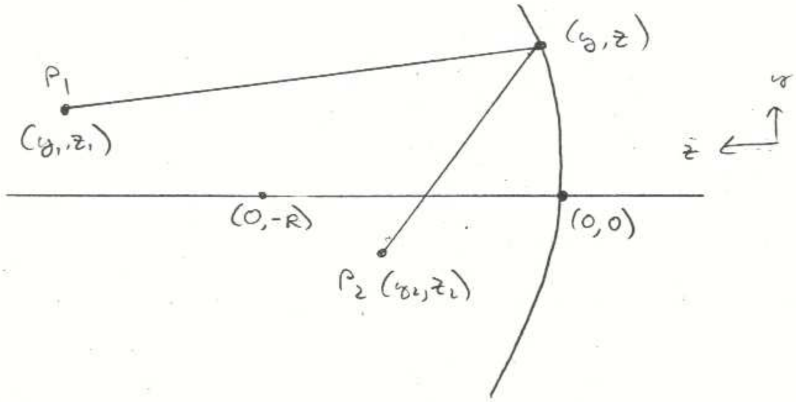
Phys 531 Asg 1 Solutions

1. Imaging with a mirror



(Drawn for $R<0$, $z_1, z_2 > 0$)

Easy to solve using Fermat's Principle

Optical path length from P_1 to P_2 is

$$\mathcal{S} = \sqrt{(y_1-y)^2+(z_1-z)^2} + \sqrt{(y-y_2)^2+(z-z_2)^2}$$

but y and z are related:

$$y^2 + (z+R)^2 = R^2$$

$$y^2 = -2zR - z^2$$

In paraxial limit, y is small compared to R, so z is also small.

Then neglect z^2, get

$$z \approx -\frac{y^2}{2R}$$

so

$$\mathcal{S} = \sqrt{(y_1-y)^2+\left(z_1+\frac{y^2}{2R}\right)^2} + \sqrt{(y-y_2)^2+\left(z_2+\frac{y^2}{2R}\right)^2}$$

Want $\mathcal{S}$ constant for all small y

So $$\frac{ds}{dy} = 0 = \frac{y - y_1 + \left(z_1 + \frac{y^2}{2R}\right)\left(+\frac{y}{R}\right)}{\sqrt{(y_1-y)^2 + \left(z_1 - \frac{y^2}{2R}\right)^2}} + \frac{y - y_2 + \left(z_2 + \frac{y^2}{2R}\right)\left(+\frac{y}{R}\right)}{\sqrt{(y_2-y)^2 + \left(z_2 - \frac{y^2}{2R}\right)^2}}$$

To leading order in y (including y_1 and y_2), get

$$\frac{ds}{dy} = \frac{y - y_1 + y\frac{z_1}{R}}{z_1} + \frac{y - y_2 + y\frac{z_2}{R}}{z_2} = 0$$

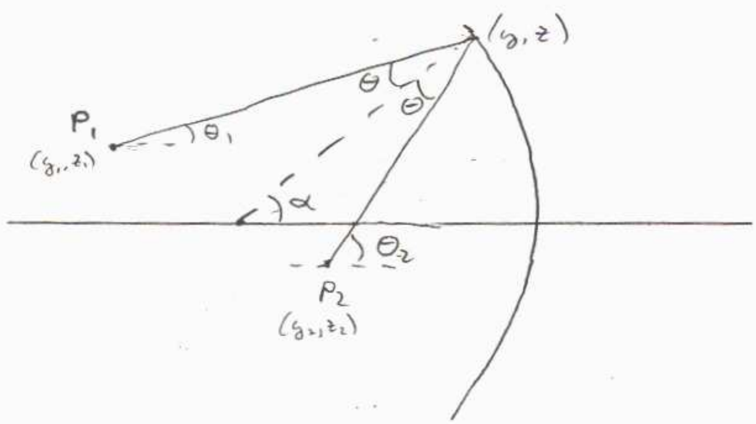
or $$y\left(\frac{1}{z_1} + \frac{1}{R} + \frac{1}{z_2} + \frac{1}{R}\right) - \left(\frac{y_1}{z_1} + \frac{y_2}{z_2}\right) = 0$$

Since this should hold for all small y, need

$$\boxed{\frac{1}{z_1} + \frac{1}{z_2} = \frac{-2}{R}} \quad \text{and} \quad \boxed{y_2 = -y_1\frac{z_2}{z_1}}$$

Alternative solution:

Use geometry:



$$\left.\begin{aligned} \theta_1 &= \alpha - \theta \\ \theta_2 &= \alpha + \theta \end{aligned}\right] \quad \theta_1 + \theta_2 = 2\alpha$$

From geometry, $\theta_1 = \frac{y-y_1}{z_1}$ $\quad \theta_2 = \frac{y-y_2}{z_2}$ $\quad \alpha = -\frac{y}{R}$ $\quad (R<0)$

(in paraxial approximation)

So $$\frac{y-y_1}{z_1} + \frac{y-y_2}{z_2} = -\frac{2y}{R}$$

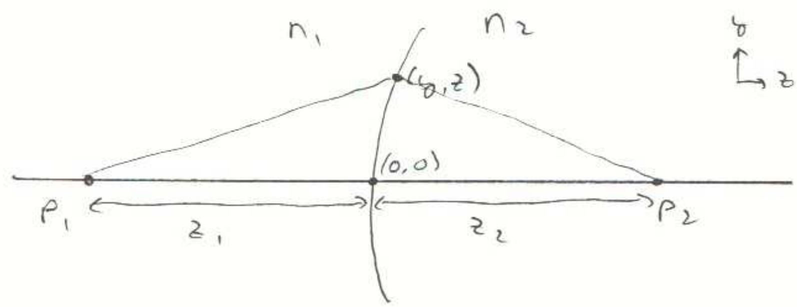
$$y\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{2}{R}\right) - \left(\frac{y_1}{z_1}+\frac{y_2}{z_2}\right) = 0$$

True for any y, as before, so

$$\boxed{\frac{1}{z_1}+\frac{1}{z_2} = -\frac{2}{R}}$$

$$\boxed{y_2 = -y_1\frac{z_2}{z_1}}$$

2. Aberration free surface



Optical path length from P_1 to P_2 is

$$\delta = n_1\sqrt{y^2+(z+z_1)^2} + n_2\sqrt{y^2+(z-z_2)^2}$$

Want $\mathcal{S} =$ constant for all (y, z).

Know that $(0,0)$ must be on surface, so

for ray passing along optic axis, have

$$\mathcal{S} = n_1 z_1 + n_2 z_2$$

So equation for surface is

$$\boxed{n_1 \sqrt{y^2 + (z+z_1)^2} + n_2 \sqrt{y^2 + (z-z_2)^2} = n_1 z_1 + n_2 z_2}$$

3. a) Output ray $\begin{bmatrix} y_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} B\Theta_1 \\ y_1 C + \Theta_1 D \end{bmatrix}$

So y_2 is independent of y_1

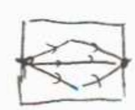
$\Rightarrow$ All parallel rays at input are focussed to single point

b) $$\begin{bmatrix} y_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} Ay_1 \\ Cy_1 + D\Theta_1 \end{bmatrix}$$

So y_2 is independent of Θ_1

$\Rightarrow$ All rays passing through an input point are imaged to the same output point

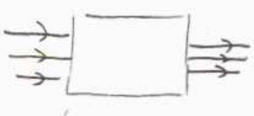
(like an imaging system)



3c) $$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} Ay_1 + B\theta_1 \\ D\theta_1 \end{bmatrix}$$

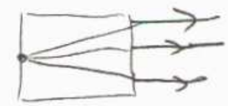
θ_2 independent of y_1

$\Rightarrow$ Parallel input rays remain parallel
(like a telescope)



d) $$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} Ay_1 + B\theta_1 \\ Cy_1 \end{bmatrix}$$

θ_2 independent of θ_1



$\Rightarrow$ All rays passing through an input point become collimated

4. Focal length of thin lens

Eq 1.2-12 gives

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

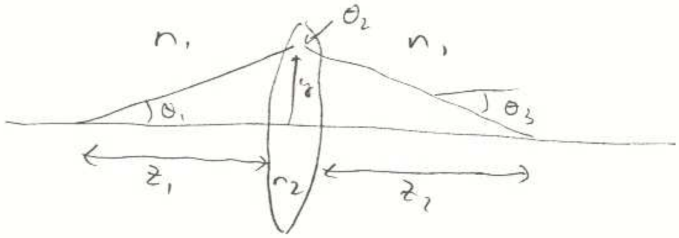
$R_1 = +20\text{ cm}$
$R_2 = -30\text{ cm}$

$$= 0.5\left(\frac{1}{20} + \frac{1}{30}\right)$$

$$= 0.0417\ \text{cm}^{-1}$$

$$\boxed{f = 24\text{ cm}}$$

In water, rederive lens equation



$$\theta_1 = \frac{y}{z_1}$$

$$\theta_2 = \frac{n_1}{n_2}\theta_1 - \frac{n_2 - n_1}{n_2 R_1} y = \frac{n_1}{n_2}\frac{y}{z_1} - \frac{n_2 - n_1}{n_2 R_1} y$$

$$\theta_3 = \frac{n_2}{n_1}\theta_2 - \frac{n_1 - n_2}{n_1 R_2} y$$

$$= \frac{y}{z_1} - \frac{n_2 - n_1}{n_1 R_1} y - \frac{n_1 - n_2}{n_1 R_2} y = -\frac{y}{z_2}$$

So $$\boxed{\frac{1}{z_1} + \frac{1}{z_2} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

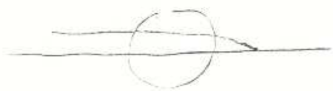
and $$\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

here $n_1 = 1.33$
$n_2 = 1.5$

$$\frac{1}{f} = \frac{1}{8}\left(\frac{1}{20} + \frac{1}{30}\right) = \frac{1}{96}$$

$$\boxed{f = 96 \text{ cm}}$$

5. a) Matrix method



$$M_{TOT} = \underbrace{\begin{bmatrix} 1 & f_0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{propagation} \\ \text{to focus}}} \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1-n}{a} & n \end{bmatrix}}_{\substack{\text{second} \\ \text{surface} \\ (R=-a)}} \underbrace{\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}}_{\substack{\text{middle} \\ \text{of sphere}}} \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{n-1}{na} & \frac{1}{n} \end{bmatrix}}_{\substack{\text{first surface} \\ (R=+a)}}$$

$$= \begin{bmatrix} 1+\left(\frac{1-n}{a}\right)f_0 & nf_0 \\ \frac{1-n}{a} & n \end{bmatrix} \begin{bmatrix} -1+\frac{2}{n} & \frac{2a}{n} \\ -\frac{n-1}{na} & \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} \left(1+\frac{1-n}{a}f_0\right)\left(-1+\frac{2}{n}\right)-(n-1)\frac{f_0}{a} & \frac{2a}{n}+\frac{2(1-n)}{n}f_0+f_0 \\ \left(-1+\frac{2}{n}\right)\left(\frac{1-n}{a}\right)-\frac{n-1}{a} & \frac{2(1-n)}{n}+1 \end{bmatrix}$$

Want $y_2 = 0$ independent of y_1, so need $A=0$

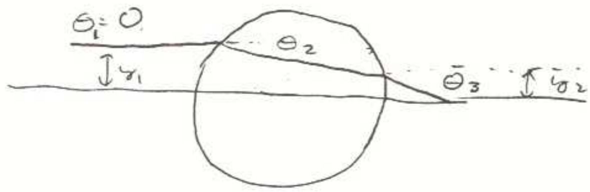
$$\left[1+(1-n)\frac{f_0}{a}\right]\left(-1+\frac{2}{n}\right)-(n-1)\frac{f_0}{a}=0$$

$$\frac{f_0}{a}(n-1)\left[-1-1+\frac{2}{n}\right] = -1+\frac{2}{n}$$

$$\frac{f_0}{a} = \frac{\left(\frac{2}{n}-1\right)\frac{n}{2}}{n-1} = \frac{1-\frac{n}{2}}{n-1}$$

$$\boxed{f_0 = \left(\frac{1-\frac{n}{2}}{n-1}\right)a}$$

Alternative solution to 5(a)



From 1.2-8 $\quad \Theta_2 = -\frac{n-1}{na} y_1$

see $\quad y_2 = y_1 + 2a\Theta_2 = y_1\left(1 - \frac{2(n-1)}{n}\right)$

$$= y_1\left(-1 + \frac{2}{n}\right)$$

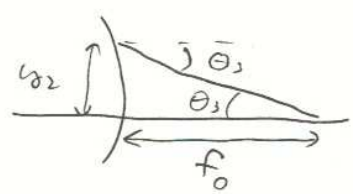
Then $\Theta_3 = n\Theta_2 + \frac{1-n}{a} y_2$

$$= \frac{1-n}{a} y_1 + \frac{1-n}{a}\left(-1+\frac{2}{n}\right) y_1$$

$$= \frac{y_1}{a}\left(1-n-1+n + \frac{2(1-n)}{n}\right) = \frac{y_1}{a}\,\frac{2(1-n)}{n}$$

f_0 is defined by $\frac{y_2}{f_0} = -\Theta_3$

or $\quad f_0 = -\frac{y_2}{\Theta_3}$



$$= \frac{y_1\left(1-\frac{2}{n}\right)}{\frac{y_1}{a}\,2\,\frac{(1-n)}{n}}$$

$$= \frac{a}{2}\,\frac{n-2}{1-n} = \boxed{a\,\frac{1-\frac{n}{2}}{n-1}}$$

5(b) From diagram & law of sines, have

$$f = b - a = a\left(\frac{\sin\beta}{\sin\alpha} - 1\right)$$

So if we can determine angles α and β, we're done.

From Snell's Law, $n\sin\theta_2 = \sin\theta_1 = \frac{y}{a} = 0.7$

So $\theta_1 = 44.427^\circ$

$\theta_2 = 27.818^\circ$

Also gives $\theta_3 = \theta_1$

Since the sum of angles in a triangle is 180°, have

at center vertex

So $\theta_1 + 180^\circ - 2\theta_2 + x = 180^\circ$

$x = 2\theta_2 - \theta_1$

Also set $\beta = 180^\circ - \theta_3 = 180^\circ - \theta_1$

Then $\alpha = 180^\circ - \beta - x$

$= 180^\circ - 180^\circ + \theta_1 - 2\theta_2 + \theta_1 = 2\theta_1 - 2\theta_2$

So, $f = a\left[\frac{\sin(180^\circ - \theta_1)}{\sin(2\theta_1 - 2\theta_2)} - 1\right] = a\left[\frac{\sin\theta_1}{\sin 2(\theta_1 - \theta_2)} - 1\right]$

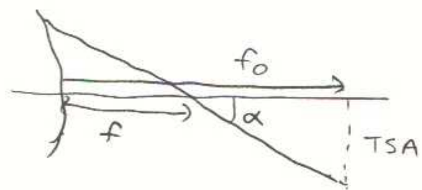
For our values, $\boxed{f = 0.278\,\text{mm}}$

5(c) For $n = 1.5$, $a = 1\,mm$ get $f_0 = 0.5\,mm$

So $\boxed{LSA = f_0 - f = 0.222\,mm}$

Since $\alpha = 2\theta_1 - 2\theta_2 = 33.218°$,

TSA is given by $(f_0 - f)\tan\alpha = \boxed{0.146\,mm}$



Note that for small y, in 5(b) we get $\theta_1 \approx \frac{y}{a}$

$\theta_2 = \frac{y}{na}$

$$f \to a\left[\frac{y/a}{2\left(\frac{y}{a} - \frac{y}{na}\right)} - 1\right]$$

$$= a\left[\frac{1}{2\left(1 - \frac{1}{n}\right)} - 1\right]$$

$$= a\left[\frac{n - 2(n-1)}{2(n-1)}\right]$$

$$= a\,\frac{2-n}{2(n-1)} = f_0 \checkmark$$