An imbalanced Fermi gas in $1 + \epsilon$ dimensions

Andrew J. A. James
A. Lamacraft

2009

Quantum Liquids

Interactions and statistics (indistinguishability)

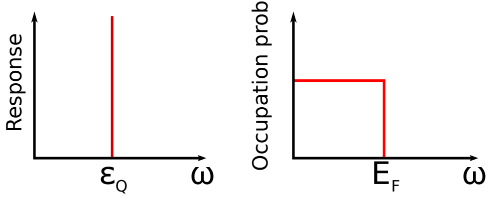
Some examples:

- ^{4}He
- ^{3}He
- Electrons in a metal
- Ultracold atomic gas

Fermi Liquids

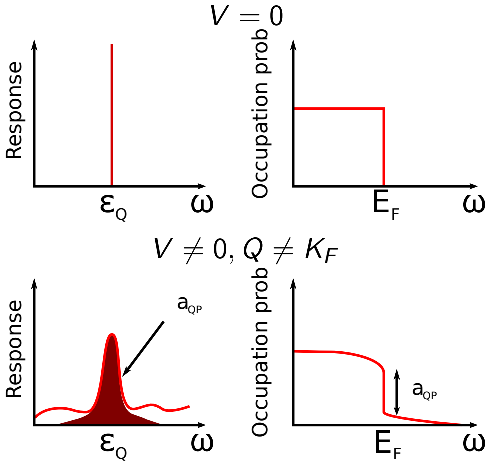
e.g. electrons in metals, ^{3}He

$V = 0$



Fermi Liquids

e.g. electrons in metals, ^{3}He



Propagator

$$G(\omega, Q) \approx \frac{a}{\omega - \xi_Q + i\eta}$$

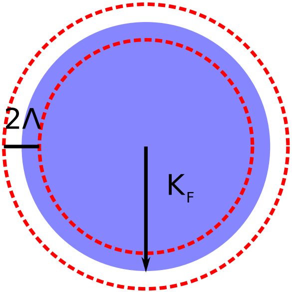
$$\xi_{K_F} = 0$$

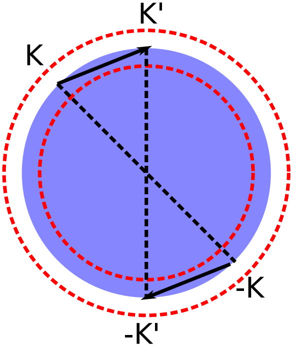
$$\eta \sim \frac{1}{\tau}$$

$$-\mathrm{Im}\, G(\omega, K_F) = a\delta(\omega)$$

Fermion pairing

BCS superconductivity



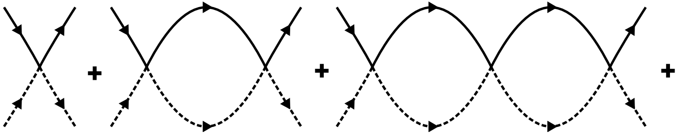


Effective Hamiltonian

$$H = \sum_{k\sigma} \xi_k \psi^\dagger_{k\sigma} \psi_{k\sigma} + V \sum_{k,k',Q} \psi^\dagger_{k'+Q\uparrow} \psi^\dagger_{-k'\downarrow} \psi_{-k+Q\downarrow} \psi_{k\uparrow}$$

Fermion pairing

BCS diagrams

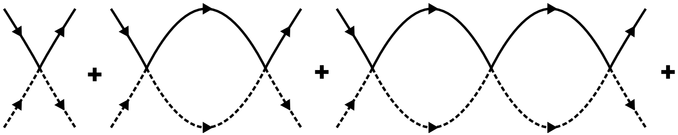


Cooper instability

$$\Gamma = V - V^2 \log(\Lambda/\mu) + \ldots \Rightarrow \Gamma = \frac{V}{1 + V \log(\Lambda/\mu)}$$

Fermion pairing

BCS diagrams

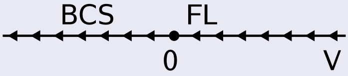


Cooper instability

$$\Gamma = V - V^2 \log(\Lambda/\mu) + \ldots \Rightarrow \Gamma = \frac{V}{1 + V \log(\Lambda/\mu)}$$

RG flow

$$\frac{d\Gamma}{d\log\Lambda} = 0 \Rightarrow -\frac{dV}{d\log\Lambda} = -V^2$$

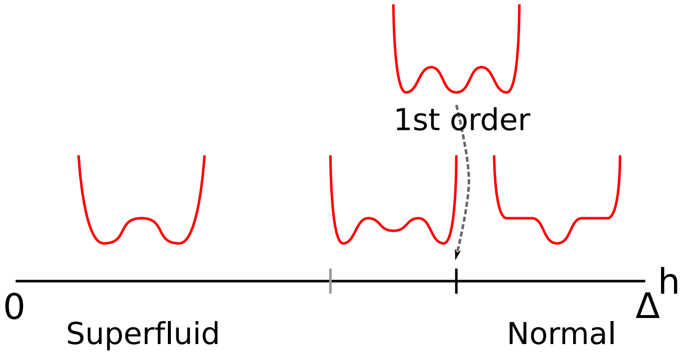


Inhomogenous Superconductors

What happens when the two spin populations are unequal?
Layered superconductor in a magnetic field...

Inhomogenous Superconductors

What happens when the two spin populations are unequal?
Layered superconductor in a magnetic field...



1st order transition when $h = \Delta/\sqrt{2}$ at $T = 0$.

Cooper's Problem

Two fermions interacting above their Fermi seas

$$\left[-\frac{\nabla_a^2}{2m_a}-\frac{\nabla_b^2}{2m_b}+V\delta(\mathbf{r}_a-\mathbf{r}_b)-E\right]\psi(\mathbf{r}_a,\mathbf{r}_b)=0$$

Cooper's Problem

Two fermions interacting above their Fermi seas

$$\left[-\frac{\nabla_a^2}{2m_a} - \frac{\nabla_b^2}{2m_b} + V\delta(\mathbf{r}_a - \mathbf{r}_b) - E\right]\psi(\mathbf{r}_a, \mathbf{r}_b) = 0$$

$$\frac{1}{V} = \int_{\substack{|\mathbf{Q}-\mathbf{p}|>K_{F,a} \\ |\mathbf{p}|>K_{F,b}}} \frac{d^d p}{(2\pi)^2} \frac{1}{\epsilon_a(\mathbf{Q}-\mathbf{p}) + \epsilon_b(\mathbf{p}) - E}$$

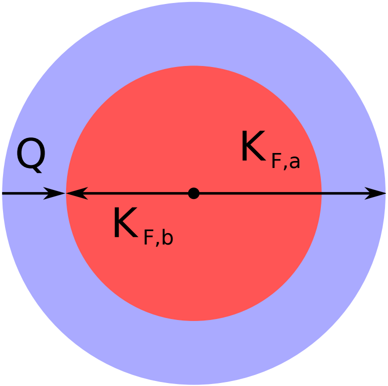
Bound state: $E = \mu_a + \mu_b + E_B, \qquad E_B < 0$

Larkin–Ovchinnikov–Fulde–Ferrell

A possibility for $(K_{F,a} > K_{F,b})$

$$Q = K_{F,a} - K_{F,b}$$

$$\Delta(\mathbf{x}) \sim \exp(i\mathbf{Q} \cdot \mathbf{x})$$

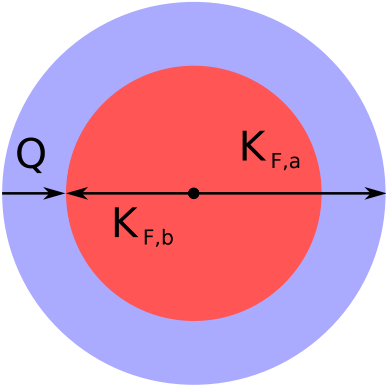


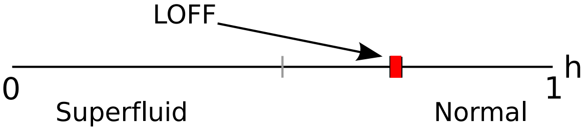
Larkin–Ovchinnikov–Fulde–Ferrell

A possibility for $(K_{F,a} > K_{F,b})$

$$Q = K_{F,a} - K_{F,b}$$

$$\Delta(\mathbf{x}) \sim \exp(i\mathbf{Q} \cdot \mathbf{x})$$



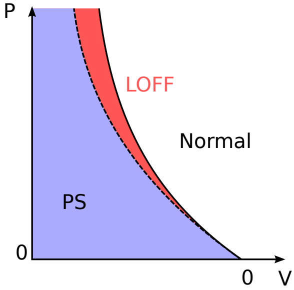


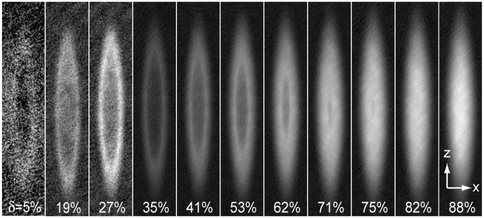
Seen in $CeCoIn_5$, heavy fermion superconductor?

Polarised Fermi gases

RF pulse through ultracold Fermi gas, alters relative spin populations (don't have to work with spins though)

Mean field and experiment suggest that





Y. Sin *et al.* PRL 2006

$$P = \frac{n_a - n_b}{n_a + n_b}$$

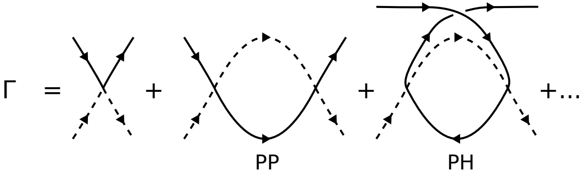
The model

Effective Hamiltonian

$$H = \sum_{K,s} (\epsilon_K - \mu_s) \psi^\dagger_{K,s} \psi_{K,s} + V \sum_{K,K',Q} \psi^\dagger_{K'+Q,a} \psi^\dagger_{-K',b} \psi_{-K+Q,b} \psi_{K,a}$$

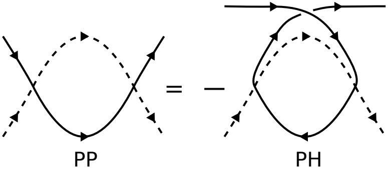
$$\epsilon_K - \mu_s \approx v_{F,s} k \quad \text{and} \quad |k| = |K| - K_{F,s}$$

One loop RG (for continuum path integral formulation)



One dimension (balanced)

Cancellation

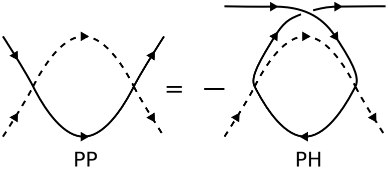


Flow vanishes

$$\Gamma = V \qquad \Rightarrow \qquad \frac{d\Gamma}{d\log\Lambda} = \frac{dV}{d\log\Lambda}$$

One dimension (balanced)

Cancellation



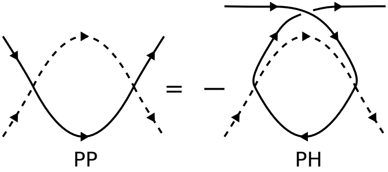
Flow vanishes

$$\Gamma = V \qquad \Rightarrow \qquad \frac{d\Gamma}{d\log\Lambda} = \frac{dV}{d\log\Lambda}$$

$$\frac{d\Gamma}{d\log\Lambda} = 0$$

One dimension (balanced)

Cancellation



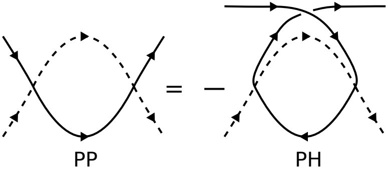
Flow vanishes

$$\Gamma = V \qquad \Rightarrow \qquad \frac{d\Gamma}{d\log\Lambda} = \frac{dV}{d\log\Lambda}$$

$$\frac{d\Gamma}{d\log\Lambda} = 0 \qquad \Rightarrow \qquad \beta(V) = 0$$

One dimension (balanced)

Cancellation



Flow vanishes

$$\Gamma = V \qquad \Rightarrow \qquad \frac{d\Gamma}{d\log\Lambda} = \frac{dV}{d\log\Lambda}$$

$$\frac{d\Gamma}{d\log\Lambda} = 0 \qquad \Rightarrow \qquad \beta(V) = 0$$

Cancellation occurs to all orders, (Ward identities, bosonization, exact solution of XXZ spin chain)

Luttinger liquid, power law behaviour with interaction dependent exponents

One + ϵ dimensions

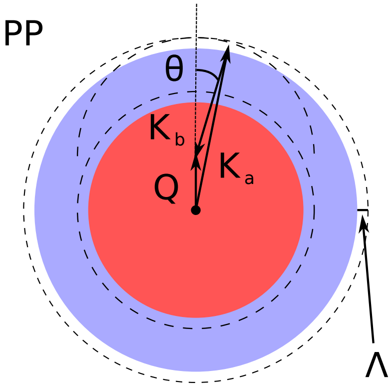
Expect flow to appear as we ‘turn on’ new dimension

One + ϵ dimensions

Expect flow to appear as we 'turn on' new dimension

Angular measure

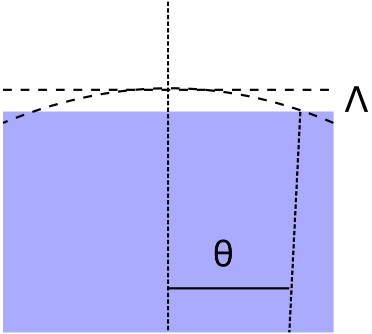
$$\int d^d\mathbf{k} \to S_{d-1} \int dk k^{d-1} \int_0^\pi d\theta (\sin\theta)^{d-2}$$



One + ϵ dimensions

Angular cut off

$$S_{d-1} \int dk k^{d-1} \int_0^{\theta_M} d\theta (\sin\theta)^{d-2} \qquad \theta_M \approx \sqrt{\frac{2K_{F,a}k}{QK_{F,b}}}$$



Γ in $1+\epsilon$

Combine PP and PH diagrams

$$\Gamma = V - V^2 \frac{2(2K_{F,b}K_{F,a})^{\epsilon/2}}{2\pi \bar{v}_F \epsilon} \left(\frac{\Lambda}{K_{F,a}} \right)^{\epsilon/2} F + \dots$$

$$F = \left(1 - \frac{K_{F,b}}{K_{F,a}} \right)^{-\epsilon/2} - \left(1 + \frac{K_{F,b}}{K_{F,a}} \right)^{-\epsilon/2}$$

1st term in F comes from PP, 2nd from PH

Flow and Fixed Point

Flow

$$\frac{d}{d\log\Lambda}\Gamma = 0$$

Flow and Fixed Point

Flow

$$\frac{d}{d\log\Lambda}\Gamma = 0 \quad \Rightarrow \quad \frac{d}{d\log\Lambda}V = \frac{(2K_{F,b}K_{F,a})^{\epsilon/2}}{2\pi\bar{v}_F}\left(\frac{\Lambda}{K_{F,a}}\right)^{\epsilon/2} FV^2 + \ldots$$

Flow and Fixed Point

Flow

$$\frac{d}{d\log\Lambda}\Gamma = 0 \quad \Rightarrow \quad \frac{d}{d\log\Lambda}V = \frac{(2K_{F,b}K_{F,a})^{\epsilon/2}}{2\pi\bar{v}_F}\left(\frac{\Lambda}{K_{F,a}}\right)^{\epsilon/2} F V^2 + \ldots$$

Define a dimensionless coupling, $\quad g = \frac{2(2K_{F,b}K_{F,a})^{\epsilon/2}}{2\pi\bar{v}_F}\left(\frac{\Lambda}{K_{F,a}}\right)^{\epsilon/2} V$

$$\beta(g) = -\frac{d}{d\log\Lambda}g = -\frac{\epsilon}{2}g - Fg^2 + \ldots$$

Flow and Fixed Point

Flow

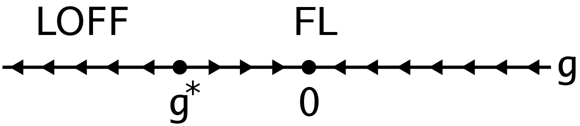
$$\frac{d}{d\log\Lambda}\Gamma = 0 \quad \Rightarrow \quad \frac{d}{d\log\Lambda}V = \frac{(2K_{F,b}K_{F,a})^{\epsilon/2}}{2\pi\bar{v}_F}\left(\frac{\Lambda}{K_{F,a}}\right)^{\epsilon/2} FV^2 + \ldots$$

Define a dimensionless coupling, $\quad g = \frac{2(2K_{F,b}K_{F,a})^{\epsilon/2}}{2\pi\bar{v}_F}\left(\frac{\Lambda}{K_{F,a}}\right)^{\epsilon/2} V$

$$\beta(g) = -\frac{d}{d\log\Lambda}g = -\frac{\epsilon}{2}g - Fg^2 + \ldots$$

Non–trivial fixed point

$$g^\star = -\frac{\epsilon}{2F}$$



Fixed Point Properties

Limits

$$P \to 0 \qquad \Rightarrow \qquad g^\star = -\frac{\epsilon}{2F} \approx -\frac{\epsilon}{2}\left(\frac{K_{F,a} - K_{F,b}}{K_{F,a}}\right)^{\epsilon/2}$$

$$P \to 1 \qquad \Rightarrow \qquad g^\star = K_{F,a}/2K_{F,b} \gg 1$$

Fixed Point Properties

Limits

$$P \to 0 \qquad \Rightarrow \qquad g^\star = -\frac{\epsilon}{2F} \approx -\frac{\epsilon}{2}\left(\frac{K_{F,a} - K_{F,b}}{K_{F,a}}\right)^{\epsilon/2}$$

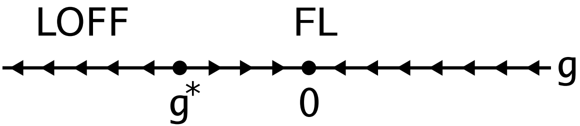
$$P \to 1 \qquad \Rightarrow \qquad g^\star = K_{F,a}/2K_{F,b} \gg 1$$

Integrated flow for weak imbalance

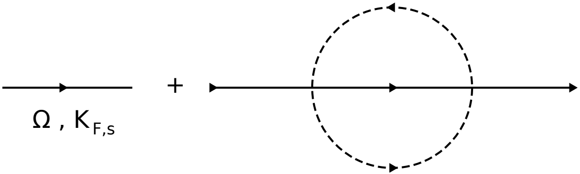
$|g| \ll |g^\star| \quad \Rightarrow \quad g = g_0(\Lambda/\Lambda_0)^{\epsilon/2} \quad \Rightarrow V$ independent of Λ

Expected result for couplings when $P = 0$ and $Q \neq 0$

In contrast $g = g^\star \quad \Rightarrow V \sim -\Lambda^{-\epsilon/2}$



Self Energy



Retarded self energy

$$\Sigma_s^R(|\mathbf{k}| = K_{F,s}, \Omega) = \frac{g^2}{4} 2\Lambda \bar{v}_F \left(\tilde{\Omega} \frac{|\tilde{\Omega}|^{\epsilon} - 1}{\epsilon} - i\pi |\tilde{\Omega}|^{1+\epsilon} \right) + \mathcal{O}(g^3)$$

$$\tilde{\Omega} = \Omega / 2\Lambda \bar{v}_F$$

Propagator and Residue

Greens function

$$G_s(|\mathbf{k}| = K_{F,s}, \Omega) = \frac{Z}{\Omega - \Sigma_s^R(|\mathbf{k}| = K_{F,s}, \Omega)}$$

$$\frac{d \ln Z}{d \ln \Lambda} = -\frac{g}{2\epsilon}\beta(g) = \frac{g^2}{4} + \mathcal{O}(g^3),$$

Propagator and Residue

Greens function

$$G_s(|\mathbf{k}| = K_{F,s}, \Omega) = \frac{Z}{\Omega - \Sigma_s^R(|\mathbf{k}| = K_{F,s}, \Omega)}$$

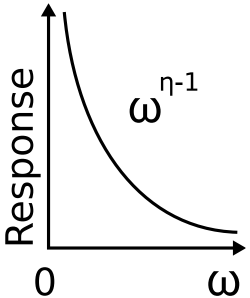
$$\frac{d \ln Z}{d \ln \Lambda} = -\frac{g}{2\epsilon}\beta(g) = \frac{g^2}{4} + \mathcal{O}(g^3),$$

At the fixed point

No pole! Residue has vanished

$$G_s(|\mathbf{k}| = K_{F,s}, \Omega) \sim \omega^{\eta-1},$$

$$\eta = \epsilon^2/16F^2$$



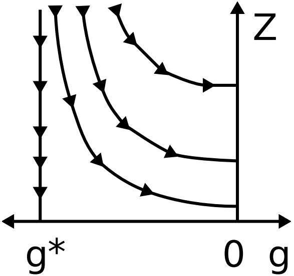
Residue

Z flow

Linearise about fixed point to get

$$Z \sim (g_0 - g^\star)^{\epsilon/8F^2}$$

Dependence on start of flow



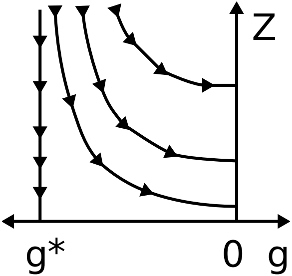
Residue

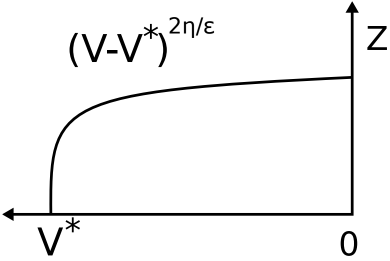
Z flow

Linearise about fixed point to get

$$Z \sim (g_0 - g^{\star})^{\epsilon/8F^2}$$

Dependence on start of flow





Two Species, with Spin

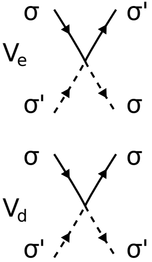
Interaction

$$H_{int} = \sum_{\sigma,\sigma'} \big[(V_{e\|}\delta_{\sigma,\sigma'} + V_{e\perp}\delta_{\sigma,-\sigma'})\psi^\dagger_{b,\sigma}\psi^\dagger_{a,\sigma'}\psi_{a,\sigma}\psi_{b,\sigma'}$$
$$+(V_{d\|}\delta_{\sigma,\sigma'} + V_{d\perp}\delta_{\sigma,-\sigma'})\psi^\dagger_{b,\sigma}\psi^\dagger_{a,\sigma'}\psi_{a,\sigma'}\psi_{b,\sigma}\big]$$

Heisenberg Interaction

Set $V_{\|} = V_{e\|} + V_{d\|}$, $V_{\|} = -V_{d\perp}$, then

$$J_z = 4V_{\|} \text{ and } J_{xy} = 2V_{e\perp}$$



Repulsive Fixed Point

Flows

$$\beta(g_{\|}) = -\frac{\epsilon}{2} g_{\|} + \frac{1}{2} g_{e\perp}^2$$
$$\beta(g_{d\perp}) = -\frac{\epsilon}{2} g_{d\perp} - \frac{1}{2} g_{e\perp}^2$$
$$\beta(g_{e\perp}) = -\frac{\epsilon}{2} g_{e\perp} - g_{e\perp}(g_{d\perp} - g_{\|})$$

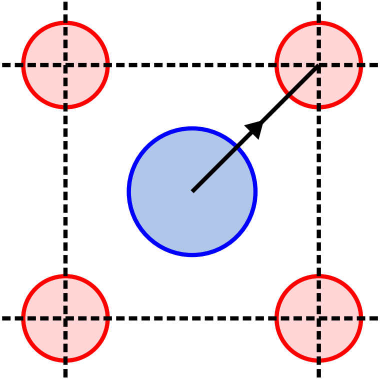
First two equations are equivalent if $V_{\|} = -V_{d\perp}$
Repulsive fixed point $(g_{\|}^{\star}, g_{e\perp}^{\star}) = (\epsilon/4, \epsilon/2)$

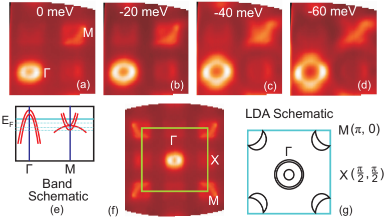
Spin density wave order

$$J_z^{\star} = J_{xy}^{\star} = \epsilon \pi \bar{v}_F / (K_{F,b} \Lambda)^{\epsilon/2}$$

Hole Pockets

e.g. Iron Pnictides





Y. Xia *et al.* 2009

Spin density wave order

$$J_z^\star = -J_{xy}^\star = -\epsilon\pi\bar{v}_F/(K_{F,b}\Lambda)^{\epsilon/2}$$

Summary

- RG flow for imbalanced Fermi gas in $d = 1 + \epsilon$
- Spinless case
  - Finite momentum pairing
  - Vanishing residue at fixed point
  - Imbalance dependent critical exponent
- Spinful cases, SDW order