Extracting the Proton Radius from Electron Scattering Data

Why the proton radius is smaller in Virginia

Douglas W. Higinbotham

How many ways can YOU determine the radius of a perfect sphere?!

Image of the sphere created to test theory of relativity on the Gravity Probe B spacecraft.

Some Answers

- Diameter = 2 r
- Area = πr^2
- Volume = $4/3 \pi r^3$ (displacement of water)
- Momentum of Inertia
  - $2/5\, m r^2$ (solid sphere)
  - $2/3\, m r^2$ (hollow sphere)

Charge Radius of the Proton

- Proton is hard as there are currently only a few ways to get the radius
  - Atomic Hydrogen Lamb Shift (~ 0.88 fm)
  - Muonic Hydrogen Lamb Shift (~ 0.84 fm)
  - And of course elastic electron scattering!
- Heavy nuclei are relatively easy (little recoil)
  - Measure charge form factor
  - Take Fourier Transform
  - Even better if you measure a diffraction minimum!

Rosenbluth Formula

From relativistic quantum mechanics one can derive the the formula electron-proton scattering where one has assumed the exchange of a single virtual photon.

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} \cdot \frac{E'}{E}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right]$$

where G_e and G_m take into account the finite size of the proton.

$G_E = G_E(Q^2)$, $G_M = G_M(Q^2)$; $G_E(0)=1$, $G_M(0) = \mu_p$

$Q^2 = 4\,E\,E'\sin^2(\theta/2)$ and $\tau = Q^2/4m_p^2$

Standard Dipole Radius: 0.81(1) fm

L.N. Hand, D.G. Miler, and R. Wilson, Rev. Mod. Physics **35** (1963) 335.

Due to its light mass, relativistic corrections make the proton radius more challenging.

For the proton, we extract the radius by determining the slope of Ge at q^2=0.

$$G_E(q^2) = G_E(0)\left[1 - \frac{1}{6}\langle r_p^2\rangle q^2 + \dots\right]$$

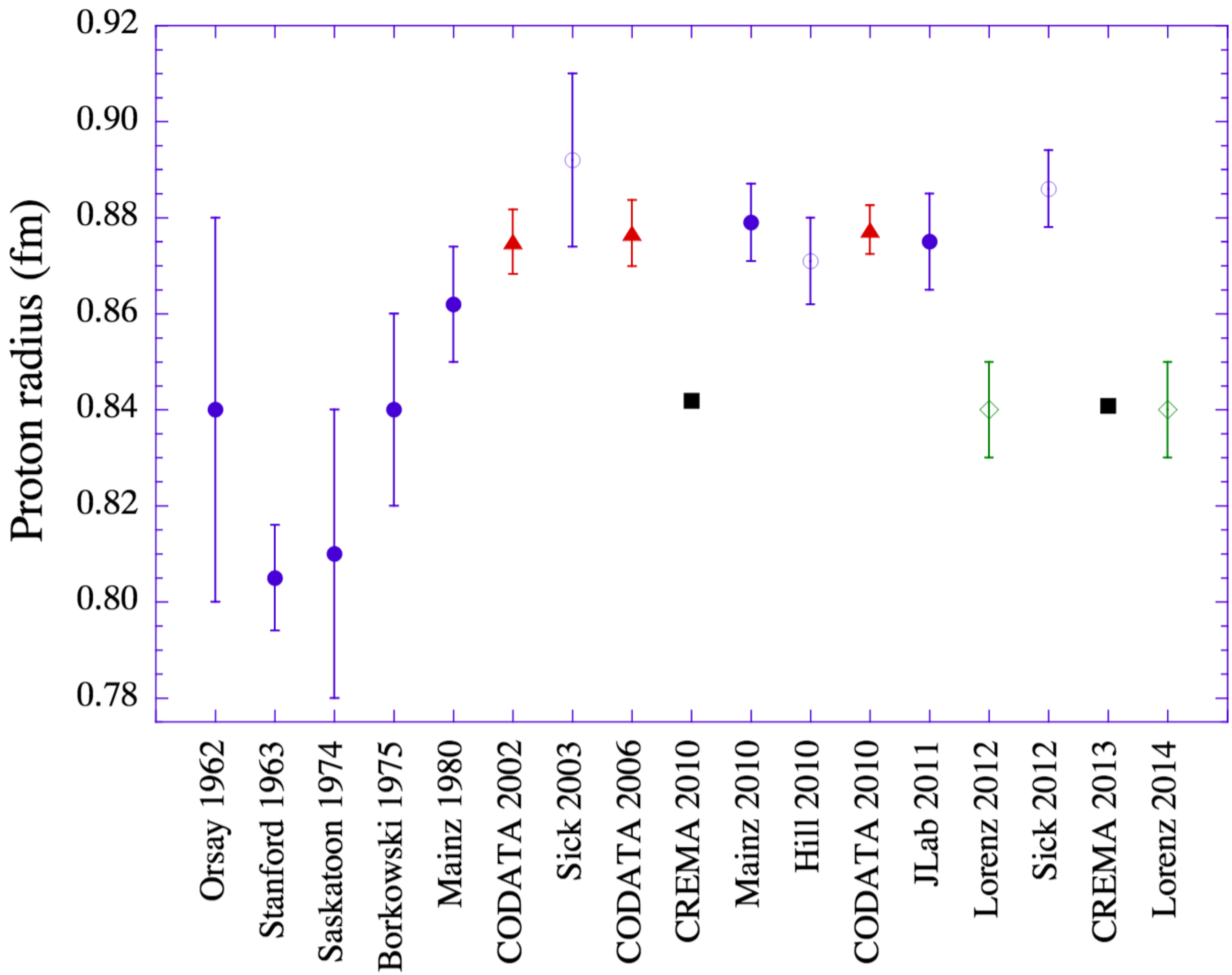
$$< r_p^2 >= -6\, \frac{dG_E(q^2)}{dq^2}\bigg|_{q^2=0}$$

Took the data that was available, fit it with Taylor N=1 and 2,, and got the slope at q^2 = 0.

DO IT YOURSELF!! If you make different choices you will get slightly different answers!

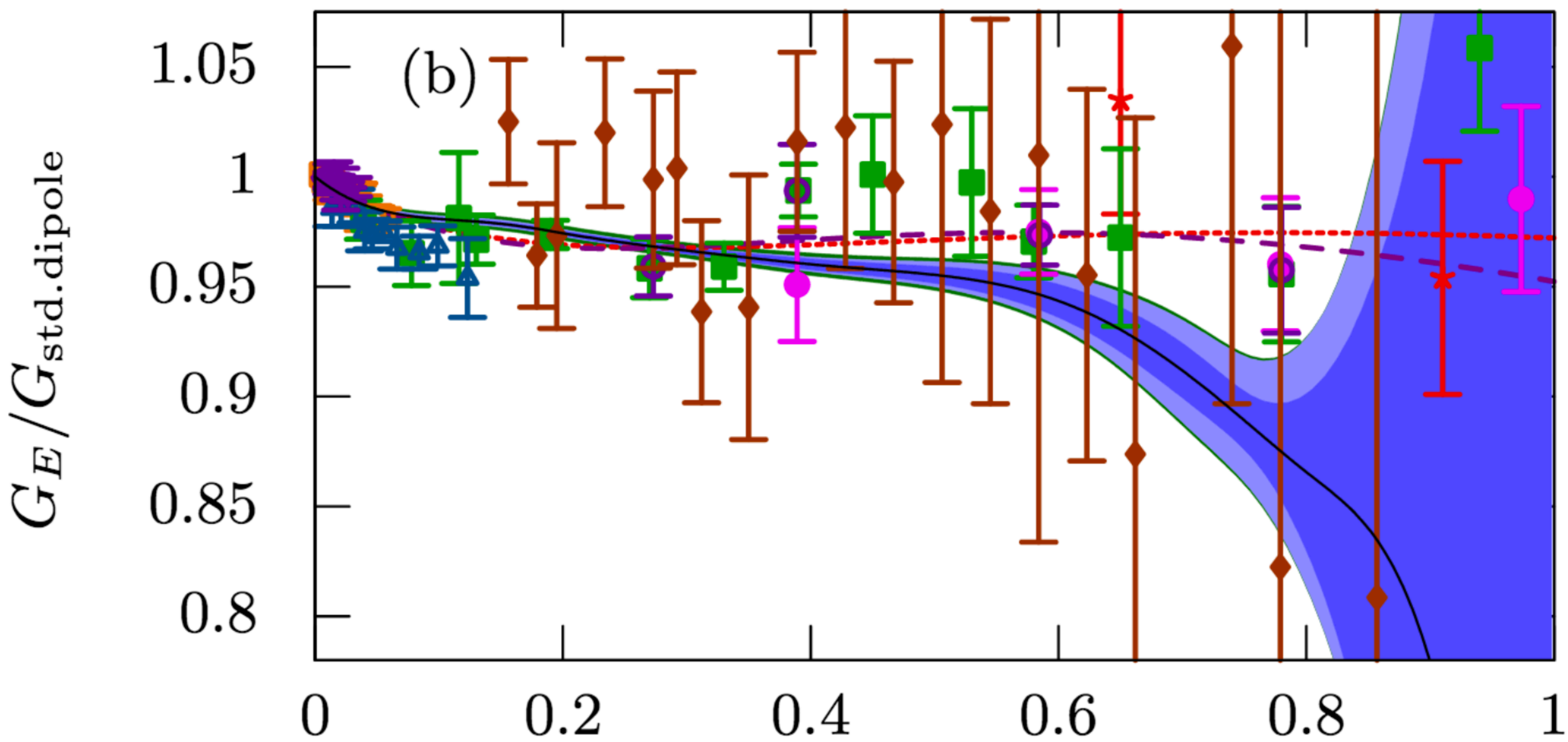
Proton Radius vs. Time

V. Punjabi et al., Eur. Phys. J. A51 (2015) 79.



Mainz 2014 G_e (Blue Band)

From J. Bernauer et al., Phys Rev. C90 (2014) 015206.



Note the spline G_e fit at high q^2 starts to fall as soon as it is not constrained by their data.

Mainz 2014 Fitting Results (G_e & G_m)

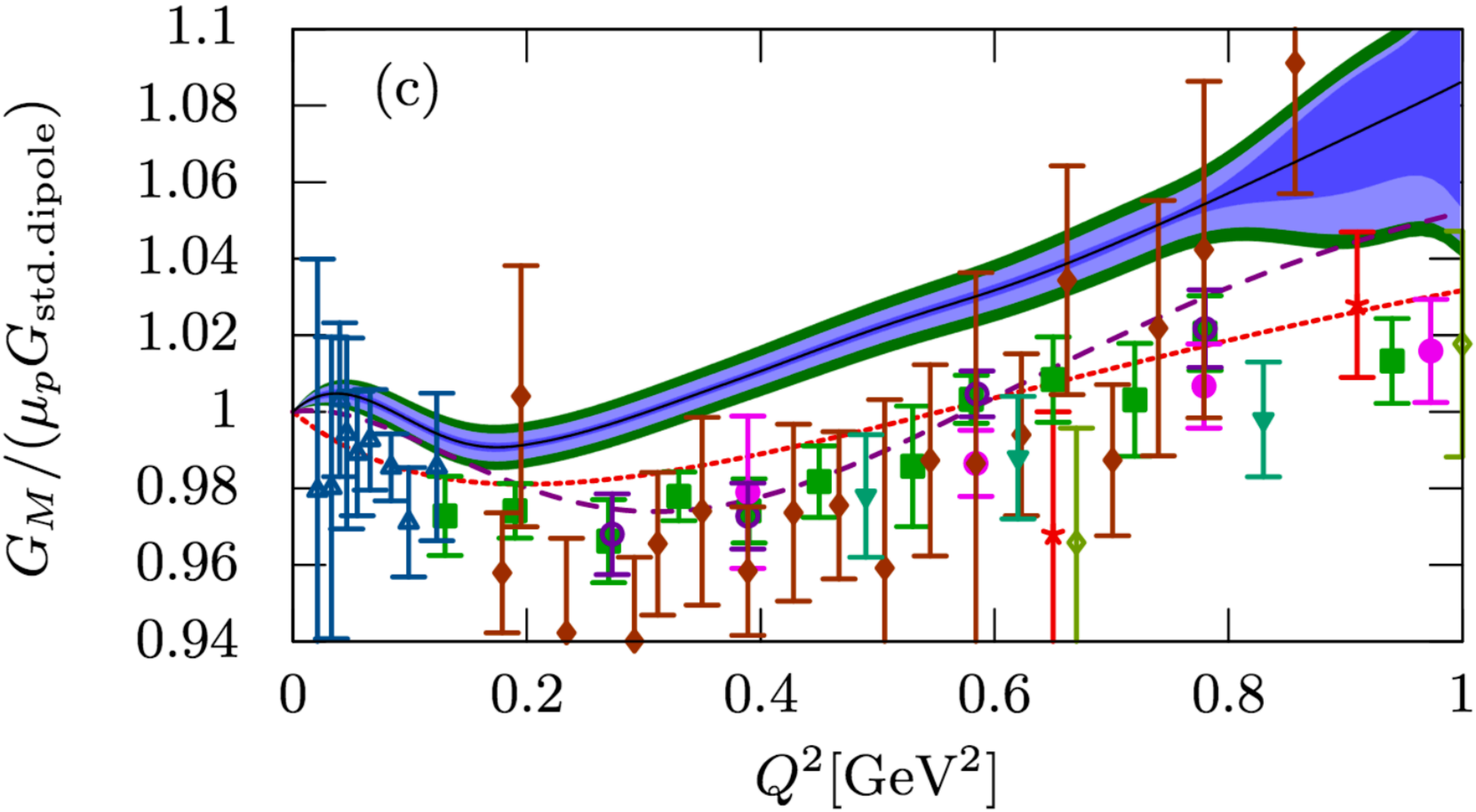
Did not follow a standard statistical method, such as an f-test, to determine number of parameters. Instead, the authors just state chi2 <1600 are "analyses with the good models."

Model	χ^2	Number of param.	χ^2_{red}
Single dipole	3422	$2 \times 1 + 31$	2.4635
Double dipole	1786	$2 \times 3 + 31$	1.2893
Polynomial	1563	$2 \times 10 + 31$	1.1399
Poly. + std. dipole	1563	$2 \times 10 + 31$	1.1400
Poly. × std. dipole	1572	$2 \times 8 + 31$	1.1436
Inv. poly.	1571	$2 \times 7 + 31$	1.1406
Spline	1565	$2 \times 8 + 31$	1.1385
Spline × std. dipole	1570	$2 \times 7 + 31$	1.1403
Friedrich-Walcher	1598	$2 \times 7 + 31$	1.1588
ext. Gari-Krümpelmann	1759	$14 + 31$	1.2777

PLEASE READ: Data Reduction and Error Analysis for the Physical Sciences by Bevington &Robinson and don't just use chi2 to judge which fit is most likely. (i.e. **probabilities**)

Tension With World G_m Results

From J. Bernauer et al., Phys Rev. C90 (2014) 015206.

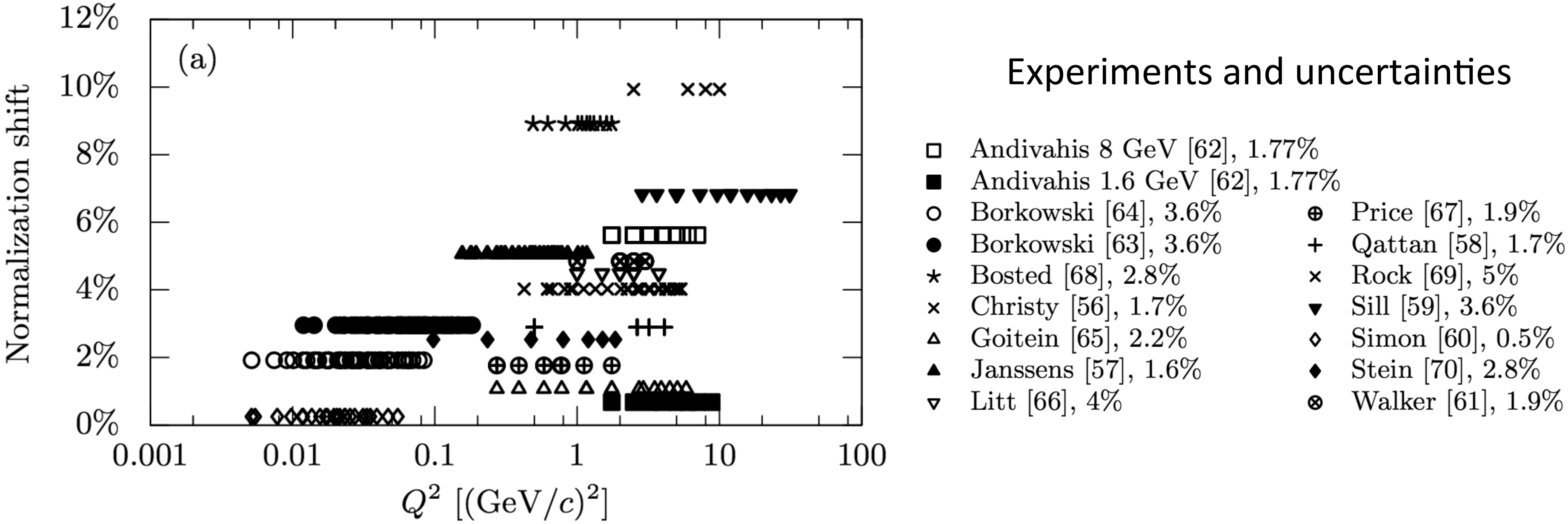


The 1.6% shift that is suggested by their "double dipole" fit would bring Mainz to the world data.

NOTE: G_e and G_m do not need to be the same function!

Spline Fit Normalization Shifts of World Data

Fig 19 of Bernauer et al., Phys. Rev. C90 (2014) 015206.



"As can be seen, all shifts are positive, i.e., the actual cross sections as reconstructed by the fit are large then the values quoted. … While it may look strange that all shifts are positive, the mean of the normalization falls together with the shift of the oldest measurement [57], …" - Bernauer *et al.*

Completely dismissing the fact their fit disagrees with world data?!

Saskatoon 1974 (elastic recoil proton)

missing from MANY global fits

As of 2 Feb. 2016, not even listed in the extensive Scholarpedia proton form factor articles :
http://www.scholarpedia.org/article/Nucleon_Form_factors

J. J. Murphy, II, Y. M. Shin, and **D. M. Skopik**, Proton form factor from 0.15 to 097 fm-2, Phys. Rev. C **9** (1974) 2125.

Please add here! ->

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Mainz80 and Saskatoon74

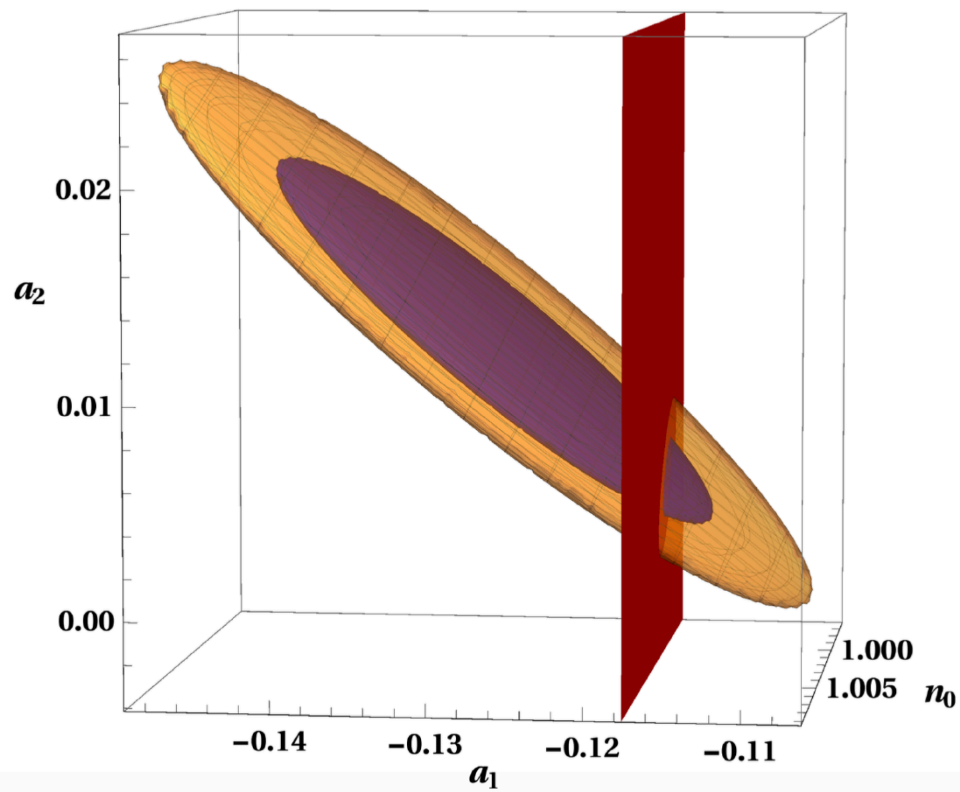
Prior to new Mainz results ~2010: Lowest, High Precision G_e Measurements

$$f(q^2) = a0(1 + a1 \times q^2 + a2 \times q^4 ... + aN \times q^{2N})$$

For a three parameter fit of the combined Mainz and Saskatoon data, we find

$$X = \begin{pmatrix} 1.0032 \\ -0.1272 \\ 0.0110 \end{pmatrix}, \Sigma = \begin{pmatrix} 0.367 & -1.05 & 0.684 \\ -1.05 & 3.54 & -2.55 \\ 0.684 & -2.55 & 2.03 \end{pmatrix} \times 10^{-5} \tag{5}$$

with a reduced χ^2 of 0.723, an a1 value that corresponds to a radius of 0.875 fm, and $\sqrt{a_{11}}$ term of 0.006 fm. However, there are clearly problems as the off-diagonal terms are larger than the diagonal elements.



Continued Fraction fit of this set of data produces a similar result.

“DISCOVERED” BY MOVING THE LAST DATA POINT ONE SIGMA AND SEEING THE EFFECT ON THE RESULT

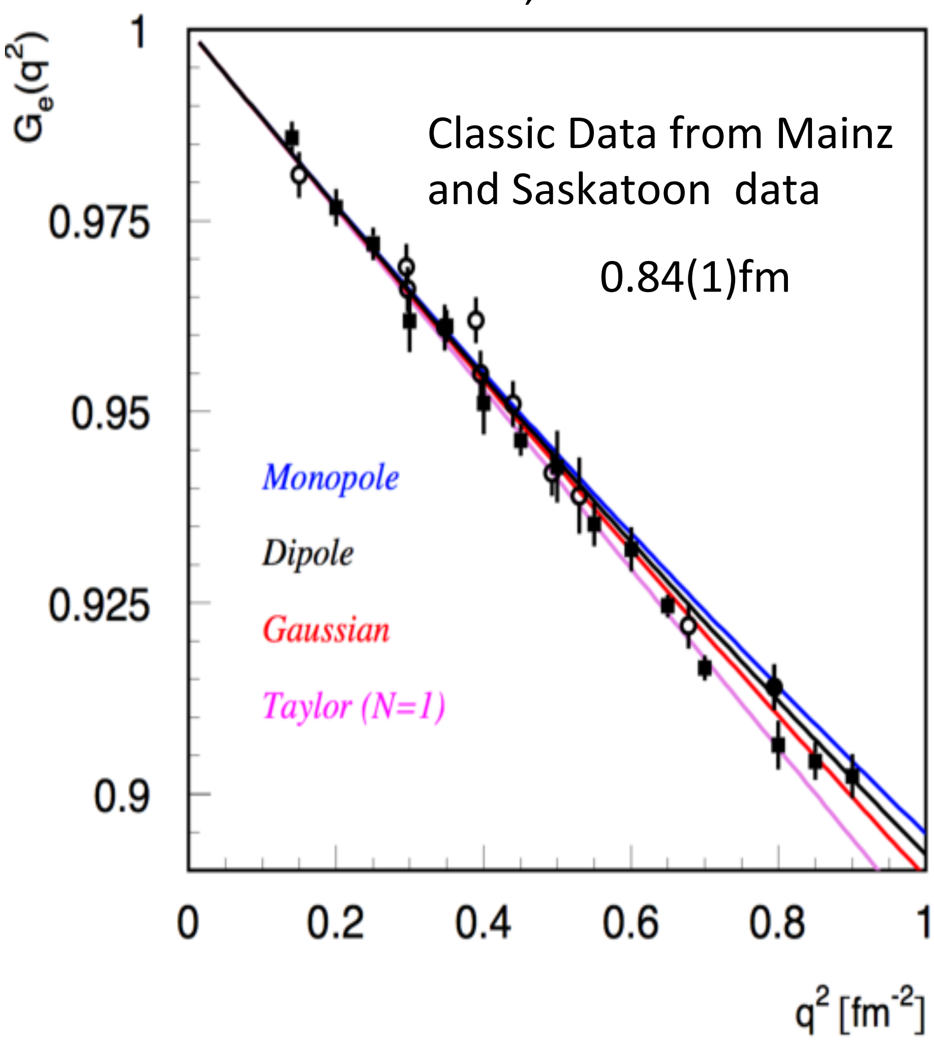
Thus, a three parameter fit of the full range of data agrees with both 0.84fm and 0.88fm radius.

Doing f-tests (see Bevington) one finds that going to second order isn’t justified by the data!

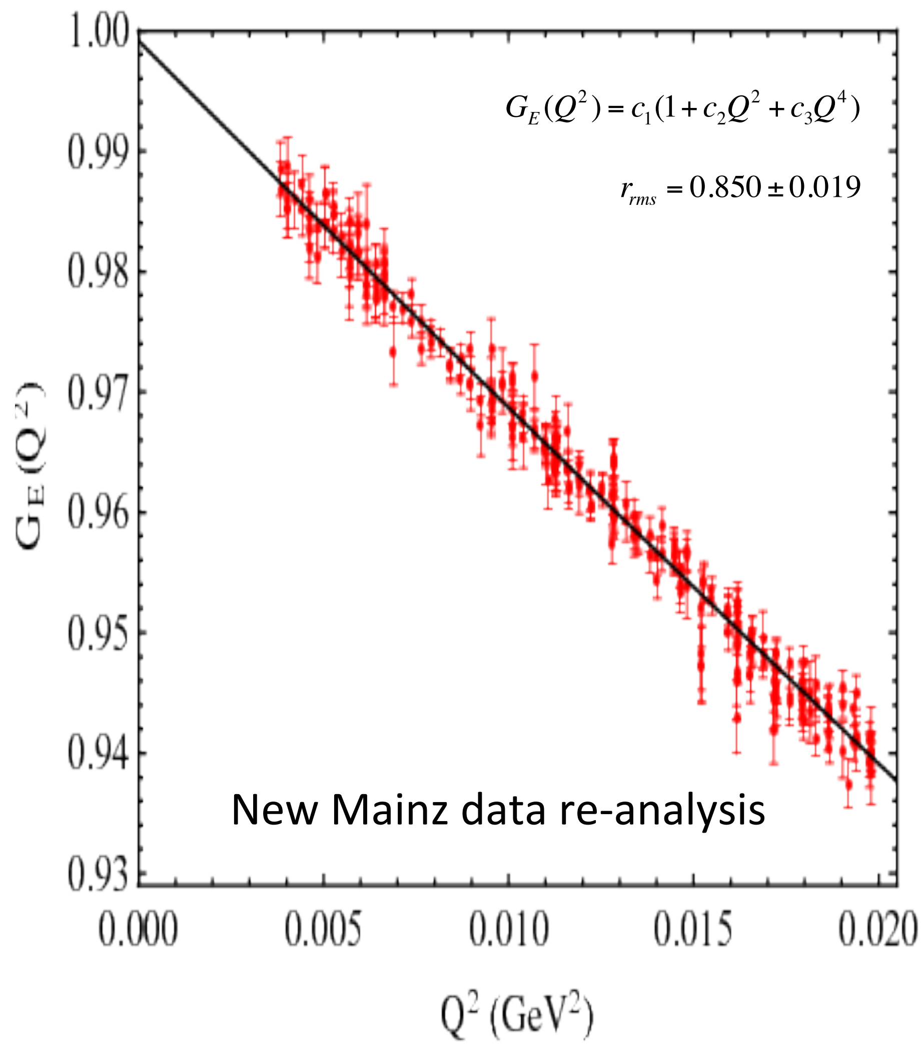
Back To The Taylor Extrapolations . . .

Hand et al.'s ideas but now with MUCH better data

D.W.H. et.al.,arXiv:1510.01293

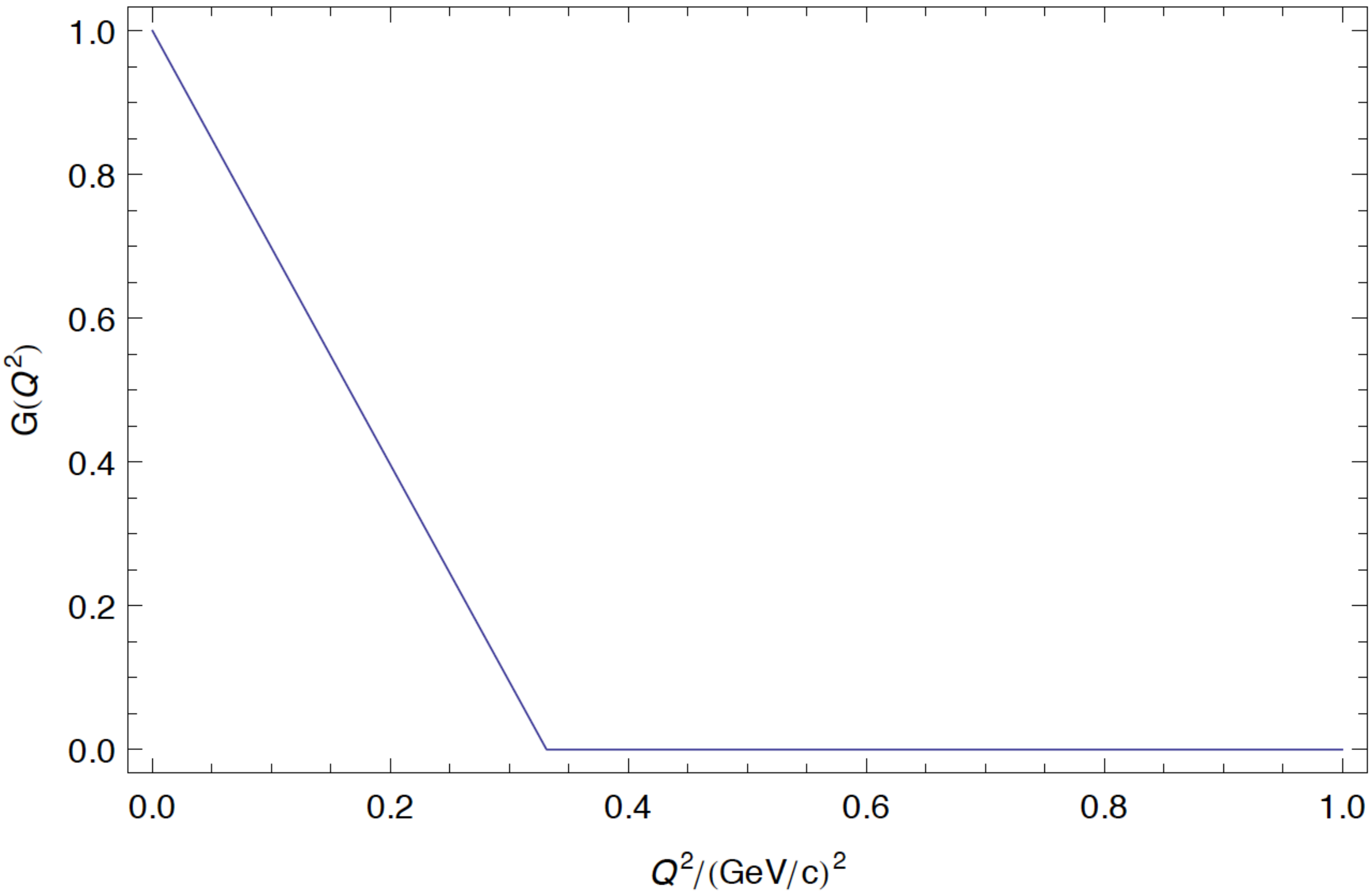


K. Griffioen et. al., arXiv: 1509.0667



1 fm $^{-2}$ = 0.0389 GeV^2

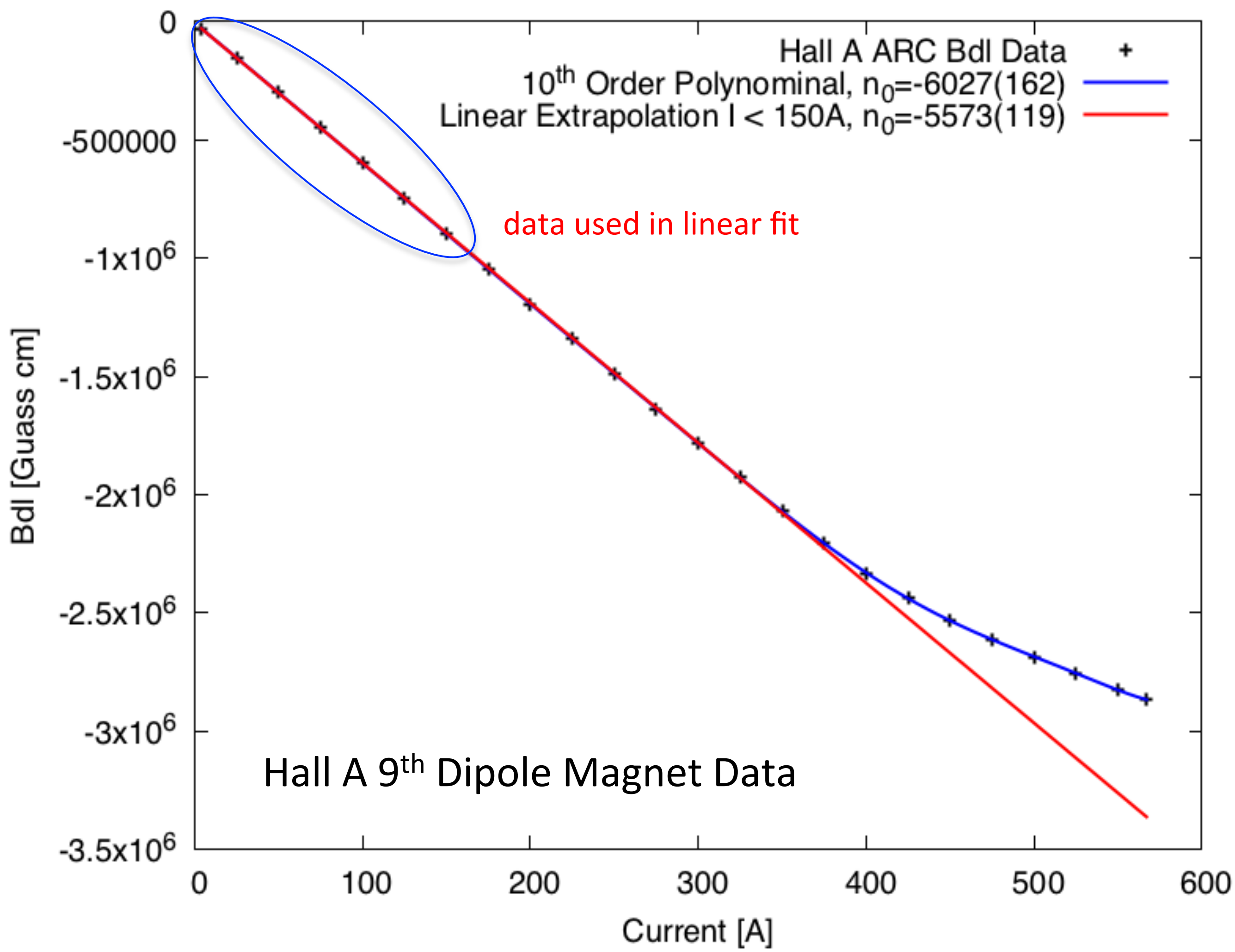
from comment arXiv:1511.00479



Of course the linear function doesn't work to all Q^2, though it is amusing to note this function does have a charge radius of 0.84 fm.

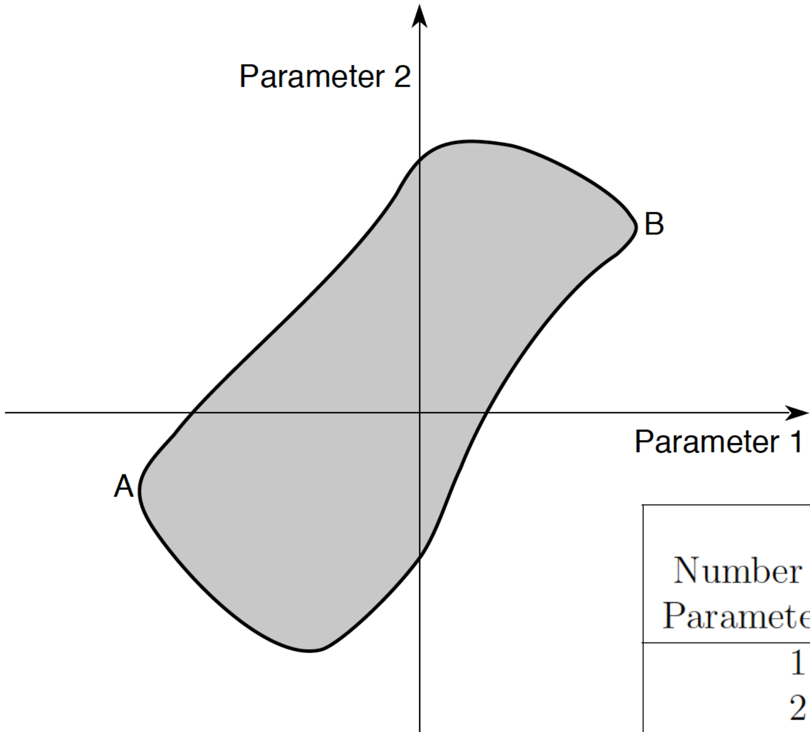
Example of Precision vs. Accuracy

Shown with simply with asymptotic standard error which can be very misleading . . .



Linear Fit of Low Current Data Accurately Extrapolates The Residual Field n_0 (also extrapolates from 150A to 300A)
The 10th Order Polynomial Fit Precisely Describes The Data But Doesn't Extrapolate Well

Multivariate Errors

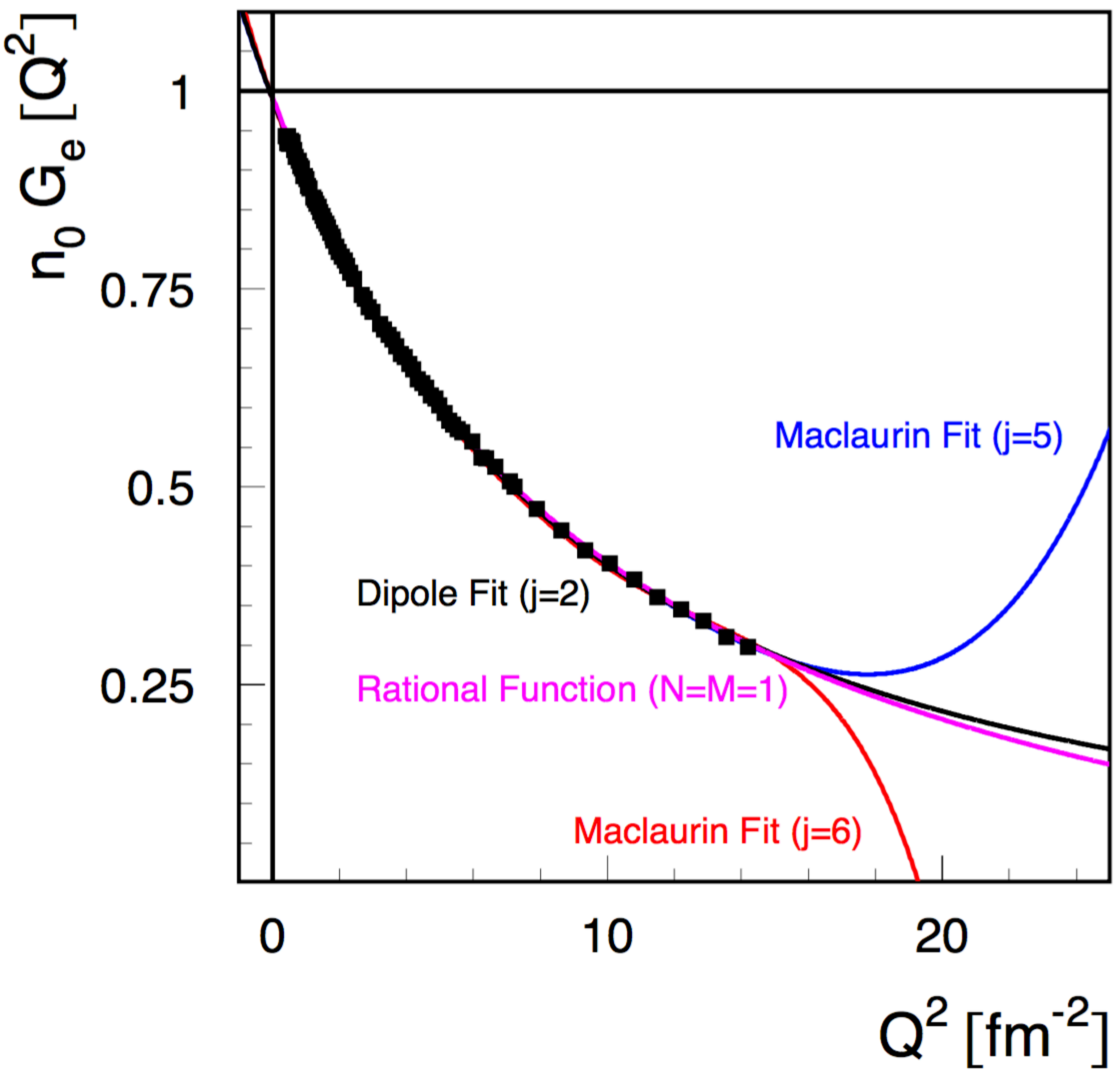


As per the particle data handbook, one should be using a co-variance matrix and calculating the probably content of the hyper-countour of the fit.   Default setting of Miniut of “up” is one.

Standard Errors often underestimate true uncertainties. (manual of gnuplot fitting has an Explicate warning about this)

Number of Parameters	Confidence level (probability contents desired inside hypercontour of $\chi^2 = \chi^2_{min} + \text{up}$)				
	50%	70%	90%	95%	99%
1	0.46	1.07	2.70	3.84	6.63
2	1.39	2.41	4.61	5.99	9.21
3	2.37	3.67	6.25	7.82	11.36
4	3.36	4.88	7.78	9.49	13.28
5	4.35	6.06	9.24	11.07	15.09
6	5.35	7.23	10.65	12.59	16.81
7	6.35	8.38	12.02	14.07	18.49
8	7.34	9.52	13.36	15.51	20.09
9	8.34	10.66	14.68	16.92	21.67
10	9.34	11.78	15.99	18.31	23.21
11	10.34	12.88	17.29	19.68	24.71
	If `FCN` is $-\log(\text{likelihood})$ instead of χ^2, all values of `up` should be divided by 2.				

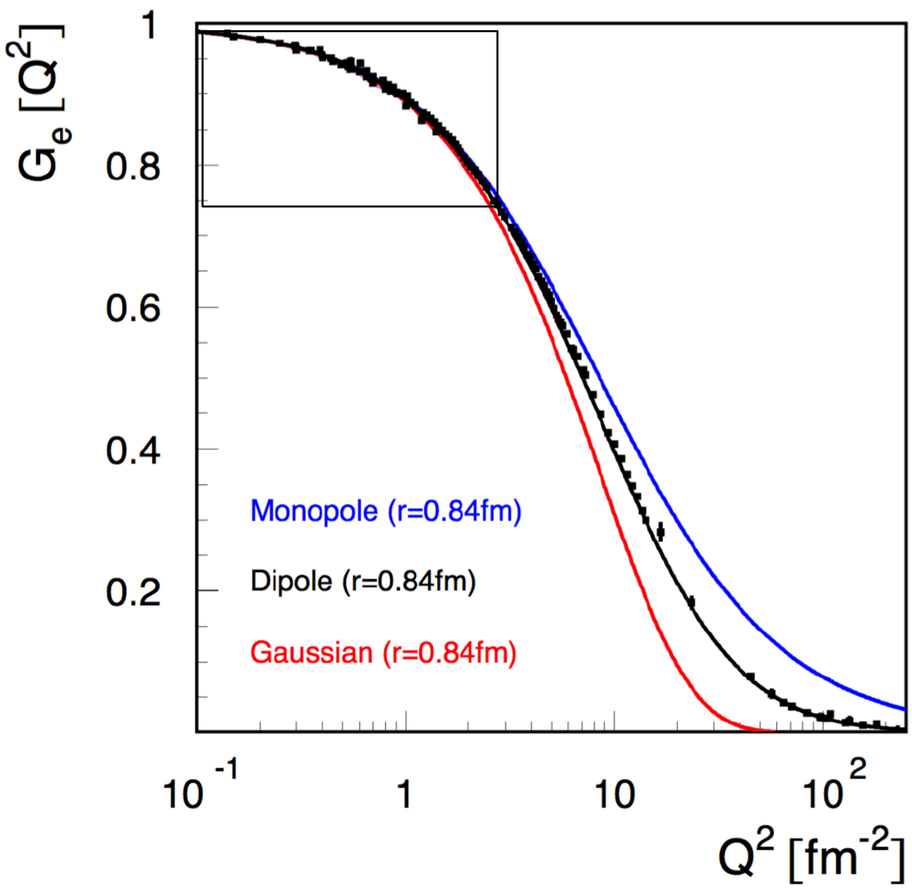
Fits of the Mainz 2014 G_E Rosenbluth Data

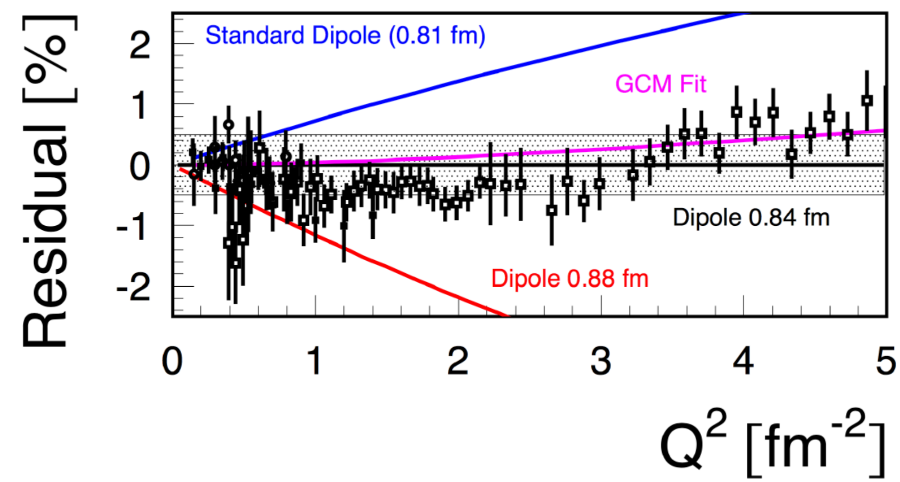


Rational Function & Dipole give radius of ~0.84 fm with Maclaurin (j=5 & 6)

Charge Form Factor with Dipole (0.84fm)

Using the classic data along with the Mainz 2014 “Rosenbluth” G_E Results



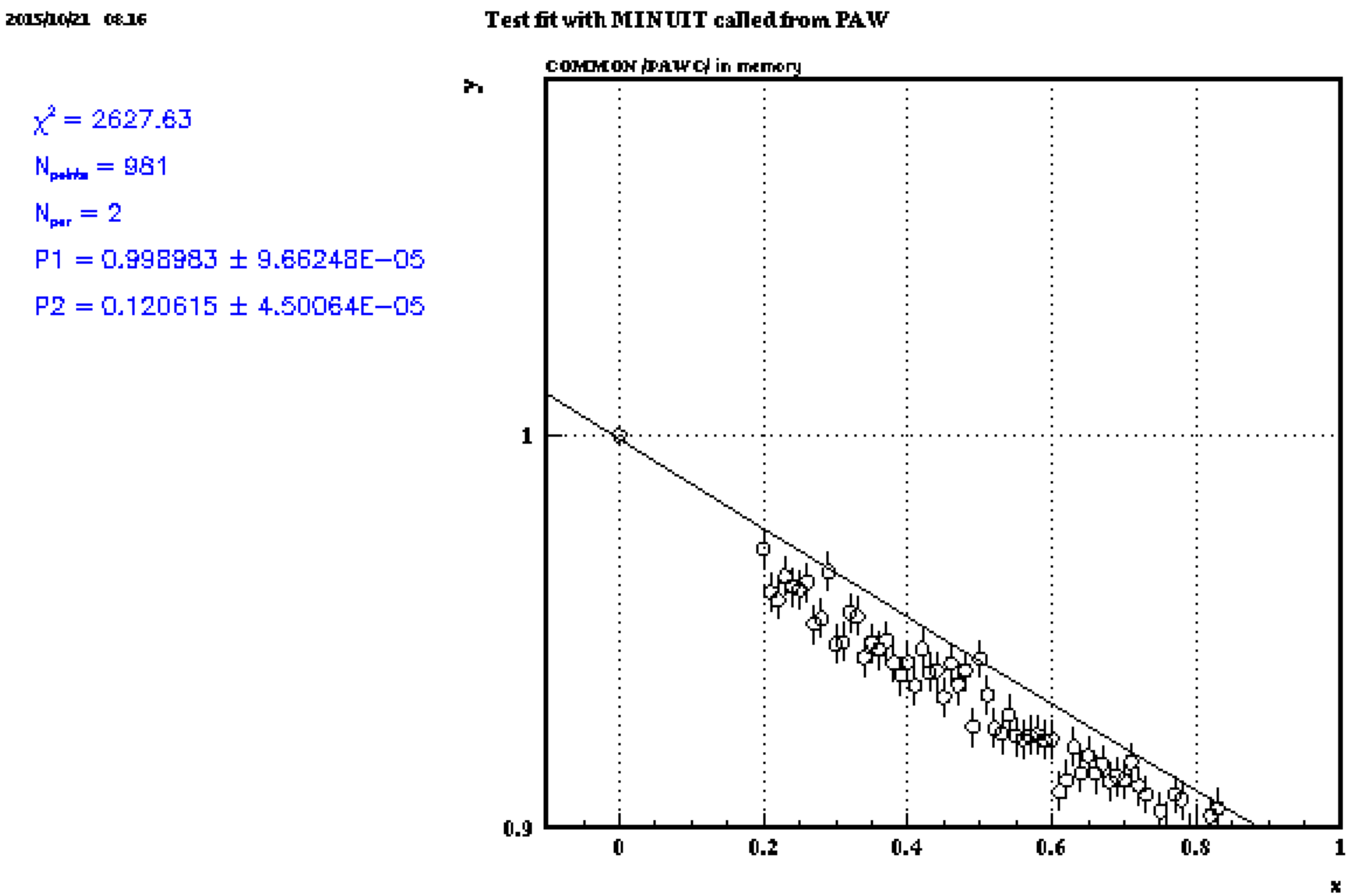


Data shown with 1/sqrt(N) errors only.
Gray error is a 0.5% systematic error band.

Within the range of the Mainz data, this result is very similar to Griffioen’s CF (N=4) fit.

The Problem with fitting the intercept

FAUX DATA: Real function y(x) = 0.985*(1 + 0.1176*x)**(-2) randomized point-to-point & systematically shifted (i.e. the normalization isn't perfect)

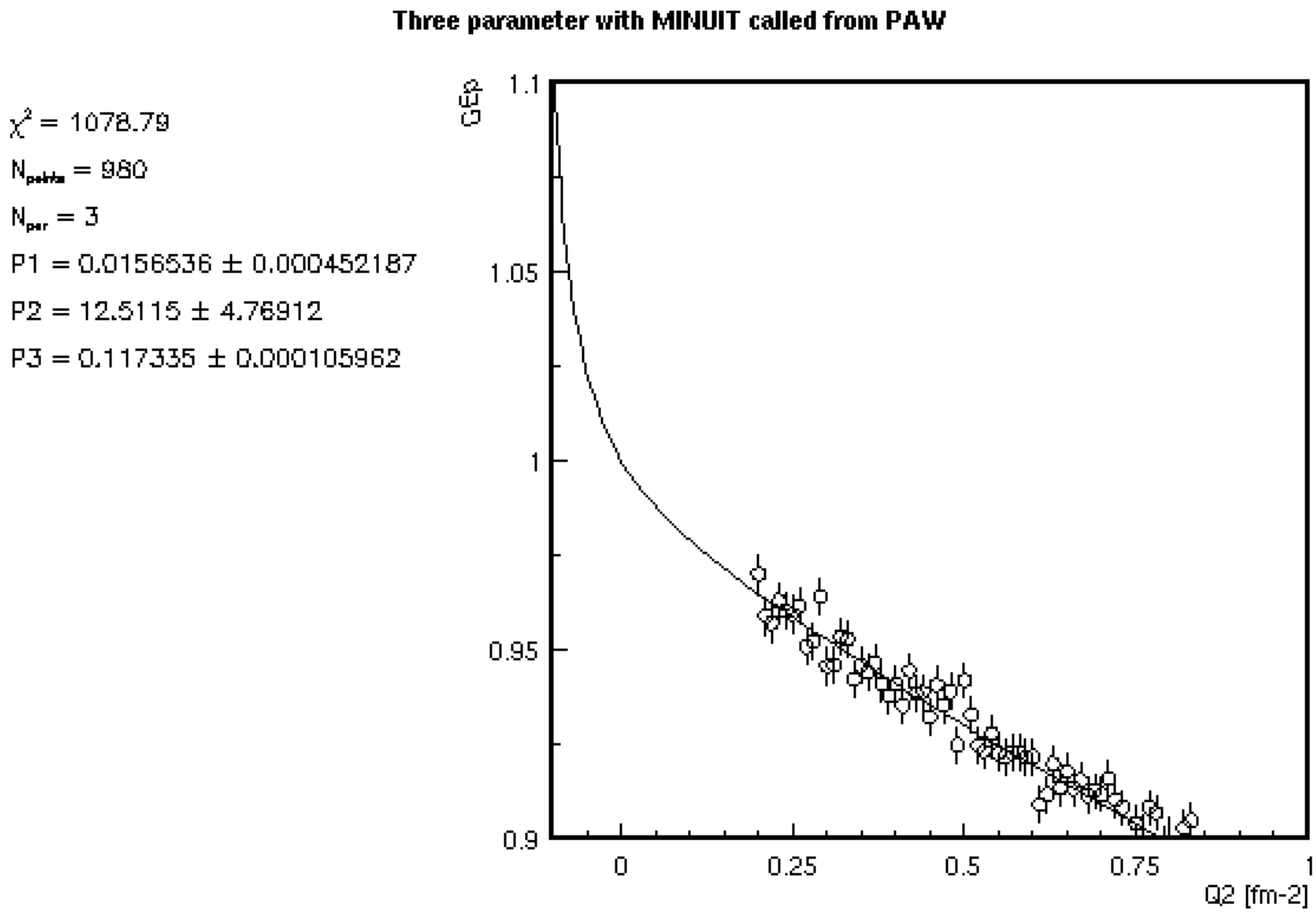


i.e. if you fit this data without letting the end point float . . . reduced chi^2 = 2.7 !

Same Faux Data Now Fit with Double Dipole

Note: Now you get the same result with or without the intercept point.

FAUX DATA: Real function y(x) = 0.985*(1 + 0.1176*x)**(-2) randomized point-to-point & systematically shifted (i.e. the normalization isn't perfect by 1.5%)



Reduced chi2 very close to one, BUT this wasn't the original function!!

Mainz 2014 Fitting Results (G_e & G_m)

Model	χ^2	Number of param.	χ^2_{red}
Single dipole	3422	$2 \times 1 + 31$	2.4635
Double dipole	1786	$2 \times 3 + 31$	1.2893
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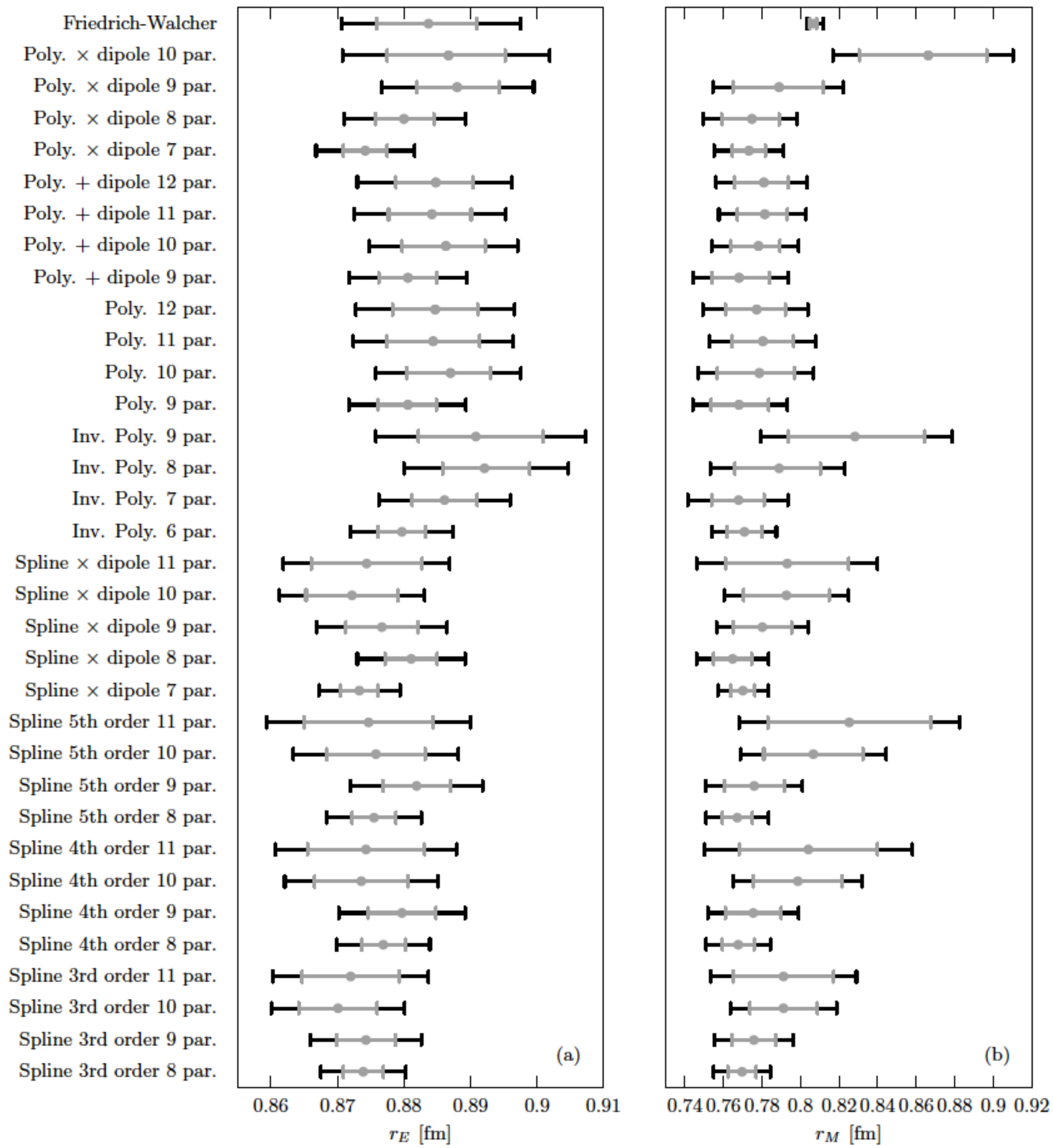
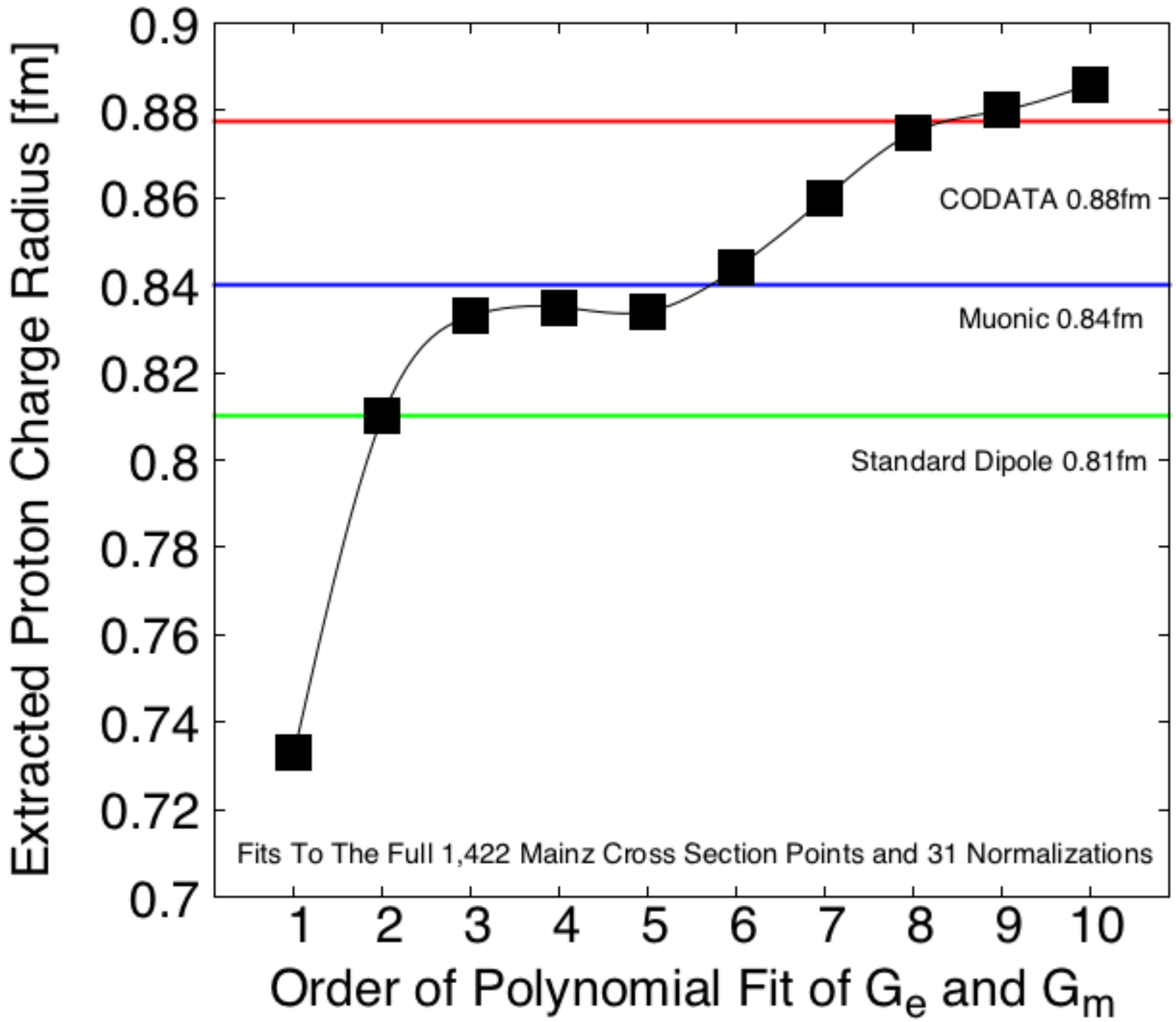
Friedrich-Walcher	1598	$2 \times 7 + 31$	1.1588
ext. Gari-Krümpelmann	1759	$14 + 31$	1.2777

The Mainz 2014 supplemental material has all the data and an example Python fitting script.

The Mainz PRC ONLY Report High Order Fits !?



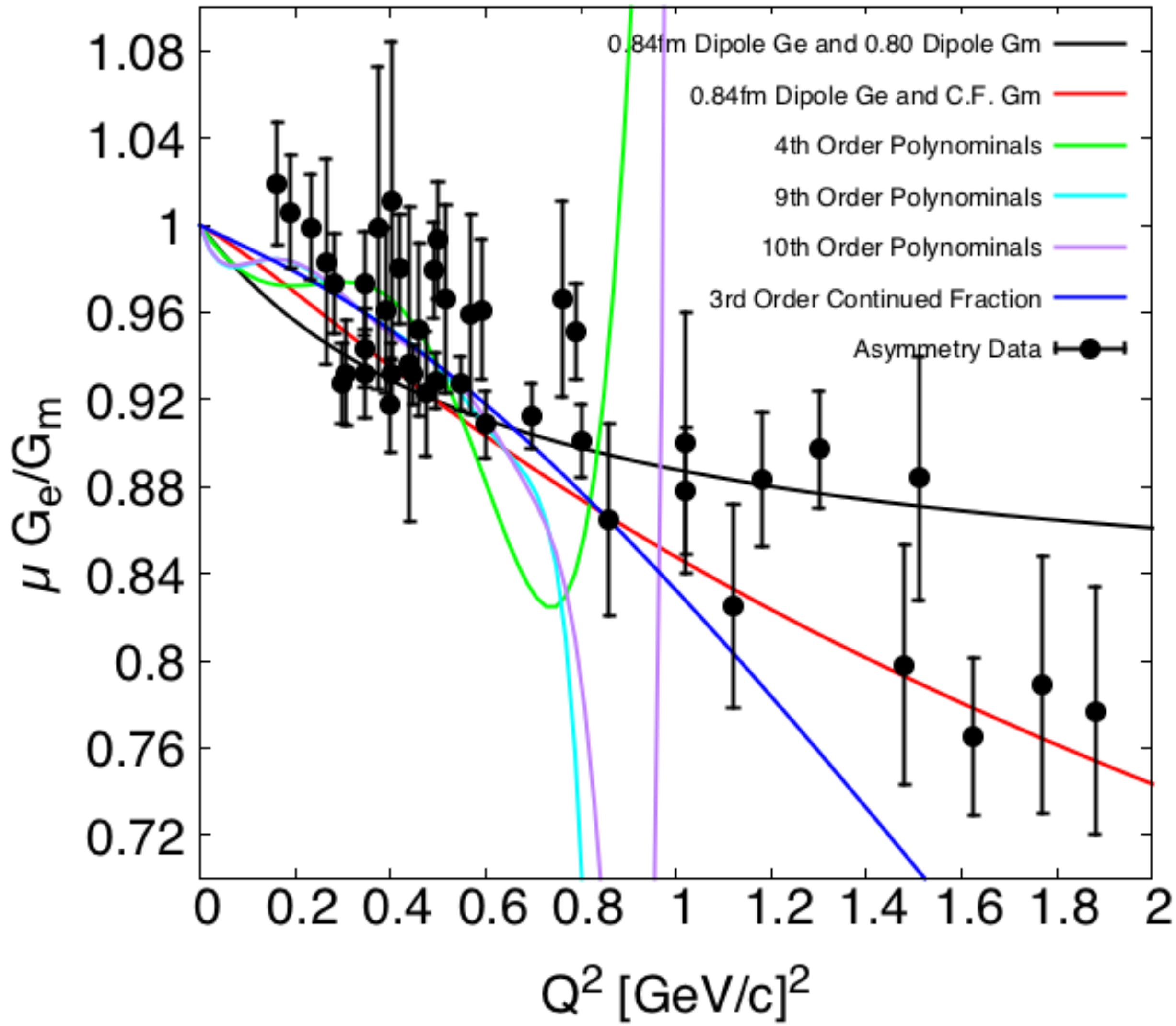
Proton Radius vs. Order of Polynomial Fits



R^2 (goodness of fit measure which runs 0 to 1) gets to 0.97 by 4th order and 0.98 by 10th….

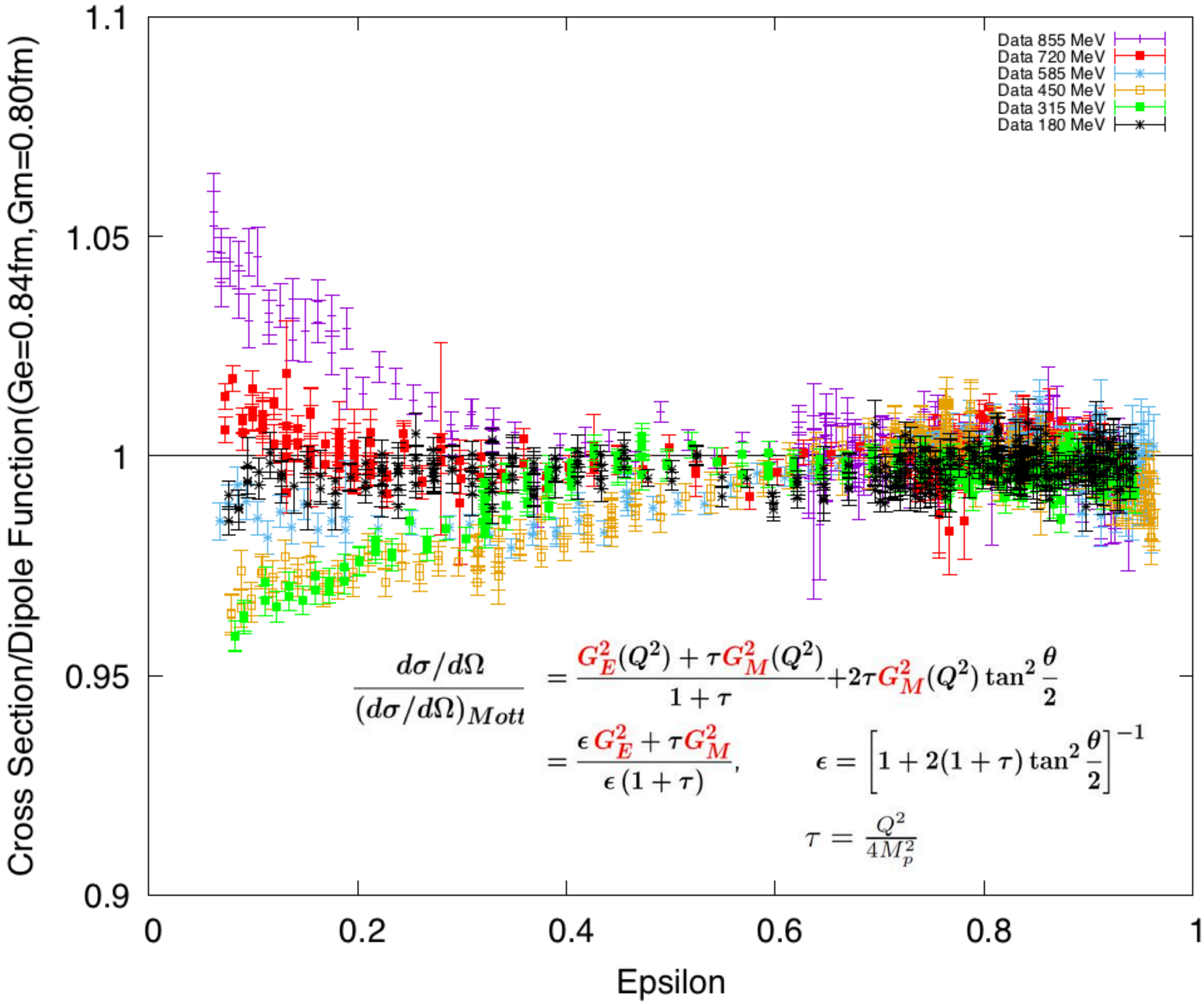
But how well do these high order fits do against data not included in the fit?!

Using the full Mainz data and a Python fitting code based on the Mainz fitting routine.



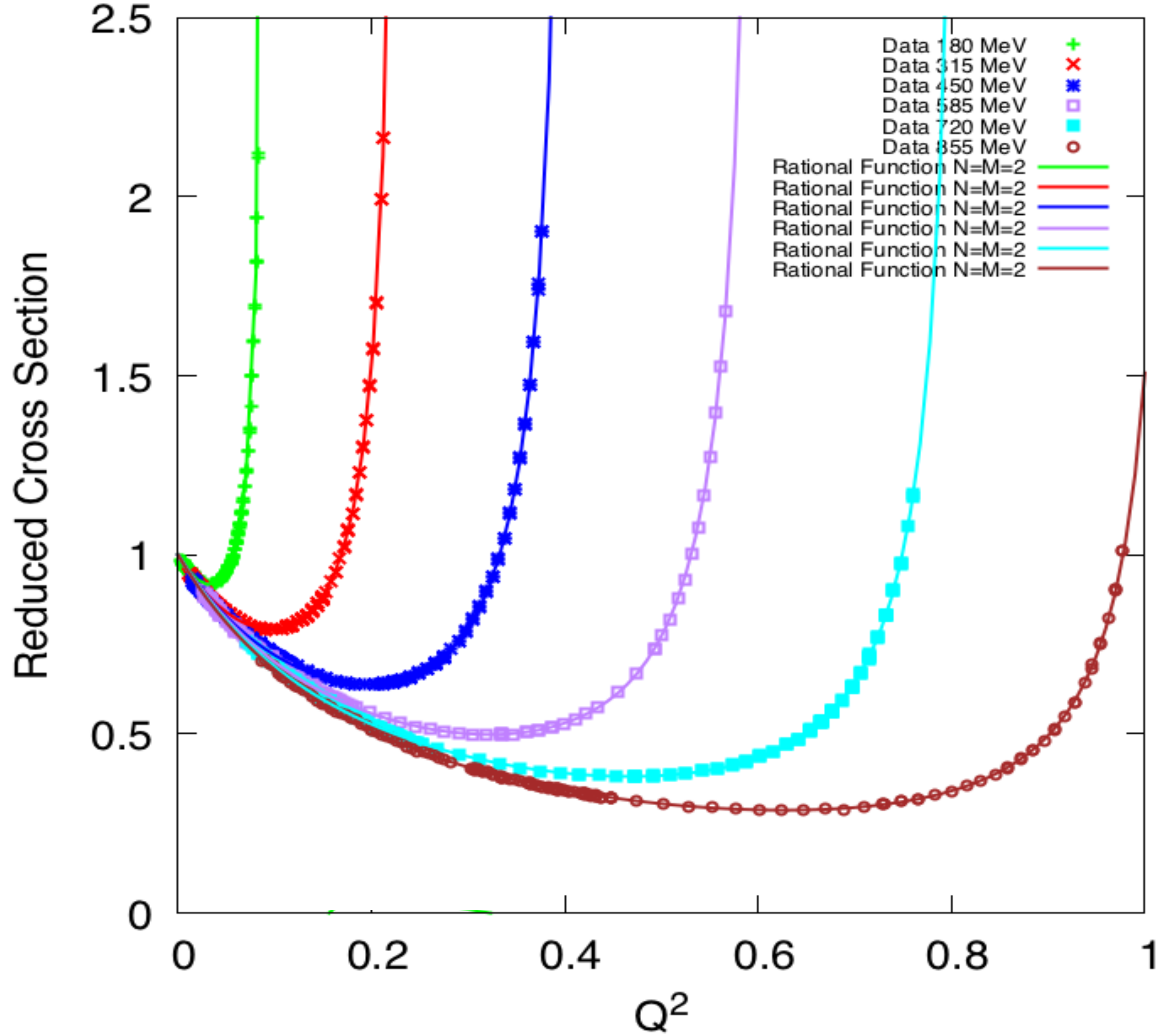
The lower order fits to the Mainz data also give agreement with world G_M & a smaller radius…

So what is going on?!



Same Data Plotted vs. Q^2

Extreme Back Angle Data That Is Approaching A Singularity



NOTE: $Q^2 = Q^2(E,\text{theta})$ has a kinematic max., $Q^2_{max}(E,180^o)$, which these fits nicely reproduce

Summary

- One can find different proton radii depending on the function used for the fit (model dependence).
- Linear Extrapolations of the lowest q^2 data (Maclaurin sereis
- N=1) give results consistent with muonic hydrogen (~0.84 fm)
- Advantages of low Q^2: floating normalization and tiny G_m contribution. (model independence)
- Do NOT just shift the low Q^2 data without redoing the normalization (i.e. if you add a correction to the cross sections, you need to start over with the normalization procedure otherwise you are biasing the result by shifting the points without moving the intercept)
- Currently working with better extrapolation functions (i.e. Rational Fractions & Chebyshev polynominals) to do our own fit to the full set Mainz 2014 published data.
- Our preliminary model independent fits results agree with Griffioen, Carlson, Maddox's fit and world G_m data... but what about all the other global fits!?

Global Fit vs Cutoff in Q^2

Gabriel Lee, John R. Arrington, and Richard Hill, Phys. Rev. D92 (2015) 013013

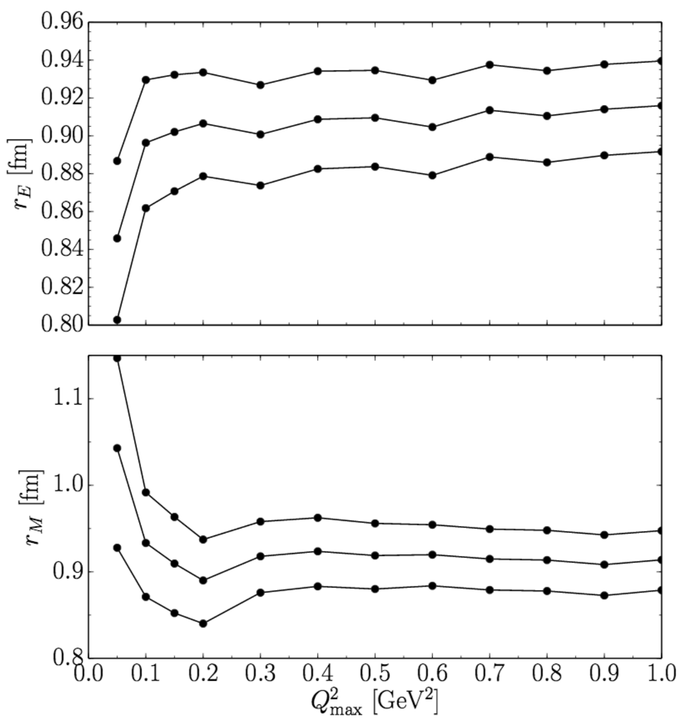


FIG. 7: Extracted electric (top panel) and magnetic (bottom panel) radii as functions of the kinematic cut $Q^2_{\max}$ on momentum transfer for the world cross section data set, using the z expansion with $t_0 = 0$, Gaussian priors with $|a_k|_{\max} = |b_k|_{\max}/\mu_p = 5$, $k_{\max} = 12$. Error bands include statistical and systematic uncertainties.

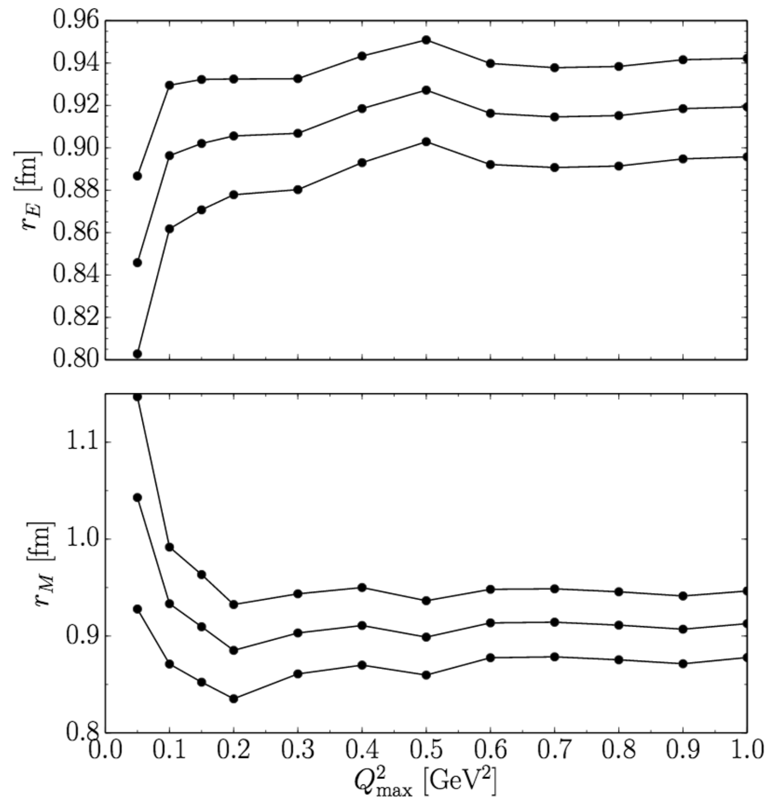


FIG. 8: Same as Fig. 7 but with both the world polarization data in addition to the world cross section data.

Very Low q^2 G_e dominates the cross section & very high q^2 G_m dominates the cross section.

(i.e. for this fit it depends on which why you look at the plot)

IV. FIXED RADIUS FITS

We also tried fixing the radius to 0.84 fm and 0.88 fm (i.e. $a_1 = 0.1176$ fm^2 and 0.1292 fm^2 respectively) and performing a five parameter fit of the Mainz14 Rosenbluth G_E data. The χ^2 is significantly better for the smaller radius.

TABLE III: Repeating the above $j = 6$ fit, but with the a_1 term fixed to the atomic hydrogen and muonic hydrogen values of the proton radius, 0.84 fm and 0.88 fm.

Fixed Radius	χ^2	χ^2/ν	n_0	a_2	a_3	a_4	a_5
0.84 fm	56.34	0.783	0.994(1)	$1.12(1) \cdot 10^{-2}$	$-0.93(2) \cdot 10^{-3}$	$5.0(1) \cdot 10^{-5}$	$1.20(5) \cdot 10^{-6}$
0.88 fm	142.1	1.97	1.003(1)	$1.62(1) \cdot 10^{-2}$	$-1.78(1) \cdot 10^{-3}$	$1.14(1) \cdot 10^{-4}$	$-2.90(7) \cdot 10^{-6}$