Phys 531 Assignment 10 Solutions

1. a) Have $q = -e \times \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi a_0^3} = -e\left(\frac{x}{a_0}\right)^3$

So $F_{coul} = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{x^2}\left(\frac{x}{a_0}\right)^3 = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{a_0^3}x$

$= -\kappa x$

for $\boxed{\kappa = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a_0^3}}$

Then $\omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{m a_0^3}}$

so $\boxed{\omega_0 = \sqrt{\frac{\hbar^2}{m^2 a_0^4}} = \frac{\hbar}{m a_0^2}}$ $\quad a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$

b) $\ddot{x} + \sigma\dot{x} + \omega_0^2 x = 0$

$x = e^{\lambda t}$

$\lambda^2 + \sigma\lambda + \omega_0^2 = 0$

$\lambda = \frac{1}{2}\left(-\sigma \pm \sqrt{\sigma^2 - 4\omega_0^2}\right) \approx -\frac{\sigma}{2} \pm i\omega_0$

So $x(t) = Ae^{-\frac{\sigma t}{2}}e^{i\omega_0 t} + Be^{-\frac{\sigma t}{2}}e^{-i\omega_0 t}$

$A + B = \xi$

$\dot{x}\big|_0 = A\left(-\frac{\sigma}{2} + i\omega_0\right) + B\left(-\frac{\sigma}{2} - i\omega_0\right) = 0$

for $\sigma \ll \omega$, get

$A - B = 0$

So $A = B = \frac{1}{2}\xi$

$\boxed{x(t) = \xi e^{-\sigma t/2}\cos\omega_0 t}$

So $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$

$$\dot{x} \approx -\alpha \omega_0 e^{-\sigma t/2} \sin \omega_0 t$$

So $E(t) = \frac{1}{2} m \omega_0^2 \xi^2 e^{-\sigma t} \sin^2 \omega_0 t + \frac{1}{2} m \omega_0^2 \xi^2 e^{-\sigma t} \cos^2 \omega_0 t$

$$= \frac{1}{2} m \omega_0^2 \xi^2 e^{-\sigma t}$$

So $\frac{dE}{dt} = -\sigma \left(\frac{1}{2} m \omega_0^2 \xi^2 e^{-\sigma t} \right) = -\sigma E$

or $\boxed{\frac{dE}{dt} = -\sigma E}$

c) Radiate power $P = \frac{p_0^2 \omega_0^4}{12\pi \varepsilon_0 c^3}$

$$p_0(t) = -e\, x(t) = -e \xi e^{-\sigma t/2} \cos \omega_0 t$$

So $p_0(t) = -e \xi e^{-\sigma t/2}$

and $P(t) = \frac{e^2 \xi^2 e^{-\sigma t} \omega_0^4}{12\pi \varepsilon_0 c^3} = -\frac{dE}{dt} = \sigma E$

$$= \sigma \left(\frac{1}{2} m \omega_0^2 \xi^2 e^{-\sigma t} \right)$$

Solve for σ:

$$\sigma = \frac{e^2 \xi^2 \omega_0^4 e^{-\sigma t}}{12\pi \varepsilon_0 c^3} \frac{2}{m \omega_0^2 \xi^2 e^{-\sigma t}}$$

$$\sigma = \frac{e^2 \omega_0^2}{6\pi \varepsilon_0 m c^3}$$

Or, with $\dfrac{e^2}{4\pi\varepsilon_0} = \dfrac{\hbar^2}{m a_0}$

$$\Gamma = \frac{2}{3}\frac{\hbar^2}{m a_0}\frac{\omega_0^2}{mc^3} = \boxed{\frac{2}{3}\left(\frac{\hbar\omega_0}{mc^2}\right)^2 \frac{c}{a_0}}$$

d) Evaluate $\omega_0 = \dfrac{\hbar}{m a_0^2} = \dfrac{1.054\times10^{-34}\,\mathrm{J\,s}}{(9.109\times10^{-31}\,\mathrm{kg})(5.29\times10^{-11}\,\mathrm{m})^2}$

$$= 4.13\times10^{16}\ \mathrm{rad/s}$$

Corresponds to $\boxed{\lambda = \dfrac{2\pi c}{\omega_0} = 45\,\mathrm{nm}}$

$$\Gamma = \frac{2}{3}\left(\frac{1.054\times10^{-34}\,\mathrm{J\cdot s}\times 4.13\times10^{16}\,\mathrm{rad/s}}{9.109\times10^{-31}\,\mathrm{kg}\times[3\times10^{8}\,\mathrm{m/s}]^2}\right)^2 \frac{3\times10^{8}\,\mathrm{m/s}}{5.29\times10^{-11}\,\mathrm{m}}$$

$$= 10.6\times10^{9}\ \mathrm{rad/s}$$

$$\boxed{\Gamma = 2\pi\times 1.7\ \mathrm{GHz}}$$

2. Calculate $\tilde{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$

Since $\tilde{E}$ is sum of plane waves, and $\hat{k}\times\hat{x}=\hat{y}$ have (S&T 5.4-5 & 5.4-6)

$$\hat{H} = \frac{\tilde{n}_1}{\mathfrak{z}_0}\hat{y} A_1 e^{i(\omega_1 t - k_1 z)} + \frac{\tilde{n}_2}{\mathfrak{z}_0}\hat{y} A_2 e^{i(\omega_2 t - k_2 z)}$$

here $\tilde{n}_1 = n_1 - i\frac{\alpha_1}{2}\frac{c_0}{\omega_1}$ etc

So $\tilde{S} = \frac{1}{2}\tilde{E}\times\vec{H}^*$

$$= \frac{1}{2\mathfrak{z}_0}\left[A_1 e^{i(\omega_1 t - k_1 z)} + A_2 e^{i(\omega_2 t - k_2 z)}\right]\left[\tilde{n}_1^* A_1 e^{-i(\omega_1 t - k_1^* z)} + \tilde{n}_2^* A_2 e^{-i(\omega_2 t - k_2^* z)}\right]\hat{z}$$

where $k_1 = \tilde{n}_1 k_{10} = \left(n_1 - i\frac{\alpha_1}{2k_{10}}\right)k_{10}$

$k_{10} = \frac{\omega_1}{c_0}$

$k_{20} = \frac{\omega_2}{c_0}$

$$= n_1 k_{10} - i\frac{\alpha_1}{2}$$

$$= \beta_1 - i\frac{\alpha_1}{2}$$

Same for k_2

$$\tilde{S} = \frac{1}{2\mathfrak{z}_0}\hat{z}\left[A_1 e^{-\frac{\alpha_1 z}{2}} e^{i(\omega_1 t - \beta_1 z)} + A_2 e^{-\frac{\alpha_2 z}{2}} e^{i(\omega_2 t - \beta_2 z)}\right] \times\left[\frac{A_1}{k_{10}}\left(\beta_1 + i\frac{\alpha_1}{2}\right) e^{-\frac{\alpha_1 z}{2}} e^{-i(\omega_1 t - \beta_1 z)} + \frac{A_2}{k_{20}}\left(\beta_2 + i\frac{\alpha_2}{2}\right) e^{-\frac{\alpha_2 z}{2}} \times e^{-i(\omega_2 t - \beta_2 z)}\right]$$

So

$$\vec{S} = \frac{1}{2\eta_0}\hat{z}\Big[\frac{A_1^2}{k_{10}}\left(\beta_1 + i\frac{\alpha_1}{2}\right)e^{-\alpha_1 z} + \frac{A_2^2}{k_{20}}\left(\beta_2 + i\frac{\alpha_2}{2}\right)e^{-\alpha_2 z}$$

$$+ \frac{A_1 A_2}{k_{20}}\left(\beta_2 + i\frac{\alpha_2}{2}\right)e^{i[(\omega_1-\omega_2)t + (\beta_1-\beta_2)z]}\ e^{-\frac{\alpha_1+\alpha_2}{2}z}$$

$$+ \frac{A_1 A_2}{k_{10}}\left(\beta_1 + i\frac{\alpha_1}{2}\right)e^{-i[(\omega_1-\omega_2)t + (\beta_1-\beta_2)z]}\ e^{-\frac{\alpha_1+\alpha_2}{2}z}\Big]$$

Real part

$$\mathrm{Re}\{\vec{S}\} = \frac{1}{2\eta_0}\hat{z}\Big[\frac{A_1^2}{k_{10}}\beta_1 e^{-\alpha_1 z} + \frac{A_2^2}{k_{20}}\beta_2 e^{-\alpha_2 z}$$

$$+ \frac{A_1 A_2}{k_{20}}\beta_2 e^{-\frac{\alpha_1+\alpha_2}{2}z}\cos(\Delta\omega t - \Delta\beta z)$$

$$- \frac{A_1 A_2}{k_{20}}\frac{\alpha_2}{2} e^{-\frac{\alpha_1+\alpha_2}{2}z}\sin(\Delta\omega t - \Delta\beta z)$$

$$+ \frac{A_1 A_2}{k_{10}}\beta_1 e^{-\frac{\alpha_1+\alpha_2}{2}z}\cos(\Delta\omega t - \Delta\beta z)$$

$$+ \frac{A_1 A_2}{k_{10}}\frac{\alpha_1}{2} e^{-\frac{\alpha_1+\alpha_2}{2}z}\sin(\Delta\omega t - \Delta\beta z)\Big]$$

Now, we're given $\omega_1 \approx \omega_2$ so have $\beta_1 \approx \beta_2 \approx \bar{\beta}$

$\alpha_1 \approx \alpha_2 \approx \bar{\alpha}$

here $\Delta\omega = \omega_1 - \omega_2$

$\Delta\beta = \beta_1 - \beta_2$

Also, $\frac{\beta_i}{k_{i0}} = n_i$ and $n_1 \approx n_2 \approx \bar{n}$

So,

$$I \approx \frac{1}{2\mathfrak{Z}_0} e^{-\bar{\alpha} z} \Big[\bar{n} A_1^2 + \bar{n} A_2^2$$

$$+ 2\bar{n} A_1 A_2 \cos(\Delta\omega t - \Delta\beta z)$$

$$+ \frac{A_1 A_2}{2} \left(\frac{\alpha_1}{k_{10}} - \frac{\alpha_2}{k_{20}} \right) \sin(\Delta\omega t - \Delta\beta z) \Big]$$

Interference pattern travels at speed

$$V = \frac{\Delta\omega}{\Delta\beta}$$

or, $\boxed{\frac{1}{V} \to \frac{d\beta}{d\omega}}$

3. Have

$$\frac{1}{v} = \frac{d\beta}{d\omega} \qquad \beta = k_o n = \frac{n\omega}{c_o}$$

$$n = \sqrt{1+\chi} \simeq 1 + \frac{\chi}{2} \qquad \text{for } \chi \ll 1$$

$$\frac{1}{v} = \frac{d}{d\omega}\left[\frac{\omega}{c_o} + \frac{1}{2c_o}\chi_o \frac{\omega_o^2 \omega}{\omega_o^2-\omega^2}\right]$$

$$= \frac{1}{c_o} + \frac{1}{2c_o}\chi_o \left[\frac{\omega_o^2}{\omega_o^2-\omega^2} + \frac{\omega_o^2\omega(2\omega)}{(\omega_o^2-\omega^2)^2}\right]$$

$$= \frac{1}{c_o} + \frac{1}{2c_o}\chi_o \frac{\omega_o^2(\omega_o^2-\omega^2) + 2\omega_o^2\omega^2}{(\omega_o^2-\omega^2)^2}$$

$$= \frac{1}{c_o} + \frac{\chi_o}{2c_o} \frac{\omega_o^2(\omega_o^2+\omega^2)}{(\omega_o^2-\omega^2)^2}$$

So $$\boxed{v \simeq c_o\left[1 - \frac{\chi_o}{2}\frac{\omega_o^2(\omega_o^2+\omega^2)}{(\omega_o^2-\omega^2)^2}\right]} \qquad \text{for } \chi_o \ll 1$$

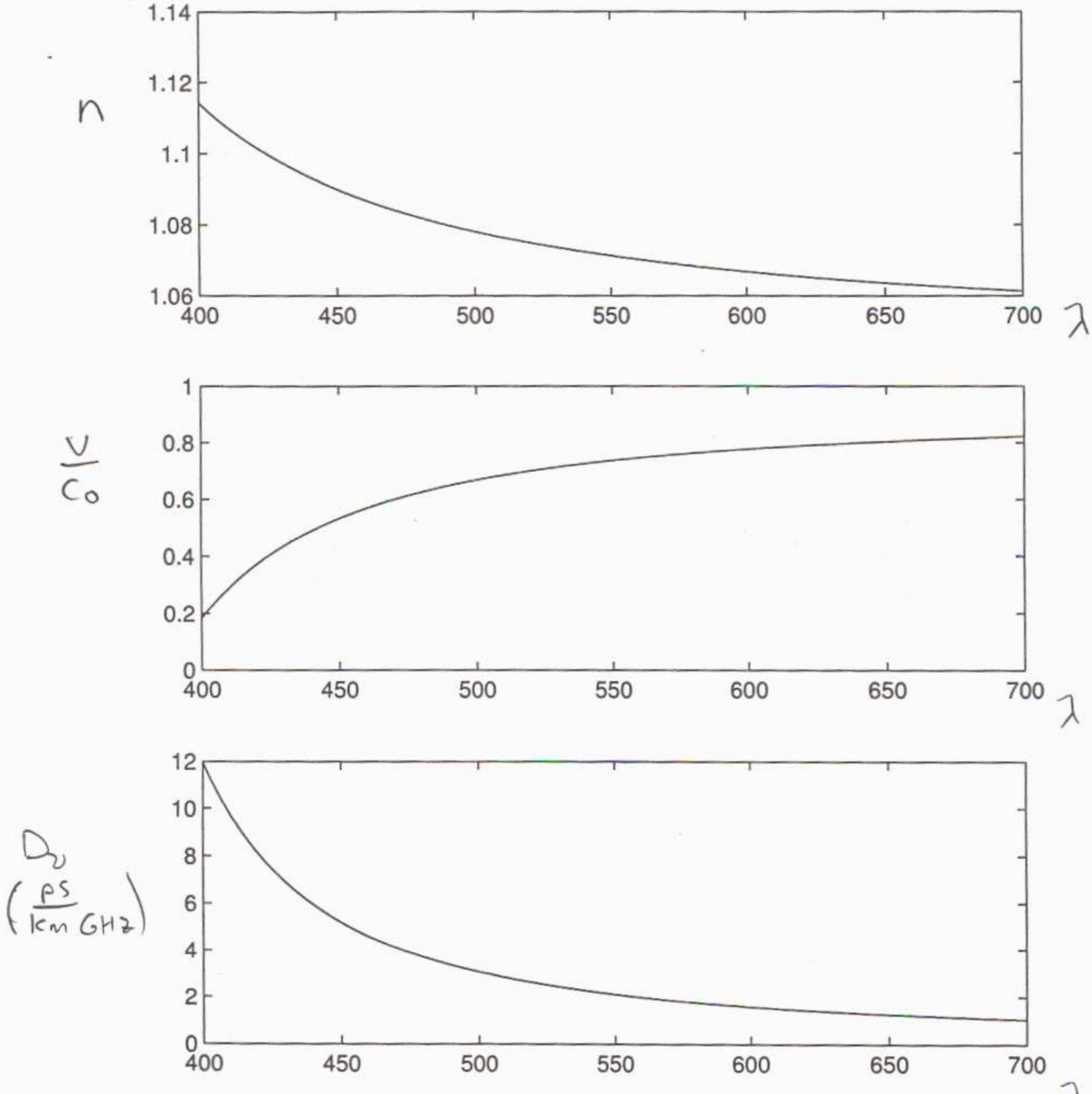
$$D_v = 2\pi \frac{d^2\beta}{d\omega^2} = 2\pi \frac{d}{d\omega}\frac{1}{v}$$

$$= \frac{2\pi}{c}\frac{d}{d\omega}\left[1 + \chi_o \frac{\omega_o^2(\omega_o^2+\omega^2)}{(\omega_o^2-\omega^2)^2}\right]$$

$$= \frac{2\pi}{c}\chi_o\left[\frac{\omega_o^2\, 2\omega}{(\omega_o^2-\omega^2)^2} + \frac{\omega_o^2(\omega_o^2+\omega^2)\,2\omega}{(\omega_o^2-\omega^2)^3}\right]$$

$$= \frac{2\pi}{c}\chi_o \frac{2\omega_o^2\omega}{(\omega_o^2-\omega^2)^3}\left[\omega_o^2-\omega^2+\omega_o^2+\omega^2\right]$$

$$\boxed{D_v = \frac{2\pi}{c}\chi_o \frac{4\omega_o^4\omega}{(\omega_o^2-\omega^2)^3}}$$



Note, approximation that $\gamma_0 \to 0$ is not really valid when $\frac{v}{c_0}$ is much less than 1, bad below ~500 nm

4. Here $\mathcal{U}(t, z=0) = [1 + m\cos(2\pi f_s t)] e^{i 2\pi \nu_0 t}$

$$= e^{i2\pi\nu_0 t} + \frac{m}{2} e^{i2\pi(\nu_0 + f_s)t} + \frac{m}{2} e^{i2\pi(\nu_0 - f_s)t}$$

Define $\nu_0 - f_s = \nu_1$
$\nu_0 + f_s = \nu_2$

$$\mathcal{U}(z=0) = e^{i2\pi\nu_0 t} + \frac{m}{2} e^{i2\pi\nu_1 t} + \frac{m}{2} e^{i2\pi\nu_2 t}$$

Sum of monochrometic waves, each of which propagates as $e^{-i\beta(\nu) z}$

$$\mathcal{U}(z,t) = e^{i(2\pi\nu_0 t - \beta_0 z)} + \frac{m}{2} e^{i(2\pi\nu_1 t - \beta_1 z)} + \frac{m}{2} e^{i(2\pi\nu_2 t - \beta_2 z)}$$

$$= e^{i(2\pi\nu_0 t - \beta_0 z)} \left\{ 1 + \frac{m}{2} e^{i[-2\pi f_s t + (\beta_0 - \beta_1)z]} + \frac{m}{2} e^{i[2\pi f_s t + (\beta_0 - \beta_2)z]} \right\}$$

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Complex envelope is

$$\boxed{A(z,t) = 1 + \frac{m}{2} e^{i[-2\pi f_s t + (\beta_0 - \beta_1)z]} + \frac{m}{2} e^{i[2\pi f_s t + (\beta_0 - \beta_2)z]}}$$

Get amplitude modulation when phase of A is constant

$$\tan\phi = \frac{\frac{m}{2}\sin[-2\pi f_s t + (\beta_0 - \beta_1)z] + \frac{m}{2}\sin[2\pi f_s t + (\beta_0 - \beta_2)z]}{1 + \frac{m}{2}\cos[-2\pi f_s t + (\beta_0 - \beta_1)z] + \frac{m}{2}\cos[2\pi f_s t + (\beta_0 - \beta_2)z]}$$

Note

$$\sin\left[-2\pi f_s t + (\beta_0-\beta_1)z\right] + \sin\left[2\pi f_s t + (\beta_0-\beta_2)z\right]$$

$$= 2\sin\tfrac{1}{2}\left[(\beta_0-\beta_1)z + (\beta_0-\beta_2)z\right]\cos\tfrac{1}{2}\left[4\pi f_s t + (\beta_1-\beta_2)z\right]$$

So, if $\frac{1}{2}(2\beta_0-\beta_1-\beta_2)z = n\pi$ integer n,

then $\phi = 0$ and signal is amplitude modulated

So, $$\boxed{z = \frac{2\pi n}{2\beta_0-\beta_1-\beta_2}}$$