Second-Order Relativistic Hydrodynamics

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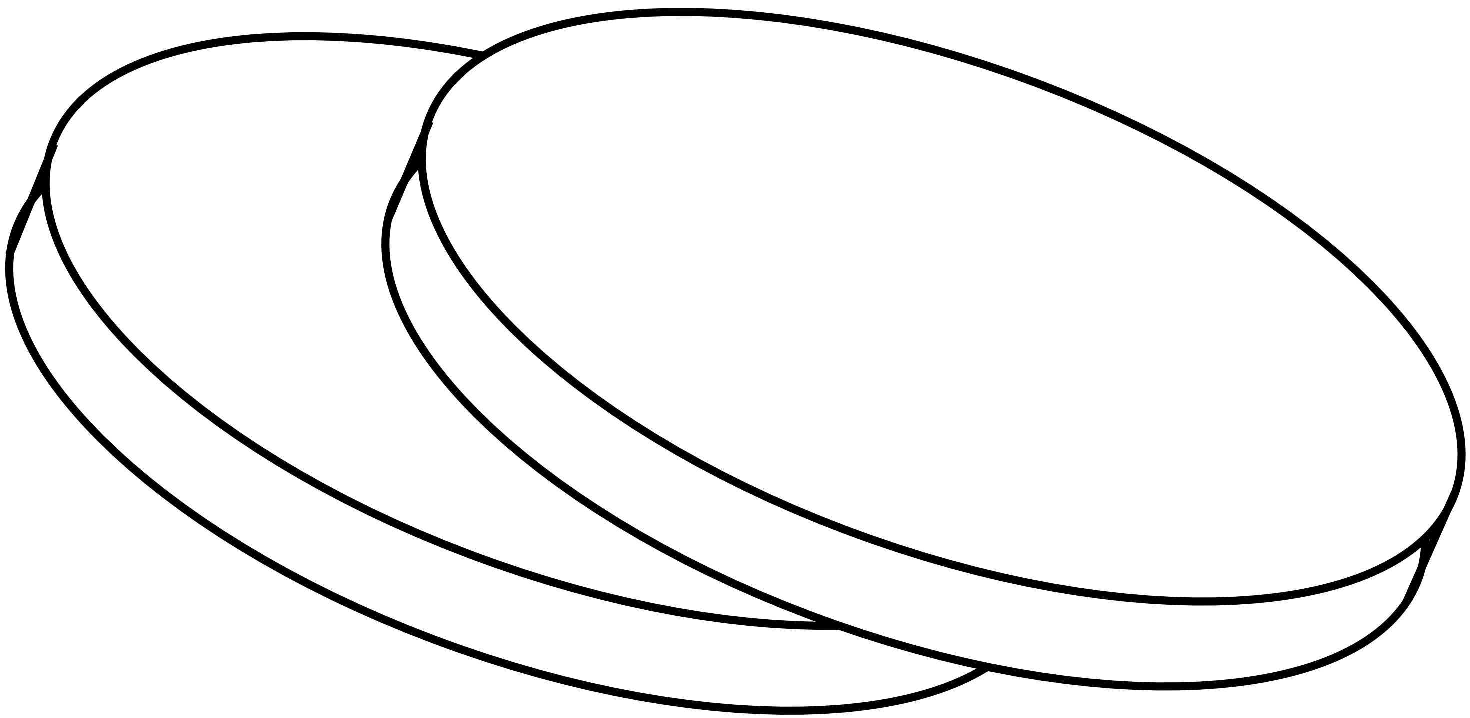
- Why do we want to do relativistic Hydro?
- Why second order hydro, and what are coefficients?
- Perturbative Calculation of Coefficients
- Kubo Relations for Coefficients
- Self-consistency: hydro's contrib. to hydro coeff.
- Conclusions

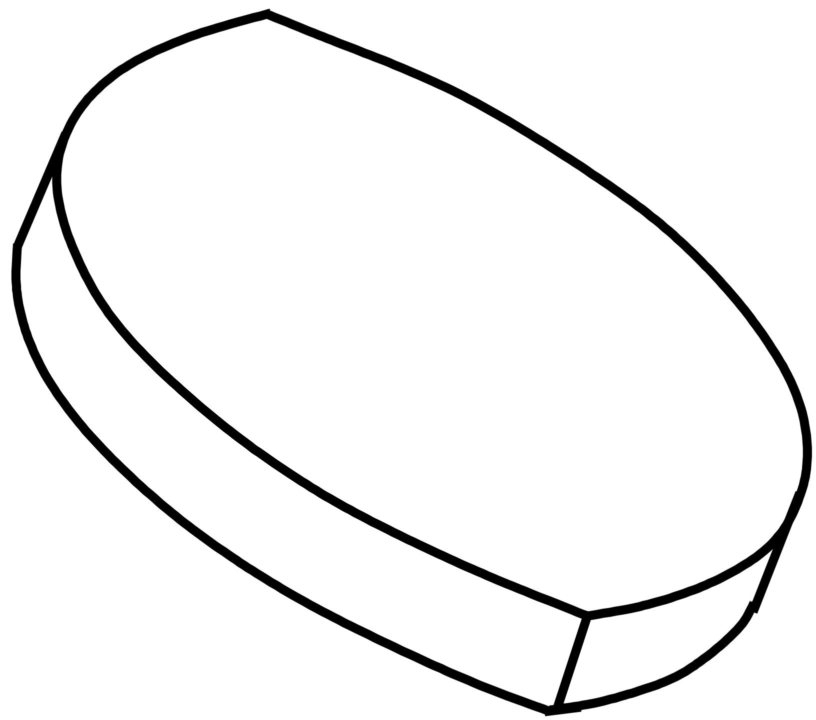
Heavy ion collisions

Accelerate two heavy nuclei to high energy, slam together.



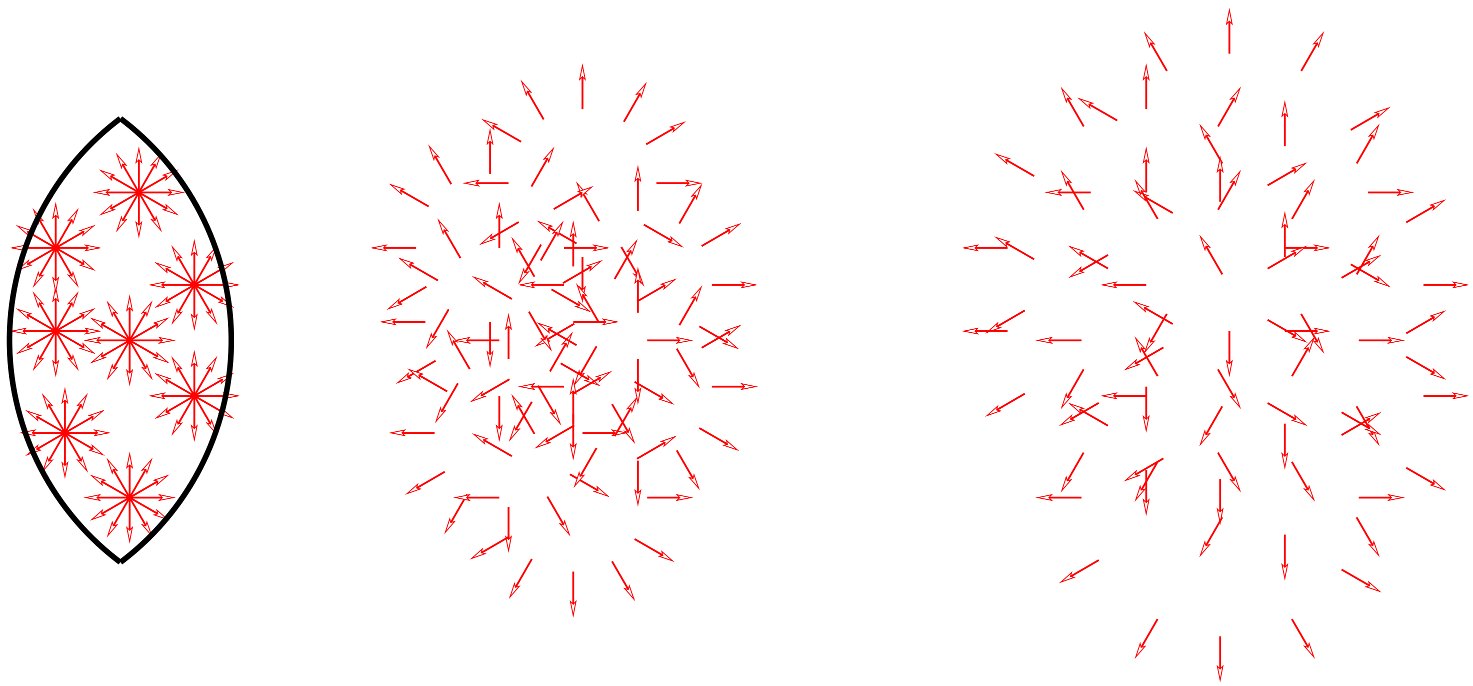
Just before: Lorentz contracted nuclei

After the scattering: region where nuclei overlapped:
"Flat almond" shaped region of $q, \bar{q}, g$ which scattered.



$\sim$2 thousand random $\mathbf{v}$ quarks$+$gluons: isotropic in xy plane

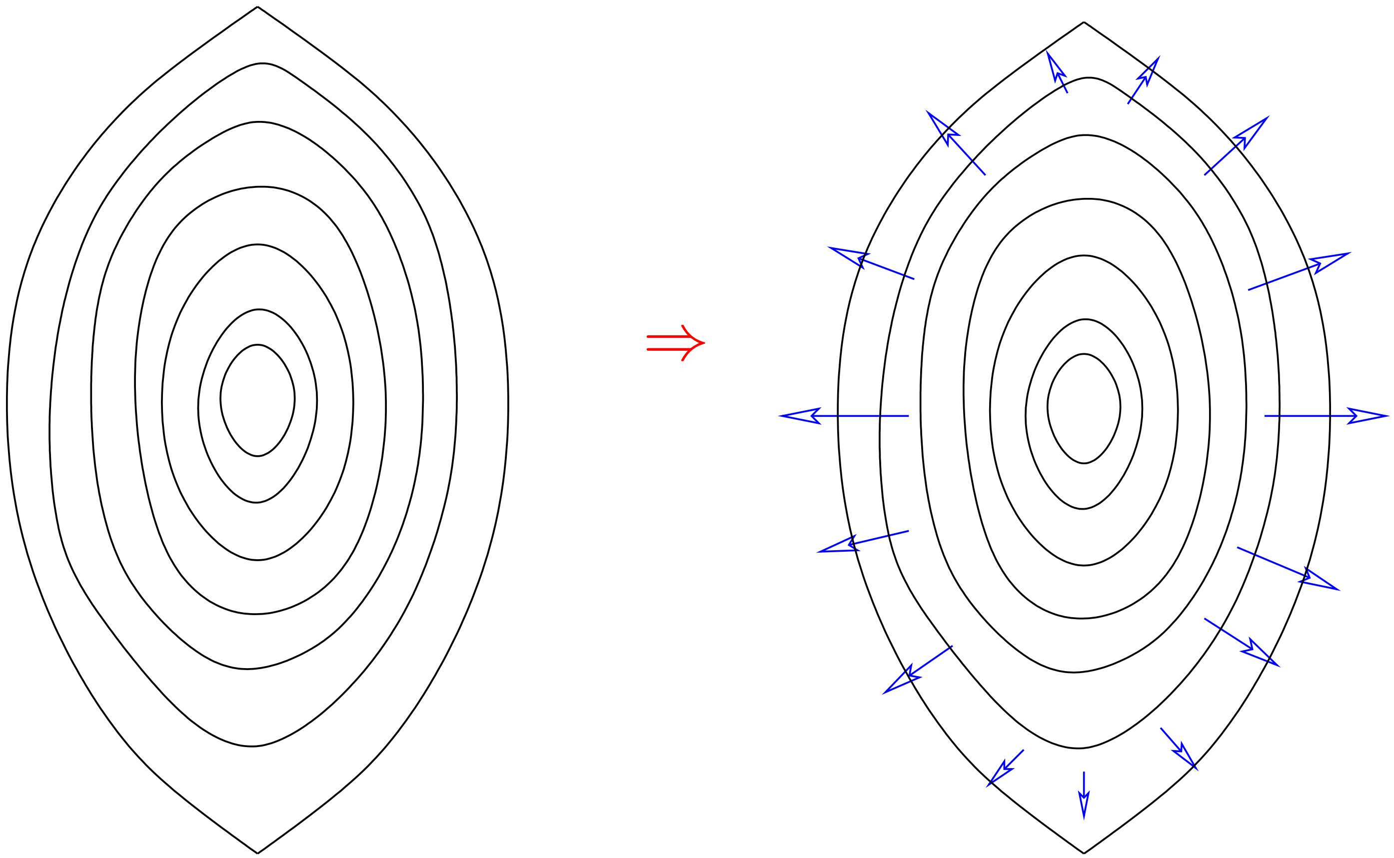
Behavior IF no re-interactions (transparency)



Just fly out and hit the detector.

Detector will see xy plane *isotropy*

local CM motions



Pressure contours $\Rightarrow$ Expansion pattern

Anisotropy leads to anisotropic (local CM motion) flow.

Free particle propagation:

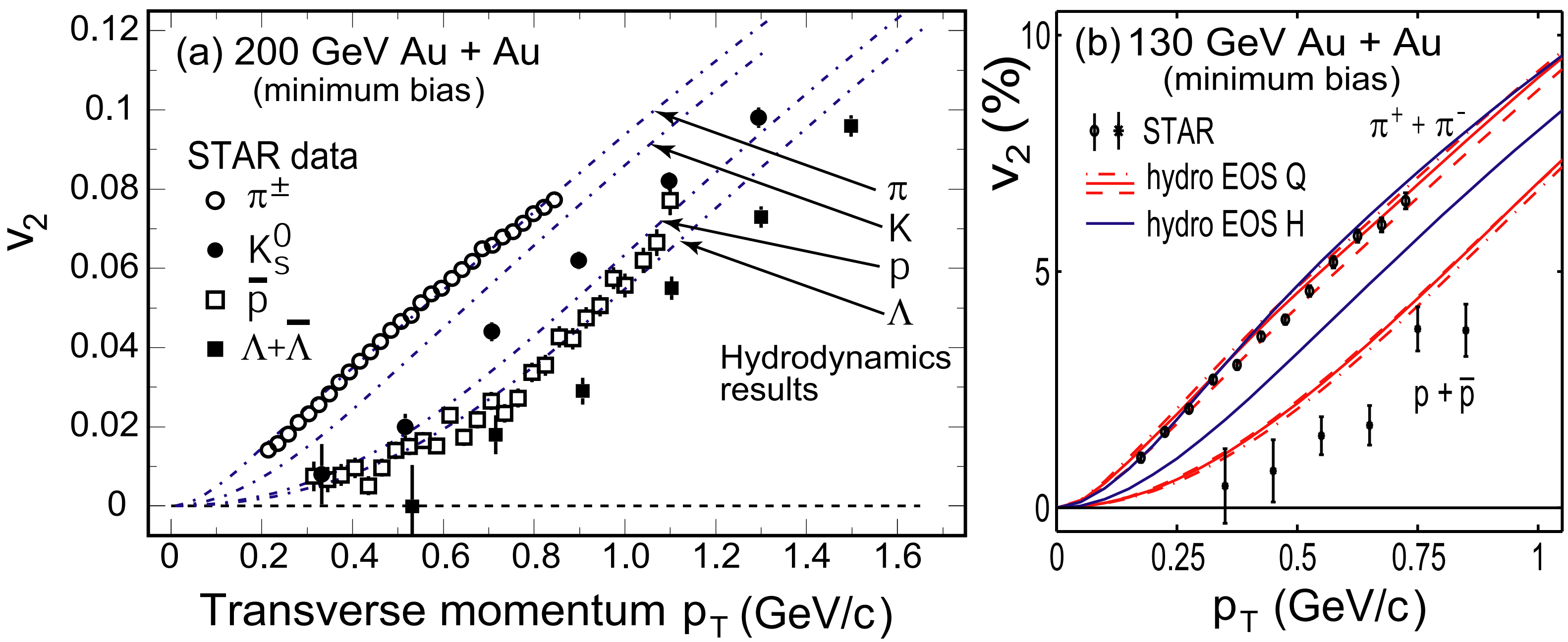
- System-average CM flow velocities $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- Must have local CM $\langle p_x^2 \rangle < \langle p_y^2 \rangle$ so total $\langle p_x^2 \rangle = \langle p_y^2 \rangle$

Efficient Equilibration:

- System-average CM flow still has $\langle v_{x,\mathrm{CM}}^2 \rangle > \langle v_{y,\mathrm{CM}}^2 \rangle$
- system changes *locally* towards $\langle T^{xx}_{\mathrm{local\ CM}} \rangle = \langle T^{yy}_{\mathrm{local\ CM}} \rangle$
- Adding these together, $\langle T^{xx}_{\mathrm{tot,labframe}} \rangle > \langle T^{yy}_{\mathrm{tot,labframe}} \rangle$

Net "Elliptic Flow" $v_2 \equiv \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2}$ measures re-interaction

Elliptic flow is measured



STAR experiment, minimum bias... .

We should try to understand it theoretically.

First try: ideal hydrodynamics (works OK!)

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{(4 equations, 10 unknowns)}$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} = \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \end{aligned}$$

depends on 4 parameters (ϵ, 3 comp of u^μ): closed.

works pretty well for heavy ions. But **quantify** corrections!

Nonideal Hydro

Assume that ideal hydro is "good starting point," look for small systematic corrections.

Near equilibrium iff $t_{\rm therm} \ll t_{\rm vary}, l_{\rm vary}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}[\partial, \epsilon, u] \\ \Pi^{\mu\nu} &= \mathcal{O}(\partial u, \partial\epsilon) + \mathcal{O}(\partial^2 u, (\partial u)^2, \ldots) + \mathcal{O}(\partial^3 \ldots) \end{aligned}$$

For Conformal theory $T^\mu_\mu = 0 = \Pi^\mu_\mu$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}\,, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right)$$

Coefficient η is shear viscosity.

So why not consider (Navier-Stokes)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \quad ?$$

Because in **relativisitc** setting, it is

- Acausal: shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976

- Unstable: $v > c$ prop + non-uniform flow velocity $\rightarrow$ propagate from future into past, exponentially growing solutions. Hiscock 1983

Problem: short length scales, $\eta|\sigma| \sim P$. Numerics must treat these scales (or there's "numerical viscosity")

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\pi\, u^\alpha \partial_\alpha \sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \to -\Pi$ in order-2 term:

$$\tau_\pi\, u^\alpha \partial_\alpha \Pi^{\mu\nu} \equiv \tau_\pi\, \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_π)

But why only one 2'nd order term???

Second order hydrodynamics

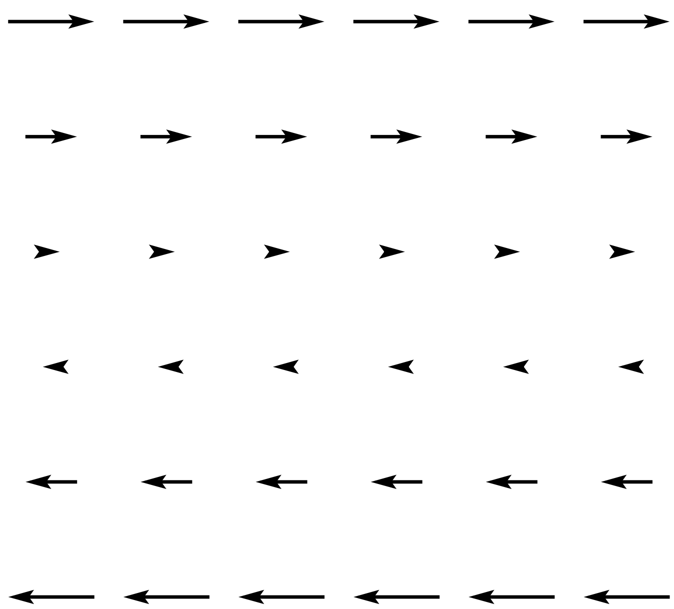
It is more consistent to include all possible 2'nd order terms.
Assume *conformality* and *vanishing chem. potentials*:
5 possible terms Baier *et al*, [arXiv:0712.2451]

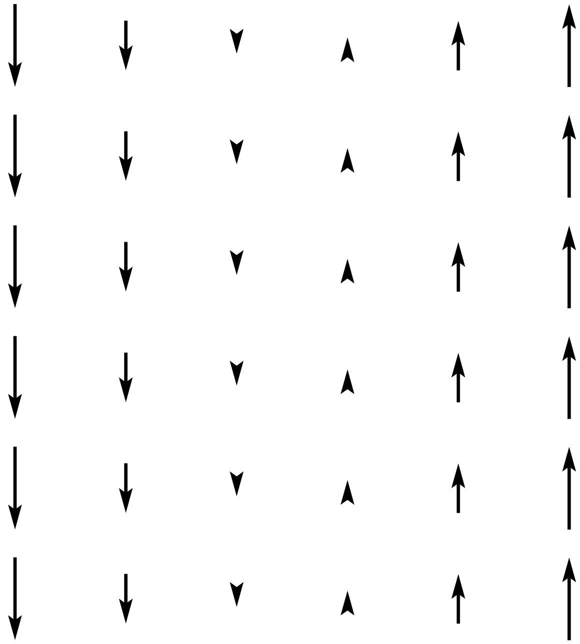
$$
\begin{aligned}
\Pi^{\mu\nu}_{2\text{ ord.}} = \; & \eta\tau_\pi \left[u^\alpha \partial_\alpha \sigma^{\mu\nu} + \frac{1}{3} \sigma^{\mu\nu} \partial_\alpha u^\alpha \right] + \lambda_1 \left[\sigma^\mu_\alpha \sigma^{\nu\alpha} - (\text{trace}) \right] \\
& + \lambda_2 \left[\frac{1}{2} (\sigma^\mu_\alpha \Omega^{\nu\alpha} + \sigma^\nu_\alpha \Omega^{\mu\alpha}) - (\text{trace}) \right] \\
& + \lambda_3 \left[\Omega^\mu{}_\alpha \Omega^{\nu\alpha} - (\text{trace}) \right] + \kappa \left(R^{\mu\nu} - \ldots \right) , \\
\Omega_{\mu\nu} \equiv \; & \frac{1}{2} \Delta_{\mu\alpha} \Delta_{\nu\beta} (\partial^\alpha u^\beta - \partial^\beta u^\alpha) \;\; [\text{vorticity}] \, .
\end{aligned}
$$

Let's learn what we can about this theory, its 6 coeff's

Step 1: What do $\sigma^{\mu\nu}$, $\Omega^{\mu\nu}$ mean?

Consider $\partial_y v_x \neq 0$ and $\partial_x v_y \neq 0$:

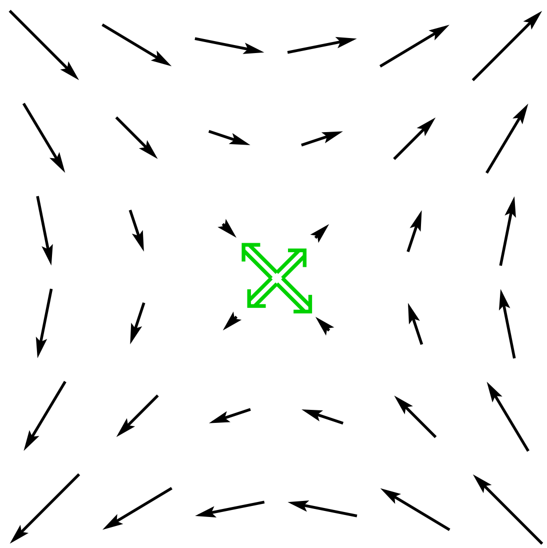




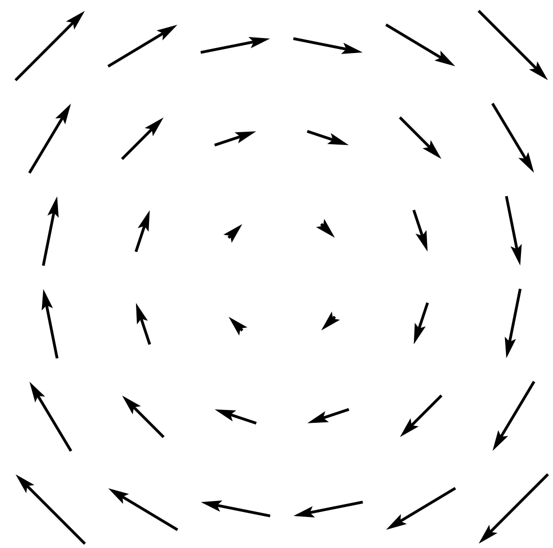
Each pattern is shear-flow. But not purely shear flow!

Step 1: What do $\sigma^{\mu\nu}$, $\Omega^{\mu\nu}$ mean?

Same-sign $\partial_x v_y = \partial_y v_x$ Opposite-sign $\partial_x v_y = -\partial_y v_x$



Shear flow



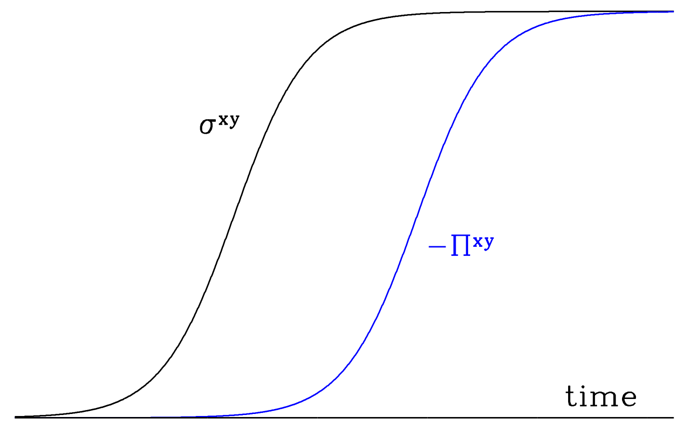
Vorticity

Two basic **local** measures of flow nonuniformity.

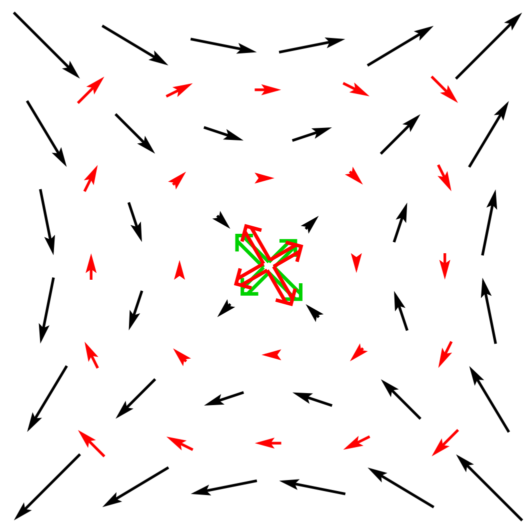
First order: $\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$ as it's symmetric!

Fluid "pushing back" against shear flow

τ_π: if shear flow $\sigma^{\mu\nu}$ "turns on", delay in $\Pi^{\mu\nu}$ "turning on"



λ_2: if shear makes $\Pi^{\mu\nu} \neq 0$, vorticity rotates $\Pi^{\mu\nu}$ axis from shear axis.
Sensible sign if $\lambda_2 < 0$ (sorry)



λ_1: some nonlinearity. λ_3: rotate about z axis$\rightarrow T^{zz}$ reduced

I would like to calculate these coefficients.

Two cases: weak coupling, strong-coupled N=4SYM

That failing, I want a rule for relating them to equilibrium field theory correlators (Kubo relation)

In any case I want to understand consistency or limitations of 2'nd order hydro theory.

Goals of remainder of the talk

Theories where I can calculate:

- Weakly coupled QCD (realistically, $\alpha_{\rm s} < 1/20!$)
- $\mathcal{N}=4$ SYM in infinite $N_{\rm c}$ and coupling limit

What I expect to find:

There should be some equilibration time scale τ.
A term with n deriv's should be $\sim P/\tau^n$.
Hence, certain ratios should be dimensionless.
Use η as my "standard" and build dim'less ratios.

If ratios robust, use as "priors": only fit η to data

QCD vs SYM comparison

η/s behaves as $1/\alpha_s^2 \ln(1/\alpha_s)$ diverges at weak coupling.

But ratios stay finite!

Ratio	QCD value	SYM value
$\frac{\eta\tau_\pi(\epsilon+P)}{\eta^2}$	5 to 5.9	2.6137
$\frac{\lambda_1(\epsilon+P)}{\eta^2}$	4.1 to 5.2	2
$\frac{\lambda_2(\epsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\epsilon+P)}{\eta^2}$	0	4
$\frac{\lambda_3(\epsilon+P)}{\eta^2}$	0	0

Good news: Not qualitatively different.

Kinetic theory relation $\lambda_2 = -2\eta\tau_\pi$ not actually general.

Kubo formulae

We want expressions which relate the transport coefficients to equilibrium correlation functions in the plasma fluct-diss

Would provide rigorous *definition* of $\eta, \lambda_{123}, \ldots$.

Example: long known that η is given by

$$\eta = \lim_{\omega \to 0} \frac{d}{d\omega} \int d^3x \, dt \, e^{i\omega t} \left\langle \left[T^{xy}(x,t) \, , \, T^{xy}(0,0) \right] \right\rangle \Theta(t)$$

Similar relations for second-order transport coefficients?

How to get Kubo relations

Find framework where I can compute $T^{\mu\nu}$ using hydro *or* using field theory, both should be valid.

Time-varying geometry does the job:

- Start at $t \ll 0$ with flat-space, equilibrium thermal system $\rho = e^{-HT}$, $g_{\mu\nu} = \eta_{\mu\nu}$
- At some time $t_0 < 0$ start deforming metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$ in such a way as to force the system to experience shear and vorticity
- Choose $h_{\mu\nu}$ small and slowly varying so you stay near equilibrium and gradient expansion, hydro are valid

Give a hydro theorist $h_{xy}(z,t)$, $h_{0x}(y)$ nonzero.

Ask them what $T^{\mu\nu}(0)$ will be.

Answer: $$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \Pi^{\mu\nu}$$

First, find ϵ, u: Hydro says

$$\nabla_\mu T^{\mu\nu} = 0 \quad \longrightarrow \quad u^\mu = (1,0,0,0) + \mathcal{O}(\partial^2)\,.$$

Then $u_\mu = (1, h_{0x}, 0, 0)$, $\Gamma^x{}_{yt}$ *etc* nonzero.
They give rise to nonzero σ^{xy}, Ω^{xy}, etc:

$$\sigma^{xy} = \partial_t h_{xy}\,, \qquad \Omega^{xy} = -\partial_y h_{0x}/2$$

Other terms $R^{\langle xy\rangle}$, $u\cdot\nabla\sigma^{xy}$ found similarly.

T^{xy} at $\mathcal{O}(h)$ and $\mathcal{O}(\partial^2)$, for $h_{xy} \neq 0$:

$$T^{xy} = -\eta \partial_t h_{xy} + \eta \tau_\pi \partial_t^2 h_{xy} - \frac{\kappa}{2} \left(\partial_t^2 h_{xy} + \partial_z^2 h_{xy} \right)$$

and T^{xy} at $\mathcal{O}(\partial^2, h^2)$ for $h_{xz}(t)$, $h_{yz}(t)$, $h_{x0}(z)$, $h_{y0}(z)$ nonzero:

$$\begin{aligned}
\Pi^{xy} = {} & \eta \partial_t (h_{xz} h_{yz}) + \frac{\kappa}{2} \left(h_{xz} \partial_t^2 h_{yz} + h_{yz} \partial_t^2 h_{xz} \right) + \lambda_1 \partial_t h_{xz} \partial_t h_{yz} \\
& + \eta \tau_\pi \left(\frac{1}{2} \partial_t h_{xz} \partial_z h_{0y} + \frac{1}{2} \partial_t h_{yz} \partial_z h_{0x} \right. \\
& \qquad \left. - \partial_t h_{xz} \partial_t h_{yz} - h_{xz} \partial_t^2 h_{yz} - h_{yz} \partial_t^2 h_{xz} \right) \\
& - \frac{\lambda_2}{4} \left(\partial_t h_{xz} \partial_z h_{0y} + \partial_t h_{yz} \partial_z h_{0x} \right) + \frac{\lambda_3}{4} \partial_z h_{0x} \partial_z h_{0y}
\end{aligned}$$

So at $\mathcal{O}(h)$ T^{xy} depends on η, τ_π, κ; at $\mathcal{O}(h^2)$, depends on all 6!

Give field theorist $h_{xy}(z,t)$, etc nonzero.

Ask them what T^{xy} will be.

$$\langle T^{\mu\nu}(t)\rangle = \mathrm{Tr}\, e^{-HT} e^{iHt} \hat{T}^{\mu\nu} e^{-iHt}\,, \quad T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\partial \sqrt{-g}\mathcal{L}}{\partial h_{\mu\nu}}$$

with $H = H[h(t')]$! Schwinger-Keldysh in $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$W \equiv \ln \int_{C=} \mathcal{D}(\Phi_1, \Phi_2, \Phi_3)\; e^{iS_1[h_1,\Phi_1] - iS_2[h_2,\Phi_2] - S_3[\Phi_3]}$$

$S_1[h_1]$, $S_2[h_2]$ depend on independent fields *and metrics!*

$$T_1 = \frac{-2i\delta W}{\delta h_1}\,, \qquad T_2 = \frac{+2i\delta W}{\delta h_2}$$

Introduce average and difference variables:

$$h_r = \tfrac{h_1+h_2}{2}\,,\ h_a = h_1 - h_2\,,\quad T_r = \tfrac{T_1+T_2}{2}\,,\ T_a = T_1 - T_2$$

Note, due to signs $e^{iS_1 - iS_2}$, $T_r = \frac{-2i\delta W}{\delta h_a}\,,\quad T_a = \frac{-2i\delta W}{\delta h_r}$. Take $\delta/\delta h_a \to \langle T\rangle$. Then set $h_a = 0$, $h_r = h$, expand in h:

$$\begin{aligned}
\langle T^{\mu\nu}\rangle_h &= G_r^{\mu\nu}(0) - \frac{1}{2}\int d^4x G_{ra}^{\mu\nu,\alpha\beta}(0,x) h_{\alpha\beta}(x) \\
&\quad + \frac{1}{8}\int d^4x d^4y G_{raa}^{\mu\nu,\alpha\beta,\gamma\delta}(0,x,y) h_{\alpha\beta}(x) h_{\gamma\delta}(y)
\end{aligned}$$

$$\begin{aligned}
G_{ra\ldots}^{\mu\nu,\alpha\beta\ldots}(0,x\ldots) &\equiv \left.\frac{(-i)^{n-1}(-2i)^n \delta^n W}{\delta g_{a,\mu\nu}(0)\delta g_{r,\alpha\beta}(x)\ldots}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \\
&= (-i)^{n-1}\left\langle T_r^{\mu\nu}(0) T_a^{\alpha\beta}(x)\ldots\right\rangle + \text{c.t.}
\end{aligned}$$

Now equate the hydro, field theorist answers:

$$
\begin{aligned}
T^{xy} &= -\eta\partial_t h_{xy} + \eta\tau_\pi\partial_t^2 h_{xy} - \frac{\kappa}{2}\left(\partial_t^2 h_{xy} + \partial_z^2 h_{xy}\right) \\
&= -\int d^4x\, G_{ra}^{xy,xy}(0,x)h_{xy}(x)
\end{aligned}
$$

Introduce Fourier transform

$$
G_{ra}^{xy,xy}(\omega,k) = \int d^4x e^{i(\omega t - kz)} G_{ra}^{xy,xy}(0,x)
$$

and use that h slowly varying, find BRSSS 0712.2451

$$
\begin{aligned}
\eta &= -i\partial_\omega G_{ra}^{xy,xy}(\omega,k)|_{\omega=0=k}\ , \\
\kappa &= -\partial_{k_z}^2 G_{ra}^{xy,xy}(\omega,k)|_{\omega=0=k}\ , \\
\eta\tau_\pi &= \frac{1}{2}\left(\partial_\omega^2 G_{ra}^{xy,xy}(\omega,k) - \partial_{k_z}^2 G_{ra}^{xy,xy}(\omega,k)\right)\Big|_{\omega=0=k}\ .
\end{aligned}
$$

Repeat for T^{xy} and $\mathcal{O}(h^2)$ terms:

$$\begin{aligned}
\Pi^{xy} &= \eta\partial_t(h_{xz}h_{yz}) + \frac{\kappa}{2}\left(h_{xz}\partial_t^2 h_{yz} + h_{yz}\partial_t^2 h_{xz}\right) + \lambda_1\partial_t h_{xz}\partial_t h_{yz} \\
&\quad +\eta\tau_\pi\left(\frac{1}{2}\partial_t h_{xz}\partial_z h_{0y} + \frac{1}{2}\partial_t h_{yz}\partial_z h_{0x}\right. \\
&\qquad \left. - \partial_t h_{xz}\partial_t h_{yz} - h_{xz}\partial_t^2 h_{yz} - h_{yz}\partial_t^2 h_{xz}\right) \\
&\quad -\frac{\lambda_2}{4}\left(\partial_t h_{xz}\partial_z h_{0y} + \partial_t h_{yz}\partial_z h_{0x}\right) + \frac{\lambda_3}{4}\partial_z h_{0x}\partial_z h_{0y} \\
&= \int d^4x d^4y \left(G_{raa}^{xy,xz,yz}(0,x,y)h_{xz}(x)h_{yz}(y)\right. \\
&\qquad +G_{raa}^{xy,xz,0y} h_{xz}(x)h_{0y}(y) + G_{raa}^{xy,yz,0x} h_{yz}(x)h_{0x}(y) \\
&\qquad \left.+G_{raa}^{xy,0x,0y}(0,x,y)h_{0x}(x)h_{0y}(y)\right)
\end{aligned}$$

Introduce Fourier transforms again:

$$G_{raa}^{xy,xz,0y}(p,q) \equiv \int d^4x d^4y e^{-i(p\cdot x+q\cdot y)} G_{raa}^{xy,xz,0y}(0,x,y) \quad etc$$

Read off 2'nd order Kubo relations:

$$\begin{aligned}
\lambda_1 &= \eta\tau_\pi - \lim_{p^t,q^t\to 0} \frac{\partial^2}{\partial p^t \partial q^t} \lim_{\mathbf{p},\mathbf{q}\to 0} G_{raa}^{xy,xz,yz}(p,q) \\
\lambda_2 &= 2\eta\tau_\pi - 4 \lim_{p^t,\mathbf{q}\to 0} \frac{\partial^2}{\partial p^t \partial q^z} \lim_{\mathbf{p},q^t\to 0} G_{raa}^{xy,xz,0y}(p,q) \\
\lambda_3 &= -4 \lim_{\mathbf{p},\mathbf{q}\to 0} \frac{\partial^2}{\partial p^z \partial q^z} \lim_{p^t,q^t\to 0} G_{raa}^{xy,0x,0y}(p,q)\,.
\end{aligned}$$

Nature of κ and λ_3

κ and λ_3 have Kubo relations $\mathbf{NOT}$ involving ∂_t's.
May (must!) set frequency $\omega = 0$ from outset:

$$\begin{aligned} \kappa &= -\lim_{\vec{q}\to 0} \frac{\partial^2}{\partial q_z^2} G_{ra}^{xy,xy}(\vec{q}, \omega = 0) \\ \lambda_3 &= -6 \lim_{\vec{p},\vec{q}\to 0} \frac{\partial^2}{\partial p_y \partial q_y} G_{raa}^{xx,0x,0x}(\vec{p}, \omega_p = 0, \vec{q}, \omega_q = 0) \end{aligned}$$

But $G_{ra\ldots}(\omega = 0) = (-)^{n-1} G_{\mathrm{E}}(\omega_{\mathrm{E}} = 0)$ Euclidean func.

Weak-coupling expansions: $\kappa, \lambda_3 = T^2(\mathcal{O}(1) + \mathcal{O}(g, g^2, \ldots))$

Leading weak-coupling values calculable and *nonzero*

But is hydro even consistent?

We said $\quad \Pi^{\mu\nu} = \mathcal{O}(\partial u) + \mathcal{O}(\partial^2 u, (\partial u)^2) + \ldots$

based on assumption thermalization is local, microscopic.

Hydro itself predicts long-lived shear,sound modes:

$$0 = \partial_\mu \Big(T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - \eta \sigma^{\mu\nu} \Big)$$

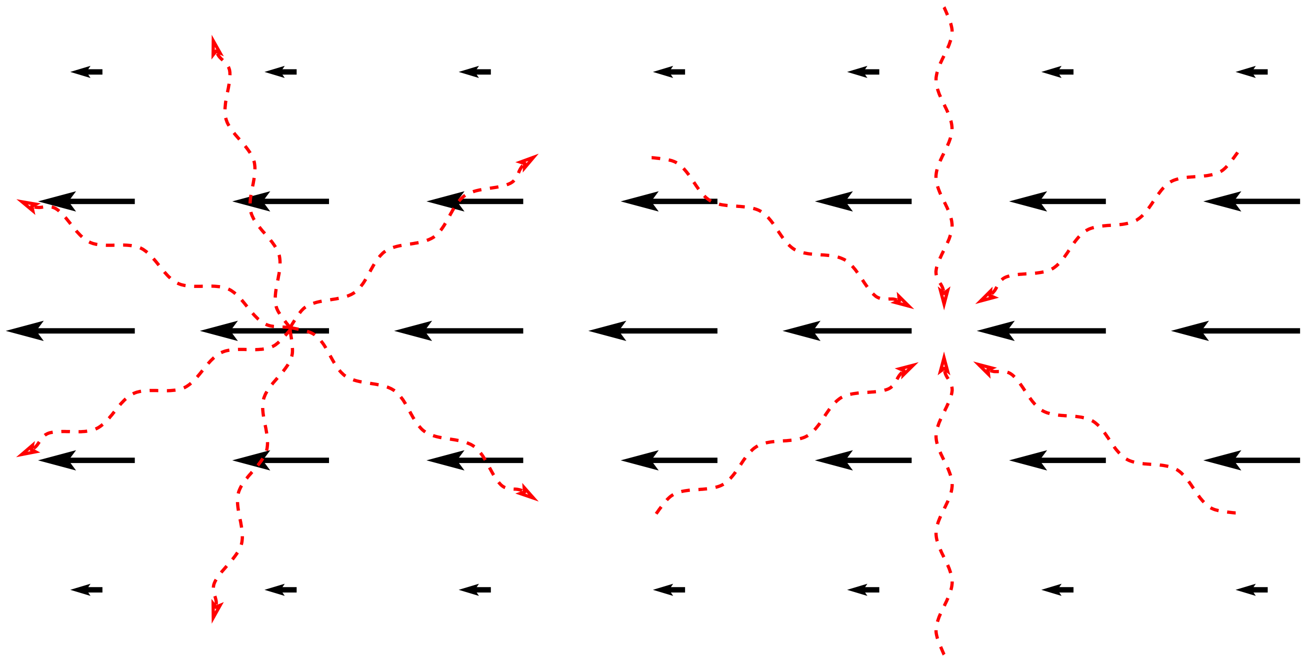
fluctuations in u^μ, ϵ: dispersion relations

$$\omega_{\rm shear} = i \frac{\eta}{\epsilon + P} k^2 \,, \qquad \omega_{\rm sound} = \pm \frac{k}{\sqrt{3}} + i \frac{2\eta}{3(\epsilon + P)} k^2$$

Small k: long lived, dissipation *not* local,microscopic

Hydro Waves Contribute to Viscosity!

Consider shear flow:



Flow decays because x-momentum leaves (diffuses from) flowing region. One mechanism: propagation of hydro (sound) waves!

How to compute hydro contribution to hydro

Above we found

$$G_{ra}^{xy,xy}(\omega) = P - i\eta\omega + \eta\tau_\pi\omega^2 + \ldots$$

Calculate contrib. of hydro modes themselves to G^{xyxy}.

$$\begin{aligned}
\text{Feynman rules:}\quad T^{ij} &= (\epsilon+P)u^i u^j + P g^{ij}\,, \\
\langle u^i u^j(k,\omega)\rangle &= \frac{T}{\epsilon+P}\frac{(\delta^{ij}-\hat{k}^i\hat{k}^j)2\gamma_\eta k^2}{(\gamma_\eta k^2 - i\omega)(\gamma_\eta k^2 + i\omega)}\text{shearwave} \\
\left[\gamma_\eta = \tfrac{\eta}{\epsilon+P}, \gamma'_\eta = \tfrac{4}{3}\gamma_\eta\right] \qquad & + \frac{T}{\epsilon+P}\frac{(\hat{k}^i\hat{k}^j)2\gamma'_\eta k^2\omega^2}{(\omega^2 - k^2/3)^2 + (\gamma'_\eta k^2\omega)^2}\text{soundwave}
\end{aligned}$$

Think of hydro as IR effective theory, η etc are Wilson coeff.

Computing $G_{ra}^{xy,xy}(\omega, k=0)$

Straightforward application of Feynman rules:

$$G_{ra}^{xy,xy}(\omega)[\text{hydro}] = -i\omega\left(\frac{17Tk_{\max}}{120\pi^2\gamma_\eta}\right) + (i+1)\omega^{\frac{3}{2}}\,\frac{7+\left(\frac{3}{2}\right)^{\frac{3}{2}}T}{240\pi\gamma_\eta^{3/2}}$$

$k_{\max}$: k-scale above which hydro incorrect/inconsistent.

- $-i\omega$ term: extra contrib. to η
- $i\omega^{3/2}$: effective ω dependence of η.
- $\omega^{3/2}$: like τ_π but *wrong* ω dependence.

Lesson: η

Small η: freer propagation of sound, shear modes.
More efficient momentum transport, raising η.

Depends on $k_{\rm max}$. Where does hydro break down?
Scale where it's no longer self-consistent.

Safe guess: $k_{\rm max} < \tau_\pi^{-1}/2$. In $\mathcal{N}{=}4$ SYM, this is about $2T$.

- $\mathcal{N}{=}4$SYM: added η/s is $\sim 1/N_{\rm c}^2$.
- Weak coupling: $\eta_{\textrm{from hydro}} \sim \alpha^4$ while $\eta_{\rm tot} \sim \alpha^{-2}$
- Real QCD: $\frac{\eta}{s} = .16$: add 0.01. $\frac{\eta}{s} = .08$: add 0.036!

Lesson: τ_π

Weak coupling and large $N_{\rm c}$: comparing

$$N_{\rm c}^0\alpha^3 T^{5/2}\,\omega^{3/2} \quad \text{vs} \quad N_{\rm c}^2\alpha^{-4}T^2\,\omega^2$$

Deep IR, $\omega^{3/2}$ term wins, 2-order hydro breaks.
But scale where $\omega^{3/2}$ term takes over is $\omega \sim N_{\rm c}^{-4}\alpha^{14}T$.

Check that ω where they equal is more IR than "your physics" and then use 2-order hydro!

- $N_{\rm c} = 3 = N_{\rm f}$ QCD, $T = 200\text{MeV}$, $\frac{\eta}{s} = .16$: $\omega \sim \frac{T}{20}$ Safe!
- $N_{\rm c} = 3 = N_{\rm f}$ QCD, $T = 200\text{MeV}$, $\frac{\eta}{s} = .08$: $\omega \sim 7T$ Problem!

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Need 2'nd order Hydro, 6 hydro coefficients!
- Pert. computation of 2'nd order Hydro: dim'less ratios same order as $\mathcal{N}=4$SYM, differ in detail
- Kubo relations for nonlinear coefficients found. κ, λ_3 special (really thermodynamic)
- Hydro waves contribute to hydro coefficients!
- Self-consistency issues if η too small, and very low freq.