The Fermion Bag Approach

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Outline

- Motivation
- Fermion path integrals and world lines
  - Free Fermions and Determinants
  - Interactions : Conventional versus Fermion Bags
- Adding fermion Chemical Potential
- Chemical Potential, Flavors, Many-Body Forces
- Conclusions

Motivation

- Sign problems hinder progress.
- QCD sign problem difficult. Try simpler theories.
- Four fermion field theories. Applications in Nuclear and Condensed matter physics.
- Conventional method : Hubbard Stratanovich approach restrictive. Three-body force difficult!
- Need fresh ideas.

Fermion Path Integrals

Partition function

$$Z = \mathrm{Tr}\left(e^{-H/T}\right) = \int [d\bar{\psi}\; d\psi]\; e^{-S(\psi,\bar{\psi})}$$

What does a Grassmann integral even mean?

Statement in the Literature :

"... there is no way to represent Grassmann variables on a computer, so we integrate them away....."

Grassmann Variables $\longrightarrow$ Fermion World lines

Free Fermions

Action :

$$S(\psi,\bar{\psi}) = -\sum_{<ij>} \{\eta_{ij}\bar{\psi}_i\psi_j + \eta_{ji}\bar{\psi}_j\psi_i\} - m\sum_i \bar{\psi}_i\psi_i$$

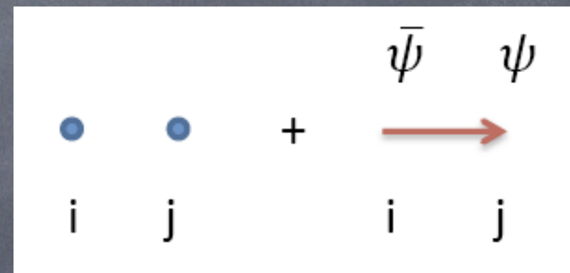
Partition function

$$Z = \mathrm{Tr}\left(e^{-H/T}\right) = \int [d\bar{\psi}\, d\psi]\, e^{-S(\psi,\bar{\psi})}$$

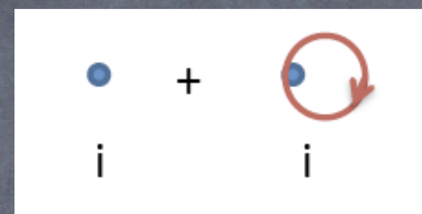
What does the partition function mean?

Pictorial Representation

$$e^{\eta_{ij}\bar{\psi}_i\psi_j} = 1 + \eta_{ij}\bar{\psi}_i\psi_j \quad = $$



$$e^{m\bar{\psi}_i\psi_i} = 1 + m\bar{\psi}_i\psi_i \quad = $$

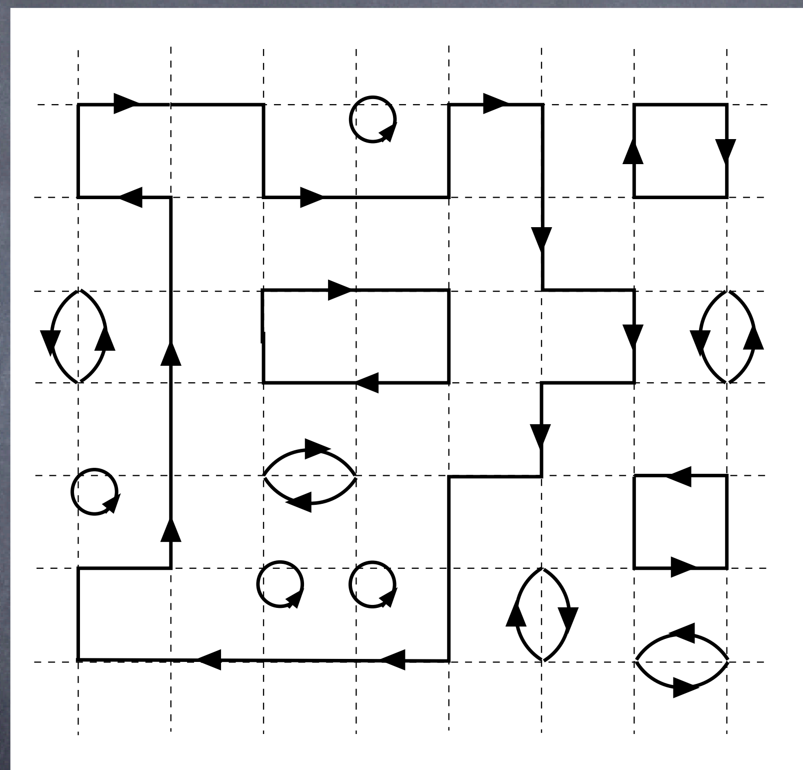


Each lattice site has one incoming and one outgoing line

Grassmann ordering leads to a negative sign for every closed loop.

Sign factors from local phases

Fermion world line configuration



Thus for free fermions

$$Z = \int [d\psi\, d\bar{\psi}]\, e^{\bar{\psi} M \psi} = \sum_{[C]} \mathrm{Sign}\,([C]) W([C]) \quad = \quad \mathrm{Det}(M)$$

Determinant of a matrix
can be thought of as a sum over
fermion world line configurations!

Massless Thirring Model

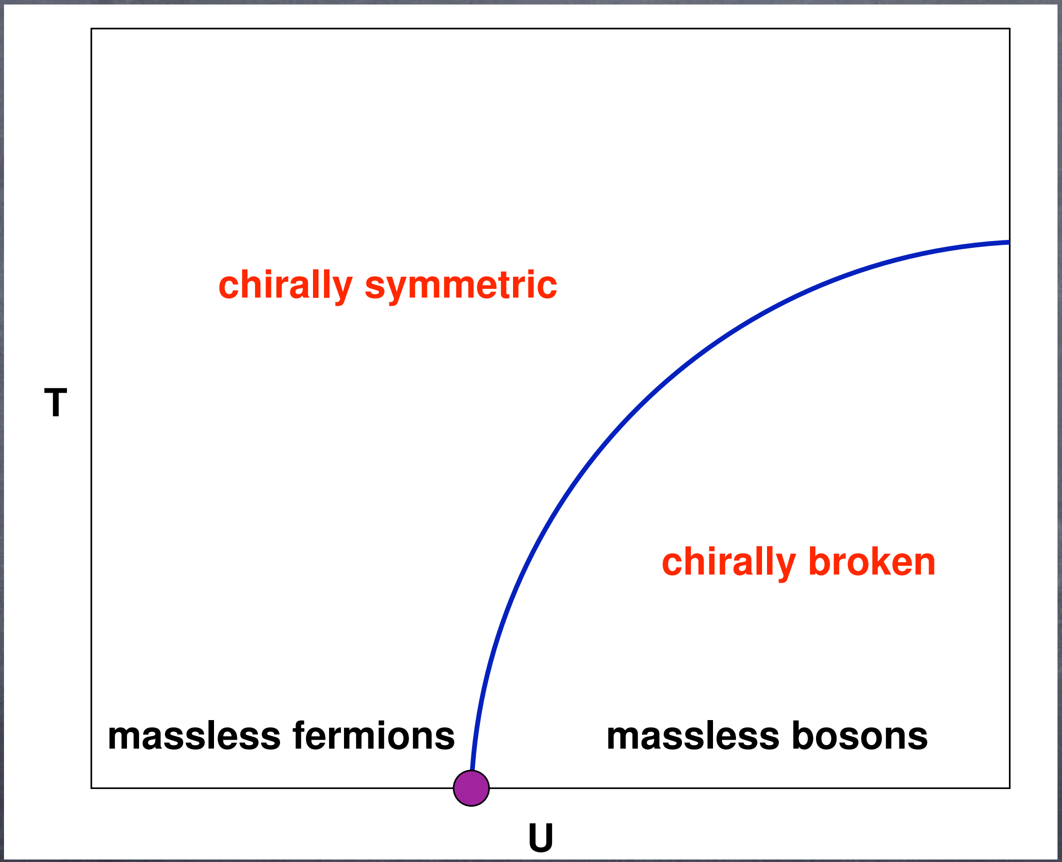
Action

$$S(\psi,\overline{\psi}) \;=\; -\sum_{\langle ij\rangle}\left[\frac{1}{2}\eta_{ij}(\overline{\psi}_i\psi_j-\overline{\psi}_j\psi_i)\;-\;U\overline{\psi}_i\psi_j\overline{\psi}_j\psi_i\right]$$

Model contains a U(1) chiral symmetry

and a quantum phase transition at $U_c \approx 0.26$

phase diagram



Conventional Approach

action

$$S(\psi,\overline{\psi}) \;=\; -\sum_{\langle ij\rangle}\left[\frac{1}{2}\eta_{ij}(\overline{\psi}_i\psi_j-\overline{\psi}_j\psi_i)\;-\;U\overline{\psi}_i\psi_j\overline{\psi}_j\psi_i\right]$$

Hubbard Stratanovich transformation

$$\mathrm{e}^{-U\;\overline{\psi}_i\psi_j\;\overline{\psi}_j\psi_i} \;=\; \int\frac{d\phi}{2\pi}\;\mathrm{e}^{\sqrt{U}\eta_{ij}[\mathrm{e}^{i\phi}\overline{\psi}_i\psi_j\;-\;\mathrm{e}^{-i\phi}\overline{\psi}_j\psi_i]}$$

Free fermions in the background field

$$S(\psi,\overline{\psi},\phi) \;=\; -\sum_{\langle ij\rangle}\frac{1}{2}\eta_{ij}\left\{\,[1+\sqrt{4U}\mathrm{e}^{i\phi_{ij}}]\;\overline{\psi}_i\psi_j-[1+\sqrt{4U}\mathrm{e}^{-i\phi_{ij}}]\;\overline{\psi}_j\psi_i\right\}$$

“integrating the fermions out”

$$Z = \int [d\phi]\, \mathrm{e}^{-S[\psi,\overline{\psi},\phi]} = \int [d\phi]\, \mathrm{Det}(M([\phi]))$$

$$M([\phi])_{ij} = -\sum_{\alpha} \frac{1}{2}\eta_{ij} \left\{ [1+\sqrt{4U}\mathrm{e}^{i\phi_{ij}}]\, \delta_{i,j+\alpha} - [1+\sqrt{4U}\mathrm{e}^{-i\phi_{ij}}]\, \delta_{i+\alpha,j} \right\}$$

M([ϕ]) is a volume x volume matrix

Example: On a 40^3 lattice M is a 64,000 x 64,000 matrix

M is anti-hermitian (purely imaginary eigenvalues)

$M \gamma_5 + \gamma_5 M = 0$ (eigenvalues come with opposite signs)

Det(M) ≥ 0

Algorithmic problems at large U

A large number of
singular points (of measure zero)
in the space of [ϕ] configurations

The matrix M develops
a density of small eigenvalues

Determinantal Monte Carlo expensive

Hybrid Monte Carlo inefficient

Fermion Bag Approach

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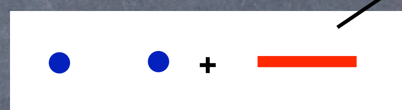
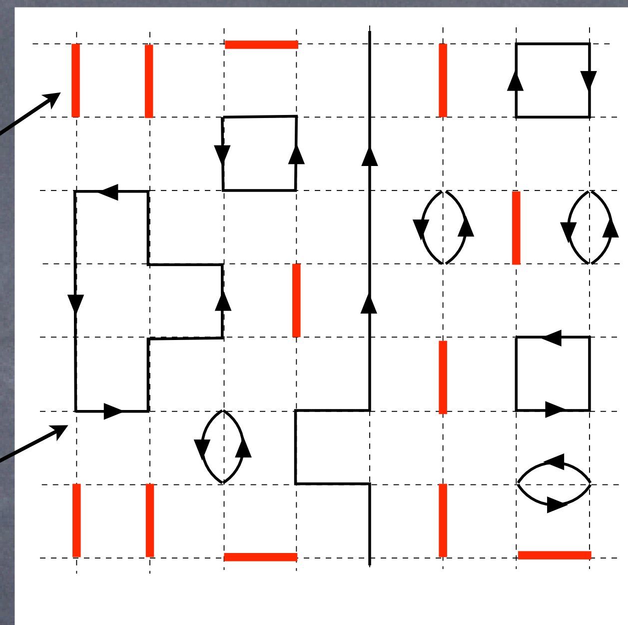
Expand the interaction terms

$$Z = \int [d\phi]\ \mathrm{e}^{-S_0[\psi,\overline{\psi}]} \prod_{\langle ij \rangle} \left(1 - U\ \overline{\psi}_i \psi_j \overline{\psi}_i \psi_j\right)$$





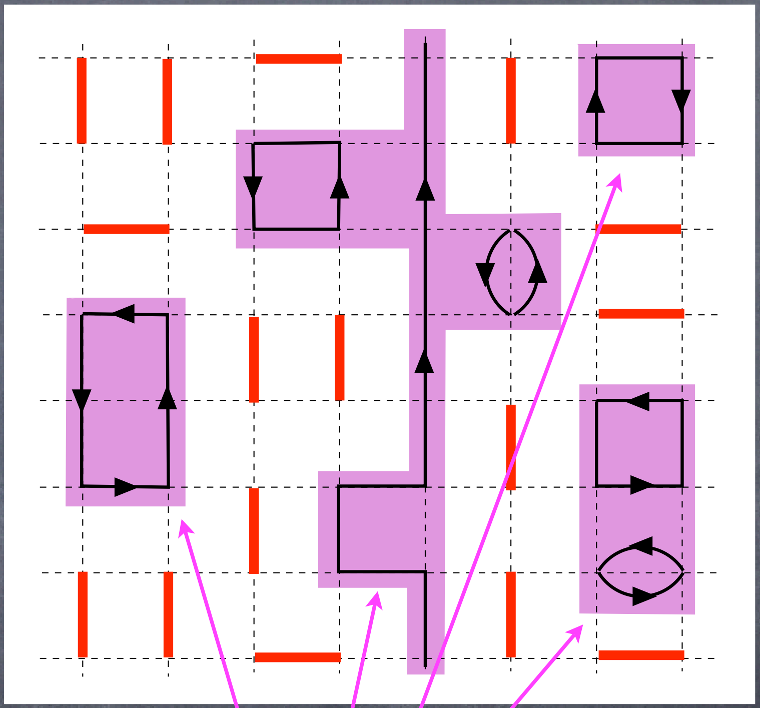
Free fermion

free fermions can only hop on lattice sites not "taken" by interactions example of a bag configuration

partition function

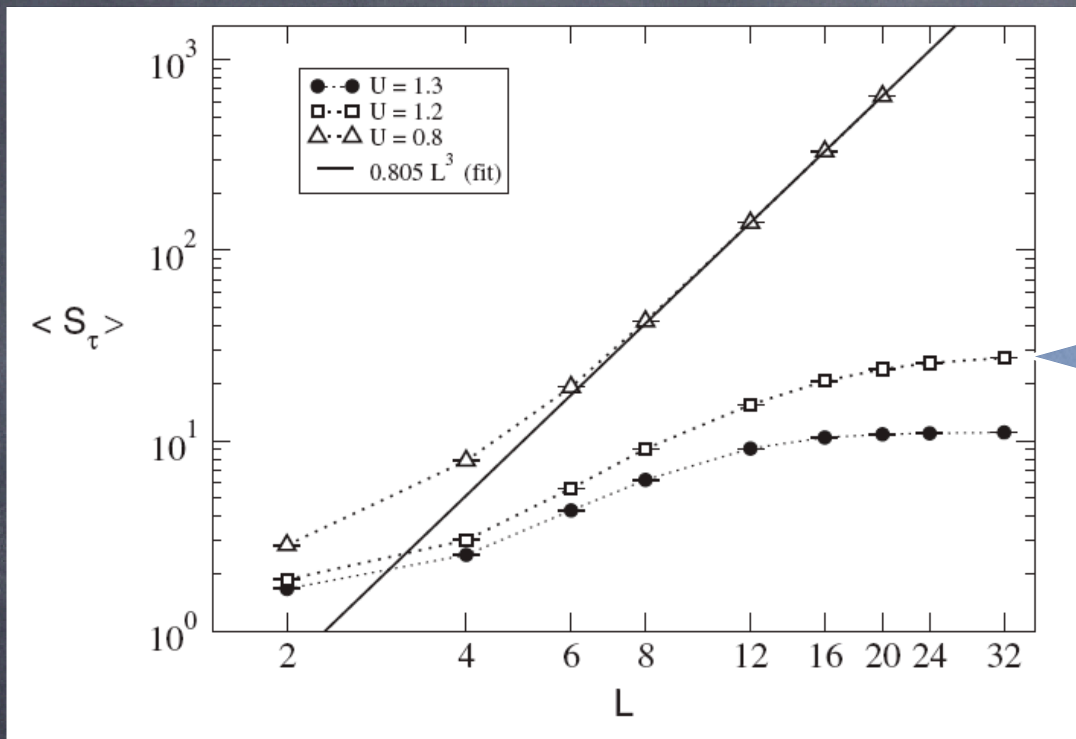
$$Z = \sum_{[b]} U^{N_B} \prod_{\text{Bags}} \text{Det}(Q([\text{Bag}]))$$

Det(Q) ≥ 0



bags

Average Bag Size



U = 1.2

30 x 30 matrices

For small U, the bag size increases with volume and the approach becomes inefficient

General Idea of the Fermion Bag Approach

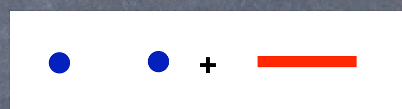
Identify the minimal set of
fermion degrees of freedom which
“interfere” and contribute
a (dominantly?) positive weight

This set of degrees of freedom we call
a “fermion bag”

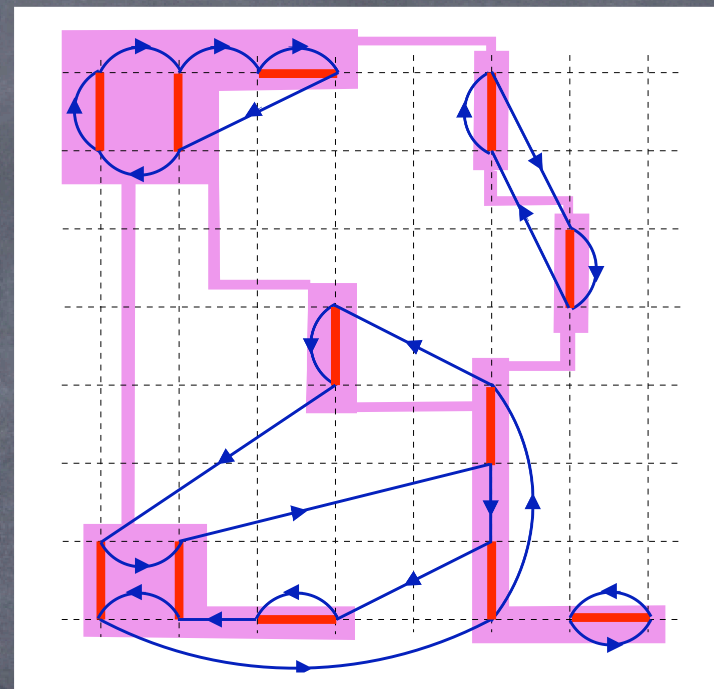
Fermion Bag Approach II

Again expand the interaction terms

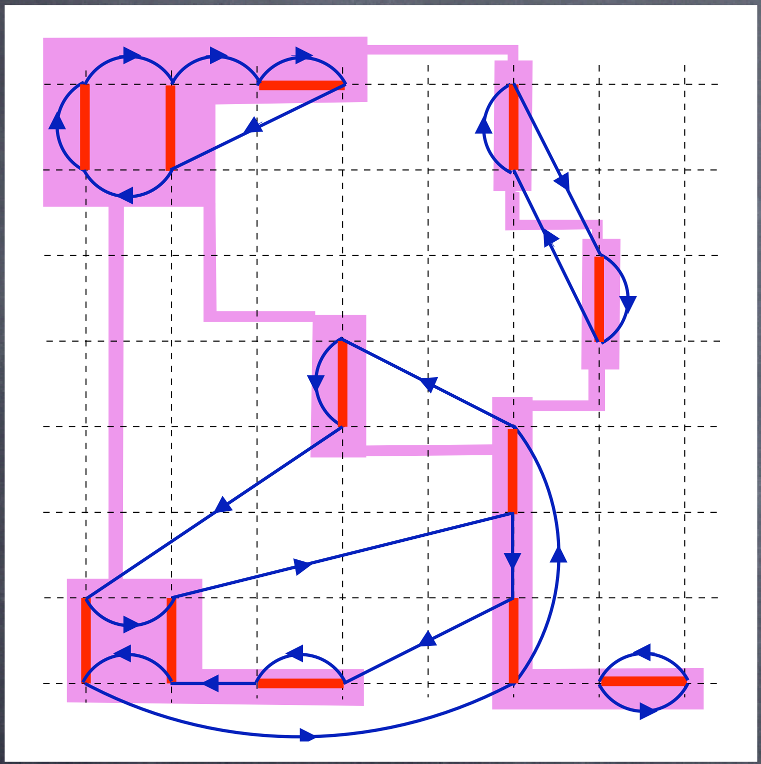
$$Z = \int [d\phi]\, \mathrm{e}^{-S_0[\psi,\overline{\psi}]} \prod_{\langle ij \rangle} \Big(1 - U\, \overline{\psi}_i \psi_j \overline{\psi}_i \psi_j\Big)$$



All fermions involved in the interaction form the bag



Use "Wick's Theorem" to compute the weight



$$Z = \sum_{[b]} U^{N_B} \left\{ \mathrm{Det}(G([b])) \right\}^2$$

$N_B \times N_B$ matrix of propagator

N_B grows with volume at a fixed value of U

At U = 0.25 and 40^3 lattice $N_B \approx 8000$

Adding Chemical Potential

Conventional approach

At large U sign problem emerges like in QCD

Det(M) becomes complex

"Silver Blaze Problem" : Even though fermions are massive
the sign problem arises even at small μ

Fermion bag approach

Det(Q) remains real but can be negative! So sign problem remains

Silver Blaze Problem Solved!
μ enters only through bags that "wrap" in temporal direction.

Chemical Potential, Flavors and Manybody forces

Action

$$S(\psi,\overline{\psi}) \;=\; -\sum_{\langle ij\rangle}\left[\frac{1}{2}\eta_{ij}(\mathrm{e}^{\mu_{ij}}\overline{\psi}_i\psi_j - \mathrm{e}^{-\mu_{ij}}\overline{\psi}_j\psi_i) \;+\Big(-U\overline{\psi}_i\psi_j\overline{\psi}_j\psi_i\Big)^{N_f}\right]$$

Conventional

$$Z \;=\; \int\,[d\phi_1\;d\phi_2,\;...]\;\mathrm{Det}(M([\phi_1,\phi_2,...]))$$

Fermion Bag

$$Z \;=\; \sum_{[b]} U^{N_B}\;\prod_{\mathrm{Bags}}\left\{\mathrm{Det}(Q([\mathrm{Bag}]))\right\}^{N_f}$$

Conclusions

- Fermion bag approach is an alternative method to interacting fermionic theories
- The essential idea : take into account interference of "fermion world-lines"
- Some thought must be put into identifying "optimal" fermion bags.
- Chemical Potential, Number of Flavors and Many-Body forces can be handled naturally
- Leads to efficient algorithms and solutions to new sign problems.
- Many new questions and opportunities ...