1. In class we discussed a "classical" model for an atom, in which a rigid cloud of negative charge (the electrons) surrounds a heavy nucleus of positive charge. We argued that the electron cloud would respond linearly to an applied force, giving our equation for a resonant medium. Develop the details of this model as follows:

(a) Suppose an atom has one electron with charge $-e$ and a nucleus of charge $+e = 1.6 \times 10^{-19}$ C, and the electronic charge is uniformly distributed over a sphere with the Bohr atomic radius $a_0 = 4\pi\epsilon_0\hbar^2/(me^2) = 5.29 \times 10^{-11}$ m. If the electron cloud is displaced a distance x, the magnitude of the attractive force between the cloud center and the nucleus is given by

$$F_{\text{Coulomb}} = \frac{1}{4\pi\epsilon_0}\frac{eq}{x^2}$$

where q is the portion of the electronic charge contained within a sphere of radius x. (In the figure below, all the charge within the dashed sphere can be effectively taken to lie at the center of the cloud, while charge outside the dashed sphere has no effect.) Using this result, show that the restoring force is linear in x and calculate the spring constant κ and the oscillation frequency ω_0.

(b) If the electron is oscillating, the accelerating charge will radiate light and thus dissipate energy. We modeled this dissipation by including a damping term in the equation of motion,

$$\ddot{x} + \sigma\dot{x} + \omega_0^2 x = 0$$

Solve this equation for initial conditions $x(0) = \xi$ and $\dot{x}(0) = 0$, in the limit $\sigma \ll \omega_0$ (but not $\sigma = 0$). Given that the mechanical energy of the electron is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}\kappa x^2$$

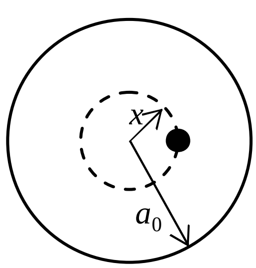
calculate dE/dt.

(c) We also know that the power radiated by a dipole is

$$P = \frac{p_0^2\omega_0^4}{12\pi\epsilon_0 c^3}$$

where the dipole moment here is $p(t) = -ex(t) = p_0(t)\cos(\omega_0 t)$. Using this formula and the result of (b), find an expression for σ in terms of the atomic parameters.

(d) Numerically compare your results to the Lyman-α resonance of hydrogen, which has a wavelength of 122 nm and a damping coefficient $\sigma = 2\pi\times 100$ MHz.



2. Suppose that two plane waves,

$$\mathbf{E}_{\text{tot}} = A_1 \hat{\mathbf{x}} e^{i(\omega_1 t - k_1 z)} + A_2 \hat{\mathbf{x}} e^{i(\omega_2 t - k_2 z)}$$

are propagating through a medium with a complex index of refraction $\tilde{n}(\omega) = n - i\alpha/2k_0$. Take $|\omega_1 - \omega_2|$ small enough that $|\tilde{n}(\omega_1) - \tilde{n}(\omega_2)|$ small (but not zero). Calculate the intensity $I(z,t) = |\text{Re}\{\mathbf{S}\}|$ and show that it exhibits interference maxima which travel in space at speed v with

$$\frac{1}{v} = \frac{d\beta}{d\omega}$$

for $\beta = kn$. This is the group velocity. (See also problem 4 from assignment 5.)

3. In a low-density medium with an optical resonance at frequency ω_0, the electric susceptability for $\omega < \omega_0$ and $\omega_0 - \omega \gg \Delta\omega$ can be approximated by

$$\chi = \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2}$$

Calculate the group velocity v and dispersion coefficient D in this case, assuming $\chi_0 \ll 1$. Plot v, D, and the index of refraction n over the wavelength range 400 nm to 700 nm, for $\chi_0 = 0.1$ and a resonance wavelength $\lambda_0 = 2\pi c_0/\omega_0 = 300$ nm.

4. Saleh and Teich Problem 5.6-1, page 192. Note that a signal

$$U(t) = A(t)\exp(i2\pi\nu_0 t)$$

is considered to be amplitude modulated if the magnitude of A varies in time but the phase of A is constant. The signal is phase modulated if $|A|$ is constant but the phase of A varies in time.