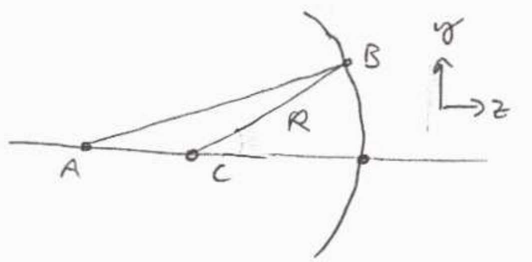
Phys 531 Midterm Exam Solutions

1. a) Could assume that B is close to axis and use parabolic expansion of mirror, but don't need to:



For coordinate system centered at C, $B = (y', z')$ with $y'^2 + z'^2 = R^2$

Point A is at $(0, z)$

So $\overline{AB} = \sqrt{y'^2 + (z - z')^2}$

$$= \sqrt{y'^2 + z'^2 + z^2 - 2zz'}$$

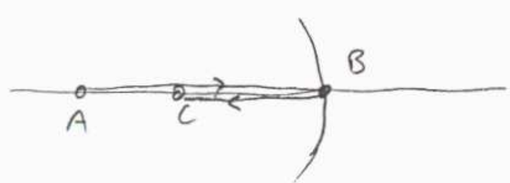
$$\overline{AB} = \sqrt{R^2 + z^2 - 2z\sqrt{R^2 - y'^2}}$$

So optical path length $\mathcal{S} = \overline{ABC} = R + \sqrt{R^2 + z^2 - 2z\sqrt{R^2 - y'^2}}$

Need $\dfrac{d\mathcal{S}}{dy'} = 0 = \dfrac{-z\dfrac{-y'}{\sqrt{R^2 - y'^2}}}{\sqrt{R^2 + z^2 - 2z\sqrt{R^2 - y'^2}}}$

Solved by $y' = 0$

So, B is on axis:



b) For small positive y', have

$$\frac{ds}{dy'} \approx \frac{z y'}{R\sqrt{R^2+z^2-2Rz}} = \frac{z y'}{R|R-z|}$$

So if $z<0$, $\frac{ds}{dy'} < 0$, s is decreasing as y' increases

So, $s(y'=0)$ is local $\boxed{\text{maximum}}$

c) If $z>0$, $\frac{ds}{dy} > 0$, s increases as y' increases

So $s(y'=0)$ is local $\boxed{\text{minimum}}$

Notes:

- Can also do in polar coordinates
- Harder to with z' as variable, because z' itself has an extremum in same place as s.

c) Have $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{F}$, if s_1 & s_2 measured from principal planes

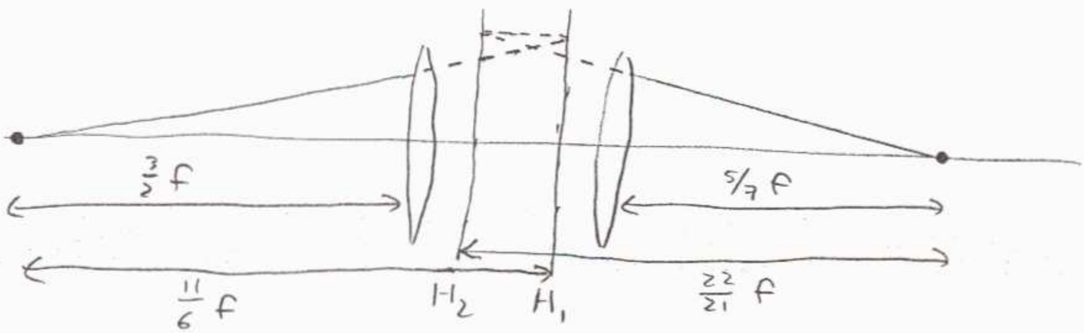
So, $s_1 = \frac{3}{2}f + \frac{1}{3}f = \frac{11}{6}f$

$$\frac{1}{s_2} = \frac{1}{\frac{2}{3}f} - \frac{1}{\frac{11}{6}f} = \left(\frac{3}{2} - \frac{6}{11}\right)\frac{1}{f} = \frac{21}{22}\frac{1}{f}$$

$$s_2 = \frac{22}{21}f$$

So measured from rear of system, image is at $s_2' = \frac{22}{21}f - \frac{1}{3}f = \frac{15}{21}f = \boxed{\frac{5}{7}f}$

Sketch:



Dashes show projections to principle planes.

2. a) Ray matrix is

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & f/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

matrix for second lens — Matrix for gap — matrix for first lens

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{f}{2} \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{f}{2} \\ -\frac{3}{2f} & \frac{1}{2} \end{bmatrix}$$

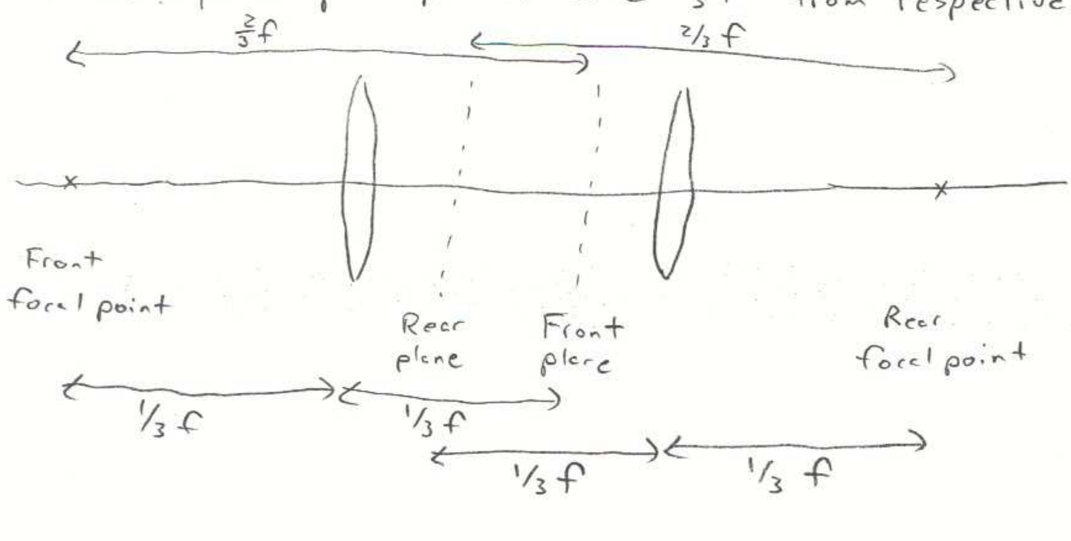
Check $AD - BC = \frac{1}{4} - \left(-\frac{3}{4}\right) = 1$ ✓

b) System focal length $F = -\frac{1}{C} = \boxed{\frac{2}{3}f}$

Front focal length $= -\frac{D}{C} = \boxed{\frac{1}{3}f}$

Back focal length $= -\frac{A}{C} = \boxed{\frac{1}{3}f}$

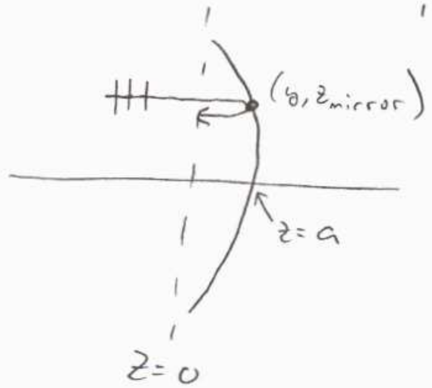
So: principal planes are $\frac{1}{3}f$ from respective lenses



3.a) For $D \ll |R|$, can use parabolic expansion of sphere:

$$z_{mirror} = a + \frac{\rho^2}{2R} \qquad \rho = \sqrt{x^2+y^2}$$
$$(R<0)$$

where a is location where mirror intercepts axis:



Have $z_{mirror} = 0$ at $\rho = \frac{D}{2}$

So

$$0 = a + \frac{D^2}{8R}$$

$$a = -\frac{D^2}{8R}$$

and $$z_{mirror} = \frac{-D^2+4\rho^2}{8R}$$

So, wave incident at point (x,y) must travel a distance $2z_{mirror}$ before returning to $z=0$.

This neglects angle of output wave:

Ok since $\Theta \approx \frac{y}{|R|} < \frac{D}{|R|} \ll 1$

So, at point (x,y) acquire phase $\phi = 2k_0 z_{mirror}$

$$\boxed{t(x,y) = e^{-i\phi} = e^{-ik_0 \frac{D^2+4\rho^2}{4R}}}$$

3(b) Output wave is

$$U_{out} = A\, e^{-ik_0 \frac{D^2}{4R}}\, e^{-ik_0 \frac{\rho^2}{R}}$$

Compare to parabolic wave at $z=0$:

$$U_{para} = \frac{A}{z-z_0}\, e^{+ik_0 z}\, e^{+ik_0 \frac{\rho^2}{2(z-z_0)}} \rightarrow -\frac{A}{z_0}\, e^{-ik_0 \frac{\rho^2}{2z_0}}$$

(note switch $k \to -k$ since travelling to left)

Same form if $R = 2z_0$, so wave is converging towards point $\boxed{z_0 = \frac{R}{2} = -\frac{|R|}{2}}$

[Converging since $z_0 < 0$ and wave travelling towards $z \to -\infty$]

Paraboloidal wave valid as long as $k \frac{\rho^2}{|R|} \ll 1$

Ok if $D^2 \ll \lambda |R|$

4. a) For either beam,

$$U_{out} = U_1 + U_2$$

$$= U_0 e^{-ikd_1} + U_0 e^{-ikd_2}$$

d_1 and d_2 are lengths of two paths

$$= U_0 e^{-ikd_1}\left(1 + e^{ik(d_1-d_2)}\right)$$

$$I_{out} = I_0 \left[1 + e^{ik(d_1-d_2)}\right]\left[1 + e^{-ik(d_1+d_2)}\right]$$

$$= I_0 \left[2 + 2\cos k(d_1-d_2)\right] \rightarrow \text{maxima when } k(d_1-d_2) = 2\pi n$$

$$= 4 I_0 \cos^2 \frac{k}{2}(d_1-d_2)$$

When mirror moves distance d, d_1 increases by $2d$, while d_2 decreases by $2d$. So, d_1-d_2 changes by $4d$

So, see N maxima, $N = \frac{4kd}{2\pi} = \frac{4d}{\lambda}$

True for either wave, so

$$\frac{\lambda}{\lambda_{ref}} = \frac{4d/N}{4d/N_r} = \frac{N_r}{N}$$

$$\boxed{\lambda = \frac{N_r}{N}\lambda_{ref}}$$

4 b) Uncertainties related by

$$\Delta\lambda^2 = \left(\frac{\partial\lambda}{\partial N}\Delta N\right)^2 + \left(\frac{\partial\lambda}{\partial N_r}\Delta N_r\right)^2$$

(Exact relation not so important.)

$$\Delta\lambda^2 = \left(\frac{N_r}{N^2}\lambda_{ref}\,\Delta N\right)^2 + \left(\frac{1}{N}\lambda_{ref}\,\Delta N\right)^2$$

$$= \lambda_{ref}^2\left(\frac{\Delta N}{N}\right)^2\left[1 + \frac{N_r^2}{N^2}\right]$$

If $d = 0.5\,m$ $\lambda_{ref} = 632.8\,nm$, $\lambda \approx 500\,nm$

Then $N_r = \frac{4d}{\lambda_{ref}} = 3.2 \times 10^6$

$N \approx \frac{4d}{\lambda} = 2.6 \times 10^6$

So

$$\Delta\lambda \approx \lambda_{ref}\left(\frac{0.5}{2.6\times10^6}\right)\sqrt{1+\left(\frac{3.2}{2.6}\right)^2}$$

$$= \lambda_{ref} \times 3\times10^{-7}$$

$$\boxed{\Delta\lambda \approx 2\times10^{-13}\,m}$$

or, $\frac{\Delta\lambda}{\lambda} \approx 2.5\times10^{-7}$

5. In general,

$$t(x,y) = h_0 e^{-i(n-1)k_0 d(x,y)}$$

$$h_0 = e^{-ik_0(d_0+a)}$$, since d_0+a = max thickness of plate

$$d(x,y) = d_0 + a\sin\left(\frac{2\pi x}{\Lambda}\right)$$

So define $h_1 = e^{-ik_0(d_0+a)} e^{-i(n-1)k_0 d_0}$

$$= e^{-ik_0(nd_0+a)}$$

Then

$$t(x,y) = h_1 e^{-i(n-1)k_0 a \sin\frac{2\pi x}{\Lambda}}$$

Note that $\beta \equiv (n-1)k_0 a = \frac{2\pi \times 0.5 \times 5\,nm}{500\,nm} = 0.031 \ll 1$

So, can expand

$$t(x,y) \approx h_1\left[1 - i\beta\sin\frac{2\pi x}{\Lambda}\right]$$

and

$$U(x,y,z=0) = U_{in}t$$

$$= Ah_1\left[1 - i\beta\sin\frac{2\pi x}{\Lambda}\right]$$

$$= Ah_1\left[1 - \frac{\beta}{2}\left(e^{i\frac{2\pi x}{\Lambda}} + e^{-i\frac{2\pi x}{\Lambda}}\right)\right]$$

This is sum of waves with form $e^{-i2\pi\nu_x x}$ for $\nu_x = 0, \pm\frac{1}{\Lambda}$

So, each propagates with $\nu_z = \sqrt{\frac{1}{\lambda^2} - \frac{1}{\Lambda^2}}$ or $\frac{1}{\lambda}$

Since $\Lambda = 2\lambda$,

$$\sqrt{\frac{1}{\lambda^2} - \frac{1}{\Lambda^2}} = \frac{1}{\lambda}\sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2\lambda}$$

So
$$U(x,y,z) = Ah_1 \left\{ e^{-i\frac{2\pi z}{\lambda}} - \frac{\beta}{2}\left[e^{i2\pi\left(\frac{x}{2\lambda} - \frac{z\sqrt{3}}{2\lambda}\right)} - e^{-i2\pi\left(\frac{x}{2\lambda} + \frac{z\sqrt{3}}{2\lambda}\right)} \right] \right\}$$

$$\boxed{U(x,y,z) \approx Ah_1 \left[e^{-ik_0 z} - i\beta e^{-ik_0 z \frac{\sqrt{3}}{2}} \sin \frac{2\pi x}{\Lambda} \right]}$$

$$h_1 = e^{-ik_0(nd_0 + a)}$$

$$\beta = (n-1)k_0 a$$

6. Fraunhofer diffraction gives

$$I(x,y,d) = \frac{1}{\lambda^2 d^2} \left| F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)\right|^2$$

for $F(\nu_x, \nu_y) =$ Fourier transform of wave in $z=0$ plane

So, $I(0,0,d) = \frac{1}{\lambda^2 d^2} |F(0,0)|^2$

$$F(\nu_x, \nu_y) = \iint f(x,y)\, e^{i2\pi(\nu_x x + \nu_y y)}\, dx\, dy$$

and $F(0,0) = \iint f(x,y)\, dx\, dy$

If $f = u_{in}$ inside hole
$= 0$ outside hole

Then $F(0,0) = \iint_{\substack{\text{Area} \\ \text{of hole}}} u_{in}\, dx\, dy = u_{in} A$

and

$$I(0,0,d) = \frac{A^2}{\lambda^2 d^2} |u_{in}|^2$$

$$\boxed{= \frac{A^2}{\lambda^2 d^2} I_{in}}$$