff frequency. Therefore

$$E(s) = \underbrace{\frac{\hbar \omega_0^2}{c} s}_{\text{Bulk, linear in size } s} + \underbrace{\hbar \omega_0}_{\text{Ends}} + \underbrace{const \frac{\hbar c}{s}}_{\text{Finite-size/Intrinsic=Cutoff-independent=Universal}}$$

line tension

phenomenologically expected

The cutoff-dependent parts have geometrical interpretation. The force on either end is cutoff-dependent , and dominated by the bulk term, $F = -dE/ds \simeq -\hbar \omega_0^2 / c$. It is *divergent* in the $\omega_0 \to \infty$ limit. Let us now insert another Dirichlet partition at $x = a$ and compute the force on it.



$$E(s) = \frac{\hbar\omega_0^2}{c}s + \hbar\omega_0 + const\,\frac{\hbar c}{s}$$

$$\begin{aligned}\mathcal{E} &= E(a) + E(s-a) \\ &= const\ \hbar c\left(\frac{1}{a} + \frac{1}{s-a}\right) + \frac{\hbar\omega_0^2}{c}(\cancel{a} + s - \cancel{a}) + \hbar\omega_0(1+1)\end{aligned}$$

- *intrinsic or universal*
- *size determined by* $\hbar, c$ *and macroscopic length scales.*
- *a-independent*
- $\omega_0 \to \infty$ *limit - infinities are subtracted*

a,s-independent

non-universal

- *if* $s \to \infty$ *then* $\mathcal{E} \approx \hbar c/a$ *is uniquely determined by dimensional analysis.*
- *although the effect is electromagnetic in origin, the charge quantum e does not appear.*
- *determines universal Casimir force on the partition,* $\mathcal{F} = -d\mathcal{E}/da$ *; the estimate is a toy version of Casimir's original calculation.*
- *determination of const requires smooth cutoff; the sign determines if it is attractive or repulsive.*

Q: Why is the force $F = -dE/ds$ cutoff-dependent while $\mathcal{F} = -d\mathcal{E}/da$ is not ?

A: The force is energy change per virtual displacement; varying s changes system size thus leading to a large non-universal force; varying a keeps system size fixed and only changes overlapping influence - the outcome is a small universal force.

Determining the numerical prefactor

- Assume a smooth cutoff function, for example *exp(-n/N)*:

$$E = \frac{\pi\hbar c}{2s}\sum_{n=1}^{\infty} n e^{-n/N} = -\frac{\pi\hbar c}{2s}\frac{\partial}{\partial(1/N)}\left(\sum_{n=1}^{\infty} e^{-n/N}\right) = \frac{\pi\hbar c}{2s}\frac{e^{-1/N}}{(1-e^{-1/N})^2}$$

$$\xrightarrow[N\gg 1]{} \frac{\pi\hbar c}{2s}\left(N^2 - \frac{1}{12}\right)_{N\simeq\frac{\omega_0 s}{c}} \rightarrow \frac{\hbar\omega_0^2}{c}s + 0\times\hbar\omega_0 - \frac{\pi\hbar c}{24s}$$

no end effect! *Can the sign be predicted?*

So $\mathcal{E} = -\frac{\pi\hbar c}{24}\left(\frac{1}{a}+\frac{1}{s-a}\right)$ - attractive.

- *Regularization route:* the Riemann ζ-function, $\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma}$, convergent for σ>1 and can be analytically continued to all complex σ≠1. Then the *regularized* energy can be defined as $E^{(R)}(\sigma) = \frac{\pi\hbar c}{2s}\zeta(\sigma)$ with the understanding that we are interested in the σ=-1 case.

Employing ς(-1)=-1/12 we find $E^{(R)} = -\frac{\pi\hbar c}{24s}$.

- *Conclusion:* ς-function regularization method only determines intrinsic piece of the effect and it shows its universality. However it does not provide an insight regarding its sign. It correctly determines the force $\mathcal{F}$ on the partition at *x=a* but overlooks the main contribution into the force *F* on the ends in the interval geometry.

Exceptions: always in calculations of self-stress

Not a complete list

- The calculation just explained is an example of a scenario common to many geometries - computations could be mathematically more involved but nothing changes in principle. However there are exceptions when regularization techniques fail to produce finite intrinsic piece of the effect:

- Bender&Milton, 1994, demonstrated that for a spherical shell in *d* spatial dimensions the Casimir pressure is *infinite* for *even d*. *Does it mean that conductive ring in two dimensions is unstable?*

- Sen, 1981, who employed the cutoff method, concluded that the Casimir energy of a Dirichlet ring in a plane (*d*=2) contains geometric terms with quadratic and logarithmic cutoff dependencies. *Perhaps the latter is responsible for failure of regularization approach to extract an intrinsic piece of the effect? Indeed regularization method would not work if analytic continuation to physically relevant situation would not be possible.*

Our contention:

Both the cases when regularization is successful (Dowker&Kennedy, 1978; Deutsch&Candelas, 1979) and those when it is not can be understood systematically through the connection of the Casimir problem to the *Weyl problem* of mathematical physics whose essence can be summarized by the title of 1966 paper by Mark Kac, *"Can one hear the shape of a drum?"*

Highly recommended for its beauty and accessibility

Calculating the Casimir energy

Temperature

Imaginary time action for a scalar field: $$S_E[w] = \frac{1}{2}\int_0^{\hbar/T} d\tau d^d x \left(c^{-2}(\frac{\partial w}{\partial \tau})^2 + (\nabla w)^2 \right)$$

Imaginary time

$w(\mathbf{r},0) = w(\mathbf{r},\hbar/T)$ - periodicity on the Matsubara circle

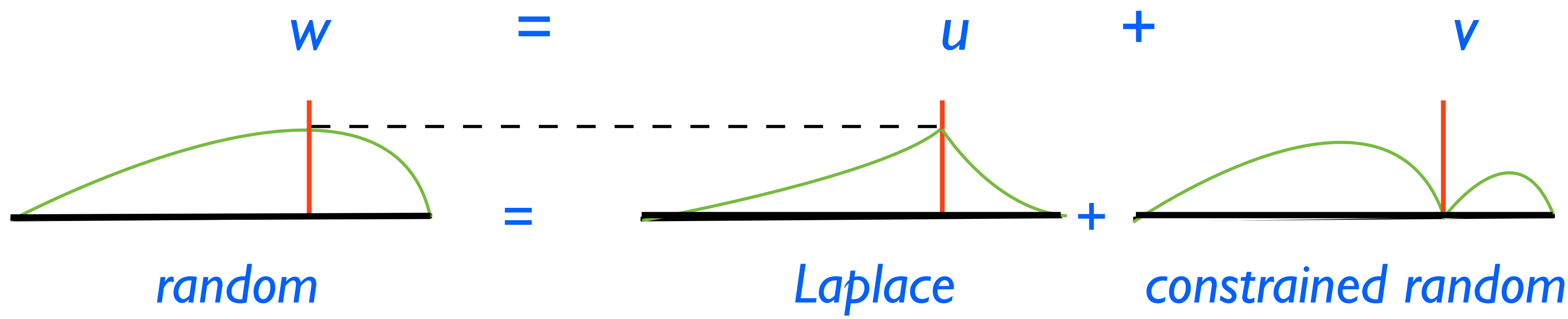
$$Z_w = \int Dw(\mathbf{r},\tau)\exp(-S_E[w]/\hbar)$$

over all possible $w(\mathbf{r},\tau)$ satisfying various boundary conditions

The Feynman path integral can be interpreted as the partition function for a classical statistical mechanics problem with the Hamiltonian S_E at a fictitious temperature equal to Planck's constant. The zero-point energy is then given by the ***T=0*** limit of the "free energy" per unit length in imaginary time direction, i.e. by ☞

$$\mathcal{E}_0 = -\hbar(\ln Z_w)/(\hbar/T) = -T\ln Z_w$$

Introduce a new Dirichlet boundary. This will constrain the field *suppressing* its fluctuations at the location of the boundary and nearby.



There is a unique way to associate the unconstrained field *w* with a constrained field *v* (satisfying new boundary condition):

$$w(\mathbf{r},\tau) = v(\mathbf{r},\tau) + u(\mathbf{r},\tau)$$

Solution to $\left(\frac{\partial^2}{c^2\partial\tau^2} + \triangle\right)u = 0$ agreeing with *w* on the boundary.

Then $S_E[w] = S_E[v] + S_E[u]$ thus implying $Z_w = Z_v Z_u$.

$$\mathcal{E} = +T \ln Z_u$$

The rule

In words: the Casimir energy due to a Dirichlet boundary is negative of the zero-point energy of the modes suppressed by this boundary.

Determination of sign: confinement is the source of the zero-point energy which is necessarily positive. Then suppression (removal) of some field fluctuations by the boundary lowers the zero-point energy.

In symbols: we need to solve the boundary-value Laplace problem:

$$(\frac{\partial^2}{c^2\partial\tau^2}+\triangle)u=0, \quad u|_{boundary}=f(\mathbf{r},\tau)$$

static *dynamical field*

After a Fourier expansion $u(\mathbf{r},\tau)=\sum_{\omega}u_{\omega}(\mathbf{r})\exp i\omega\tau$ we arrive at the boundary-value Helmholtz problem $(\triangle-\frac{\omega^2}{c^2})u_{\omega}=0, \quad u_{\omega}|_{boundary}=f_{\omega}(\mathbf{r})$ - put into $S_E(u)$:

$$S_E[u(f)]=\frac{1}{2}\int_0^{\hbar/T}d\tau\int[u\nabla u]d\mathbf{s}=\frac{\hbar}{2T}\sum_{\omega}\int f_{\omega}[\nabla u_{-\omega}]d\mathbf{s}\overset{u_\omega\propto f_\omega}{=}\frac{\hbar}{2T}\sum_{\omega,\nu}\frac{|f_{\omega\nu}|^2}{\lambda_{\nu}(|\omega|/c)}$$

boundary *discontinuity* *modes* *geometry*

implicit cutoff

$$\mathcal{E}=\frac{\hbar}{2\pi}\sum_{\nu}{}'\int_0^{\infty}d\omega\ln\frac{\lambda_{\nu}(\omega/c)}{\lambda_{\nu}(\infty)}$$

reference free field geometry

Gaussian

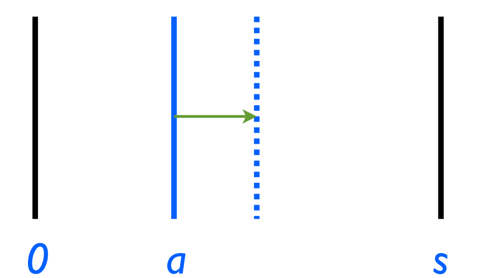
Geometrical interpretation of ultraviolet divergences

Let us assume that the physical boundary is characterized by a frequency cutoff ω_0 : the boundary is impenetrable to low-energy field modes but invisible to the modes whose energy significantly exceeds $\hbar\omega_0$. Such a boundary can be modeled by a Dirichlet surface. Let us estimate the coefficient of fluctuation-induced surface tension γ_0 of a single Dirichlet plane immersed in a d-dimensional vacuum.

The problem is only characterized by microscopic energy and length scales, $\hbar\omega_0$ and c/ω_0 , respectively. As a first step, dimensional analysis will suffice:

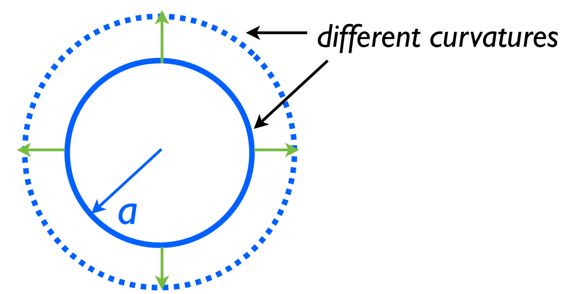
$$\gamma_0 \sim \frac{energy}{(length)^{d-1}} \sim \frac{\hbar\omega_0}{(c/\omega_0)^{d-1}} = \hbar c(\omega_0/c)^d$$ diverges as $\omega_0 \to \infty$

Deutsch&Candelas,1979; Jaffe et. al. 2002+, Barton, 2004: *this is a formal divergence that may have measurable consequences:*



The area does not change, the force is small and cutoff-independent - similar to 1d example analyzed earlier.

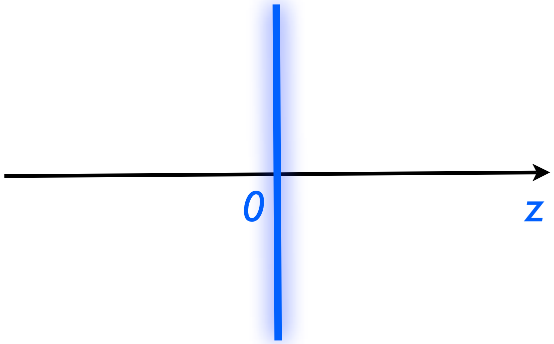
versus



The area changes, the force is large and cutoff-dependent. What if (like in 1d) the surface tension vanishes? Still the force could be large and cutoff-dependent because curvature changes.

To see the role of the geometry an explicit calculation is needed!

Surface energy of a plane in d dimensions



In-plane translational symmetry: $u_\omega(\mathbf{r}) = \sum_{\mathbf{q}} u_{\omega\mathbf{q}}(z) \exp i\mathbf{q}\mathbf{r}_\perp$

Boundary-value problem: $\left(\frac{d^2}{dz^2} - q^2 - \frac{\omega^2}{c^2}\right) u_{\omega\mathbf{q}}(z) = 0, \;\; u_{\omega\mathbf{q}}(0) = f_{\omega\mathbf{q}}$

Solution: $u_{\omega\mathbf{q}}(z) = f_{\omega\mathbf{q}} \exp\left(-\sqrt{q^2+\omega^2/c^2}\,|z|\right)$ - localized at the boundary.

Gaussian action: $S_E = \frac{\hbar}{2T} \sum_{\omega,\mathbf{q}} 2\mathcal{A}\sqrt{q^2+\omega^2/c^2}\,|f_{\omega\mathbf{q}}|^2$ - conforms with $S_E = \frac{\hbar}{2T} \sum_{\omega,\nu} \frac{|f_{\omega\nu}|^2}{\lambda_\nu(|\omega|/c)}$.

area

Geometrical coefficient: $\lambda_{\mathbf{q}}(\omega/c) = 1/(2\mathcal{A}\sqrt{q^2+\omega^2/c^2})$ becomes small for large q. The disturbance

u introduced by the boundary is localized within a length that is proportional to λ itself.

Surface energy: $\mathcal{E} = -\frac{\hbar}{4\pi} \sum_{\mathbf{q}}{}' \int_0^\infty d\omega \ln \frac{\omega^2 + c^2 q^2}{\omega^2} = -\frac{1}{2} \sum_{\mathbf{q}}{}' \frac{\hbar c q}{2}$ - negative of a fraction of the

zero-point energy of a harmonic field in *d-1* dimensions! Why?

(i) If u would be infinitely localized, the surface energy would be exactly negative of d-1- dimensional zero-point energy. *However the surface energy is merely dominated by highly-localized field modes - the fraction is less than unity.*

(ii) It is negative because the effect is due to field modes eliminated by the boundary.

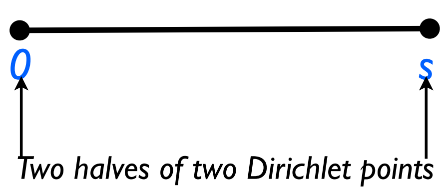
Surface energy of a plane in d dimensions, continued...

$$\mathcal{E} = -\frac{1}{2}\sum_{\mathbf{q}}{}' \frac{\hbar c q}{2} \underset{\textit{macroscopic limit}}{\longrightarrow} -\frac{\hbar c \mathcal{A} K_{d-1}}{4}\int_0^{\infty}{}' q^{d-1}dq \sim -K_{d-1}\hbar c(\omega_0/c)^d\mathcal{A}$$

coefficient of surface tension; agrees with dimensional estimate

$$K_d = \frac{area\ of\ d-dimensional\ unit\ sphere}{(2\pi)^d}$$

Coefficient of surface tension is *negative* except for *d=1* where it is *zero* ($K_0 = 0$). Does the latter contradict the argument that introduction of the Dirichlet surface lowers the vacuum energy? No, in fact, it explains the sign of the intrinsic piece of the effect:



One-dimensional Dirichlet interval again: $E = \frac{\hbar\omega_0^2}{c}s - \underbrace{\frac{\pi\hbar c}{24s}}_{\textit{now understood}}$

Compared to boundary-free segment of vacuum, insertion of two halves of two Dirichlet points still lowers the vacuum energy. This decrease manifests itself in the intrinsic piece of the effect since surface tension (edge energy) is zero.

Is there more to understand? Yes, there is a fundamental feature built into the cutoff-dependent part of the effect!

Surface energy of a plane in d dimensions and the Weyl problem

Let us make explicit the fact that the surface energy has its origin in zero-point motion:

$$\mathcal{E} = -\frac{\hbar c \mathcal{A} K_{d-1}}{4} \int_0^{\infty}{}' q^{d-1} dq \equiv \int_0^{\infty}{}' \frac{\hbar c q}{2} g_{area}(q) dq$$

number of vibrations of wavevector between q and q+dq

$$g_{area}(q) = -\frac{1}{2} \mathcal{A} K_{d-1} q^{d-2}$$

areal density of states (DOS), purely geometrical (cutoff-independent) quantity

Dowker&Kennedy, 1978; Deutsch&Candelas, 1979: all cutoff-dependent contributions into the Casimir self-energy have a geometrical nature interpretable in terms of some DOS!

Scalar Casimir effect: asymptotic limit of the density of eigenvalues of the Laplacian, the Weyl problem.

In 1910 Lorentz conjectured that $g(q) = \mathcal{V} K_d q^{d-1}$ for a field confined to a volume *V* in the *large q limit independent of the shape of the volume.* Hilbert predicted that the proof will not be supplied during his lifetime. In 1911-1913 Weyl proved the statement and conjectured next order term, proportional to the area *A*, essentially areal DOS above. Lorentz-Weyl result is easy to demonstrate for rectangular parallelepiped shape (Jeans, 1905) and we use it all the time when macroscopic limit is taken:

$$\sum_{\mathbf{q}} \to \mathcal{V} \int d^d q/(2\pi)^d$$

In fact, I used it already when areal DOS was derived

Weyl DOS and the formally divergent part of the Casimir energy

For a field confined to a region the zero-point energy is the sum of zero-point energies of the field oscillators:

$$\mathcal{E} = {\sum_\nu}' \frac{\hbar c q_\nu}{2} \equiv \int_0^\infty{}' \frac{\hbar c q}{2} G(q)dq$$

exact DOS

$-q_\nu^2$ are eigenvalues of the Laplacian: $(\triangle + q^2)w = 0$; the spectrum is determined by $w|_{boundary} = 0$.

$$G(q) = \sum_\nu \delta(q - q_\nu) \equiv g(q) + [G(q) - g(q)]$$

histogram

smooth Weyl DOS, average of G over scales exceeding distance between neighboring peaks of G, large-q behavior, origin of cutoff-dependent part of the Casimir energy

oscillatory remainder, origin of intrinsic part of the Casimir effect

Assume separability of the zero-point energy into cutoff-dependent and intrinsic pieces:

$$\mathcal{E} = \int_0^\infty{}' \frac{\hbar c q}{2} g(q)dq + \int_0^\infty \frac{\hbar c q}{2}[G(q) - g(q)]dq$$

Not a universally applicable rule

additive Weyl energy of local origin

non-additive intrinsic part

Weyl expansion and geometry

The smooth part of the exact DOS, $g(q)$, can be represented as a large-q expansion and each term of this Weyl expansion can be interpreted geometrically. Indeed, for a d-dimensional volume V enclosed by a (d-1)-dimensional Dirichlet boundary of area A the Weyl expansion starts out as

$$g(q) = \mathcal{V} K_d q^{d-1} - \frac{1}{4} \mathcal{A} K_{d-1} q^{d-2} + \ldots$$

conjectured by Lorentz, proved by Weyl (1911-13)

half of areal DOS derived earlier, conjectured by Weyl, proved by Brownell (1957), Ivrii (1980), Melrose (1980) and others.

curvature, perimeter, edge, corner etc. terms

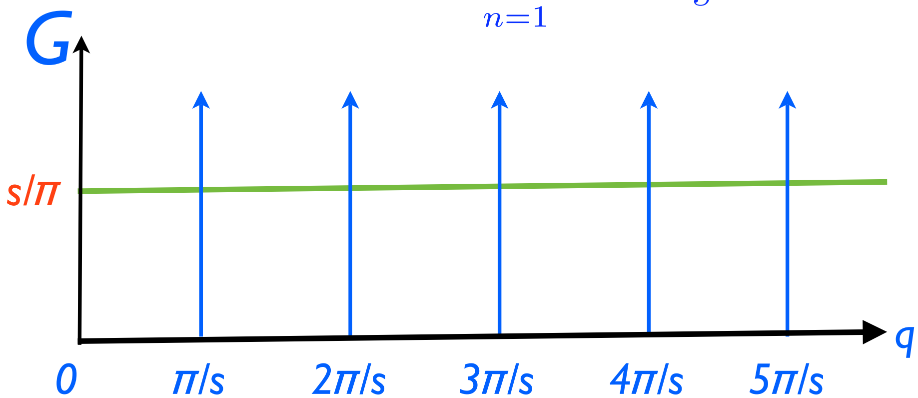
Spectral information encoded in the Weyl DOS could be used to extract at least partial information about the volume, area and shape, thus explaining Mark Kac's question: *Can one hear the shape of a drum?*

Although the Weyl DOS is a purely geometrical concept having little to do with physics, its relationship to the Casimir problem explains the sign of the surface term.

Examples and applications

One-dimensional Dirichlet interval of length s

Exact DOS: $G(q)=\sum_{n=1}^{\infty}\delta(q-\frac{\pi n}{s})$



every interval of length Δq=π/s (except for q=0) contains exactly one eigenvalue: *the Weyl DOS is g(q)=s/π*.

In the macroscopic limit $\omega_0 s/c \gg 1$ the zero-point energy can be computed with desired accuracy with the help of the Euler-Maclaurin summation formula:

$$\sum_{n=1}^{\infty}{}' F(n) \approx \int_0^{\infty}{}' F(x)dx - \frac{1}{2}F(0) - \frac{1}{12}F'(0)$$

The zero-point energy is given by:

$$\mathcal{E} = \frac{\pi\hbar c}{2s}\sum_{n=1}^{\infty}{}' n \to \frac{\pi\hbar c}{2s}\left(\int_0^{\infty}{}' x dx - \frac{1}{12}\right) \overset{q=\pi n/s}{=} \int_0^{\infty}{}' \frac{\hbar c q}{2}\frac{s dq}{\pi} - \frac{\pi\hbar c}{24 s}$$

Weyl energy *Intrinsic piece*

One-dimensional periodic interval of length s: effect of topology

Exact DOS: $G(q) = \sum_{n=-\infty}^{\infty} \delta(q - \frac{2\pi n}{s})$ - twice the distance between the peaks of the Dirichlet case. However *$|n|>0$ eigenvalues are doubly degenerate - same Weyl DOS $g(q)=s/\pi$ as in the Dirichlet case.*

The zero-point energy is given by:

$$\mathcal{E} = \frac{\pi\hbar c}{s} \sum_{n=-\infty}^{\infty} {}' n \to \frac{2\pi\hbar c}{s} \left(\int_0^{\infty'} x dx - \frac{1}{12} \right) \overset{q=2\pi n/s}{=} \int_0^{\infty'} \frac{\hbar c q}{2} \frac{s dq}{\pi} - \frac{\pi\hbar c}{6s}$$

Weyl energy is insensitive to topology (its origin is local)

intrinsic piece is sensitive to topology (its origin is non-local)

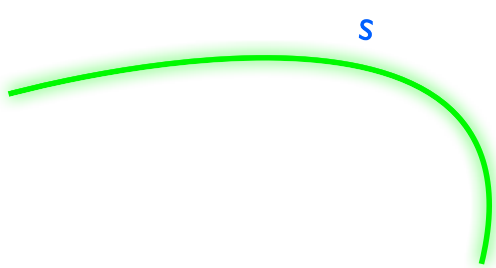
Although this is the case without physical boundary, we can still understand it geometrically:

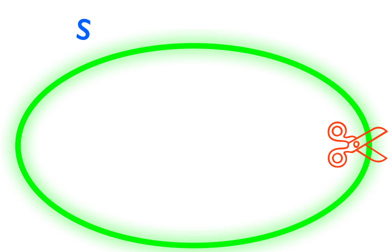
- The cutoff is provided by deviation of the spectrum from *$\omega=cq$ at large q*.
- Edge term cannot be present since the interval is periodically bound.
- Intrinsic term is negative because periodically binding the interval turns continuum spectrum into a discrete spectrum - removed field modes no longer contribute into the zero-point energy. As a result the latter goes down.

The difference between $-\pi\hbar c/(24s)$ (Dirichlet) and $-\pi\hbar c/(6s)$ (periodic) intrinsic pieces is well-known: Johnson (1975), Lüscher et al. (1980), Blöte et al.(1986), Affleck (1986).

Smooth boundary in two dimensions

It was demonstrated earlier that the zero-point energy due to a Dirichlet plane inserted in a *d*-dimensional space is *negative half of the (d-1)-dimensional zero-point energy*. The same will remain true for a finite-size piece of the plane and *approximately true* for sufficiently smooth surface. Thus one-dimensional results explained earlier have interesting implications on what is going on in two dimensions. Let us consider two Dirichlet curves of length s, open and closed...





twice Weyl areal DOS

$$\mathcal{E}_{open} = {\int_0^\infty}' \frac{\hbar c q}{2}\left(-\frac{s\,dq}{2\pi}\right) + \frac{\pi \hbar c}{48 s} \qquad \mathcal{E}_{closed} = {\int_0^\infty}' \frac{\hbar c q}{2}\left(-\frac{s\,dq}{2\pi}\right) + \frac{\pi \hbar c}{12 s}$$

Cutting the loop at a point lowers the energy and this is determined by the intrinsic part of the effect!

This neglects the effects of curvature but accounts for circumference.

Boundary as a membrane

The Weyl energy of a boundary separating media with the same speed of light is given by a surface integral of an "even" combination of curvature invariants that does not depend on the sense of local normal (contributions from "odd" terms cancel). In three dimensions we have (Deutsch&Candelas, 1979):

$$\mathcal{E}(\omega_0) = \int ds(\underbrace{\gamma_0}_{\sim \hbar\omega_0^3/c^2} + \underbrace{\gamma_{1a}}_{\sim \hbar\omega_0}\underbrace{(\mathcal{C}_1 - \mathcal{C}_2)^2}_{\text{mean}} + \underbrace{\gamma_{1b}}_{\sim \hbar\omega_0}\underbrace{\mathcal{C}_1\mathcal{C}_2}_{\text{Gaussian}})$$

surface tension — γ_0; *curvature stiffnesses* — γ_{1a}, γ_{1b}; *principal curvatures* — $\mathcal{C}_1$, $\mathcal{C}_2$

Canham-Helfrich Hamiltonian of a biological membrane, 1970,1983

- This is an expansion in powers of the cutoff - no need to take into account invariants beyond those displayed.
- Since the boundary is made of real material, the shape constants γ's should be interpreted as contributions into elastic properties of the boundary viewed as a flexible membrane.
- Can be written down phenomenologically without referring to the Weyl problem.
- Applicable to any harmonic field and boundary conditions.

Spherical shell of radius a

$$\mathcal{E} = \underbrace{4\pi\gamma_0 a^2 + 4\pi\gamma_{1b}}_{\text{Weyl energy}} + \underbrace{\#\frac{\hbar c}{a}}_{\text{intrinsic}}$$

Only for the case of electromagnetic field when *surface tension is zero* (Boyer, 1968) is Casimir self-stress determined by small intrinsic part of the effect.

Long cylindrical shell of radius a

$$\frac{\mathcal{E}}{L} = \underbrace{2\pi a\gamma_0 + \frac{2\pi\gamma_{1a}}{a}}_{\text{Weyl energy}} + \underbrace{\#\frac{\hbar c}{a^2}}_{\text{intrinsic}}$$

Casimir self-stress is *always* dominated by large Weyl part of the effect

Why are even space dimensions special?

Examples and applications described so far assumed separability of the Weyl and intrinsic pieces of the Casimir effect. This assumption breaks down in even space dimensions. Indeed, let us *assume* separability and estimate the Weyl energy of a spherical shell of radius *a* in *d* dimensions:

$$\mathcal{E}(\omega_0) \sim \sum_{n=0}^{M} \gamma_n \underbrace{a^{-2n}}_{\text{even powers of curvature}} \underbrace{a^{d-1}}_{\text{surface area}} = \sum_{n=0}^{M} \gamma_n a^{d-1-2n} \overset{\text{dimensional analysis}}{\sim} \hbar c \sum_{n=0}^{M} \left(\frac{\omega_0}{c}\right)^{d-2n} a^{d-1-2n}$$

The number of terms *M+1* of the Weyl series is fixed by the condition *d-2M≥0*.

- *d* is odd →*(d+1)/2* terms → the least divergent is *linear* in ω_0.

- *d* is even → *(d/2)+1* terms → the least divergent is *cutoff-independent*. This however contradicts the expectation that the Weyl energy only contains the cutoff-dependent parts of the Casimir effect.

Phenomenological resolution: *allow logarithmic cutoff dependence.* Then in addition to the usual cutoff-dependent and intrinsic contributions the Casimir energy would have a contribution of the form

$$\mathcal{U}_{even} \sim \underbrace{\frac{\hbar c}{a}}_{\text{universal}} \underbrace{\ln \frac{\omega_0 a}{c}}_{\text{no longer local}}$$

such terms cannot be removed by formal regularization

For even d the Weyl and intrinsic parts of the effect are entangled!

Main result: Casimir energy due to a smooth Dirichlet boundary Γ

length of the boundary

$$\mathcal{E} = \int_0^{\infty}{}' \frac{\hbar c q}{2}\left(-\frac{\mathcal{A} dq}{2\pi}\right)$$

$$+ \int_{1/S}^{\infty}{}' \frac{\hbar c q}{2}\left(-\frac{dq}{128\pi q^2}\int_\Gamma \mathcal{C}^2(s)ds\right)$$

curvature square

have their origin in the Weyl DOS ☞ $g(q) = -\frac{\mathcal{A}}{2\pi} - \frac{1}{128\pi q^2}\int_\Gamma \mathcal{C}^2(s)ds$

Stewartson&Waechter, 1971,
can be anticipated phenomenologically

inverse macroscopic length scale, sensitive to topology

Euler's constant

$$- \frac{\gamma \hbar c}{256\pi}\int_\Gamma \mathcal{C}^2(s)ds$$

☜ *geometric and cutoff-independent, unique to two dimensions*

$$+ \mathcal{U}_{na}$$

non-additive, intrinsic, sensitive to topology

☜ contains circumference terms $\pi\hbar c/(48\mathcal{A})$ (open curve) or $\pi\hbar c/(12\mathcal{A})$ (closed curve) and curvature corrections.

The bulk of the effect has geometrical origin:

$$\mathcal{E}_{geom} = \int_0^{\infty'} \frac{\hbar c q}{2}\left(-\frac{\mathcal{A} dq}{2\pi}\right) - \left(\frac{\hbar c}{256\pi}\int_\Gamma \mathcal{C}^2(s)ds\right)\ln\frac{\omega_0 S}{c}$$

logarithmic accuracy

universal

Summary

- Solution of the problem of the Casimir self-energy that invokes transmission properties of the boundary inevitably encounters the Weyl problem of mathematical physics.

- The intrinsic part of the Casimir effect is interesting because it does not depend on the material properties of the boundary; the physical effect is however small.

- The cutoff-dependent part of the Casimir effect is also interesting because it can lead to large measurable stress and because its origin can be traced back to the universal Weyl DOS, the fundamental concept of geometry.

- In most cases there is clear separation of the Weyl and intrinsic contributions into the energy and cutoff-dependent part of the effect has entirely local geometrical origin.

- This fails in even space dimensions because the Weyl DOS expansion contains a marginal $1/q^2$ term. However even in such cases the concept of the Weyl DOS continues to play a prominent role. It is expected that the mystery of divergent Casimir self-stress in general even space dimension is solved similarly to our solution of the two-dimensional case.