Phys 531 Assignment 11 Solutions

1. (a) If $|\chi| \ll 1$, then $\tilde{n} = \sqrt{1+\chi} \simeq 1 + \frac{1}{2}\chi$

$$\tilde{n} = 1 + \frac{1}{4}\chi_0 \nu_0 \frac{(\nu_0-\nu) - i\,\Delta\nu/2}{(\nu_0-\nu)^2 + (\Delta\nu/2)^2}$$

So

$$n = \mathrm{Re}\,\tilde{n} = 1 + \frac{1}{4}\chi_0\nu_0 \frac{\nu_0-\nu}{(\nu_0-\nu)^2 + (\Delta\nu/2)^2}$$

$$\beta = \frac{2\pi\nu}{c_0} n = \frac{2\pi}{c_0}\left[\nu + \frac{1}{4}\chi_0\nu_0 \frac{\nu(\nu_0-\nu)}{(\nu_0-\nu)^2 + (\Delta\nu/2)^2}\right]$$

Then

$$\frac{1}{v} = \frac{1}{2\pi}\frac{d\beta}{d\nu} = \frac{1}{c_0}\left\{1 + \frac{1}{4}\chi_0\nu_0\left[\frac{\nu_0 - 2\nu}{(\nu_0-\nu)^2 + (\Delta\nu/2)^2} - \frac{\nu(\nu_0-\nu)(2)(\nu_0-\nu)(-1)}{(\nu_0-\nu)^2 + (\Delta\nu/2)^2}\right]\right\}$$

Evaluated at $\nu = \nu_0$ gives

$$\frac{1}{v(\nu_0)} = \frac{1}{c_0}\left\{1 - \frac{1}{4}\chi_0\nu_0 \frac{\nu_0}{(\Delta\nu/2)^2}\right\}$$

$$\boxed{\frac{1}{v} = \frac{1}{c_0}\left[1 - \chi_0\left(\frac{\nu_0}{\Delta\nu}\right)^2\right]}$$

We need $|\chi(\nu_0)| = \chi_0 \frac{\nu_0}{\Delta\nu} \ll 1$

But, we could have $\frac{\nu_0}{\Delta\nu} \gg 1$, so

$\chi_0\left(\frac{\nu_0}{\Delta\nu}\right)^2$ can be arbitrarily large

If $\chi_0 \left(\frac{\nu_0}{\Delta\nu}\right)^2 = 1 - \delta$ for $\delta > 0$,

then

$$\frac{1}{v} = \frac{1}{c_0}\left[1 - (1-\delta)\right] = \frac{\delta}{c_0}$$

or $v = \frac{c_0}{\delta} \to \infty$ for $\delta \to 0$

Also, if $\delta < 0$, get $v < 0$.

(b) $\tau = L\left(\frac{1}{c_0} - \frac{1}{v}\right) = \frac{L}{c_0}\chi_0\left(\frac{\nu_0}{\Delta\nu}\right)^2$

But $\alpha(\nu_0) = \frac{2\pi\nu_0}{c_0}\chi''(\nu_0) = \frac{2\pi}{c_0}\chi_0\frac{\nu_0^2}{\Delta\nu}$

$$= \frac{2\pi\Delta\nu}{c_0}\chi_0\frac{\nu_0^2}{\Delta\nu^2}$$

$$= 2\pi\Delta\nu\left(\frac{\tau}{L}\right)$$

Then $L\alpha = 2\pi\Delta\nu\tau \lesssim 1$

so we need $\tau \lesssim \frac{1}{2\pi\Delta\nu}$

Band width requirement says that we need

$$\Delta t \gg \frac{1}{2\pi\Delta\nu}$$

So in combination, have

$$\tau \lesssim \frac{1}{2\pi\Delta\nu} \ll \Delta t$$

So $\boxed{\tau \ll \Delta t}$

2. If polarizations are orthogonal, then

$$A_{1x}A_{2x}^* + A_{1y}A_{1y}^* = 0 \qquad (S+T\ 6.1\text{-}7)$$

or

$$\frac{A_{1x}}{A_{1y}} = -\frac{A_{2y}^*}{A_{2x}^*}$$

If $A_{1x} = a_{1x}e^{i\phi_{1x}}$ etc, then

$$\frac{a_{1x}}{a_{1y}}e^{-i\phi_1} = -\frac{a_{2y}}{a_{2x}}e^{-i\phi_2}$$

for $\phi_1 = \phi_{1y} - \phi_{1x}$

$\phi_2 = \phi_{2y} - \phi_{2x}$

Take magnitude of both sides, see

$$\frac{a_{1x}}{a_{1y}} = \frac{a_{2y}}{a_{2x}}$$

So that $\phi_2 = \phi_1 \pm \pi$

Equation for ellipse is $\dfrac{x^2}{a_x^2} + \dfrac{y^2}{a_y^2} - 2\cos\phi\,\dfrac{xy}{a_xa_y} = \sin^2\phi$

Find axes:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} \frac{1}{a_x^2} & -\frac{\cos\phi}{a_xa_y} \\ -\frac{\cos\phi}{a_xa_y} & \frac{1}{a_y^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \sin^2\phi$$

Eigenvalues

$$\lambda = \frac{1}{2}\left[\frac{1}{a_x^2} + \frac{1}{a_y^2} \pm \sqrt{\left(\frac{1}{a_x^2} - \frac{1}{a_y^2}\right)^2 + \frac{4\cos^2\phi}{a_x^2 a_y^2}}\,\right]$$

$$= \frac{1}{2}\left[\frac{1}{a_x^2} + \frac{1}{a_y^2} \pm \sqrt{\left(\frac{1}{a_x^2} + \frac{1}{a_y^2}\right)^2 - \frac{4\sin^2\phi}{a_x^2 a_y^2}}\,\right]$$

Eccentricity of ellipse:

$$\Sigma = \frac{\lambda_+}{\lambda_-} = \frac{\frac{1}{a_x^2} + \frac{1}{a_y^2} + \sqrt{\left(\frac{1}{a_x^2} + \frac{1}{a_y^2}\right)^2 - \frac{4\sin\phi}{a_x^2 a_y^2}}}{\frac{1}{a_x^2} + \frac{1}{a_y^2} - \sqrt{\left(\frac{1}{a_x^2} + \frac{1}{a_y^2}\right)^2 - \frac{4\sin^2\phi}{a_x^2 a_y^2}}}$$

Define $\rho = \left(\frac{1}{a_x^2} + \frac{1}{a_y^2}\right)\frac{a_x a_y}{2} = \frac{1}{2}\left(\frac{a_y}{a_x} + \frac{a_x}{a_y}\right)$

$$\boxed{\Sigma = \frac{\rho + \sqrt{\rho^2 - \sin^2\phi}}{\rho - \sqrt{\rho^2 - \sin^2\phi}}}$$

Angle of major axis: eigenvector $\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$ with

$$\begin{bmatrix} \frac{1}{a_x^2} & -\frac{\cos\phi}{a_x a_y} \\ -\frac{\cos\phi}{a_x a_y} & \frac{1}{a_y^2} \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \lambda_+\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

$$\frac{1}{a_x^2}\alpha - \frac{\cos\phi}{a_x a_y}\beta = \lambda_+ \alpha$$

$$\tan\theta = \frac{\alpha}{\beta} = \frac{\cos\phi}{a_x a_y}\left(\frac{1}{a_x^2} - \lambda_+\right)^{-1}$$

$$= \frac{\cos\phi}{a_x a_y}\;\frac{1}{\frac{1}{2}\left(\frac{1}{a_x^2} - \frac{1}{a_y^2} - \sqrt{\left(\frac{1}{a_x^2} - \frac{1}{a_y^2}\right) + \frac{4\cos^2\phi}{a_x^2 a_y^2}}\,\right)}$$

So if $q = \frac{1}{2}\left(\frac{1}{a_x^2} - \frac{1}{a_y^2}\right) a_x a_y = \frac{1}{2}\left(\frac{a_y}{a_x} - \frac{a_x}{a_y}\right)$, then

$$\boxed{\tan\Theta = \frac{\cos\phi}{q - \sqrt{q^2 + \cos^2\phi}}}$$

Now in our problem, we have

$$\rho_1 = \frac{1}{2}\left(\frac{a_{1y}}{a_{1x}} + \frac{a_{1x}}{a_{1y}}\right) = \rho_2$$

$$\phi_1 = \phi_2 \pm \pi$$

so $\sin^2\phi_1 = \sin^2\phi_2$

and $\boxed{\varepsilon_1 = \varepsilon_2}$, eccentricities are the same

But

$$q_1 = \frac{1}{2}\left(\frac{a_{1y}}{a_{1x}} - \frac{a_{1x}}{a_{1y}}\right) = -q_2$$

and $\cos\phi_1 = -\cos\phi_2$

So
$$\tan\Theta_2 = \frac{-\cos\phi_1}{-q_1 + \sqrt{q_1^2 + \cos^2\phi_1}} = \frac{-\cos\phi_1\left(-q_1 - \sqrt{q_1^2 + \cos^2\phi_1}\right)}{q_1^2 - \left(q_1^2 + \cos^2\phi_1\right)}$$

$$= -\frac{q_1 + \sqrt{q_1^2 + \cos\phi_1}}{\cos\phi_1}$$

$$= -\frac{1}{\tan\Theta_1} = -\cot\Theta_1$$

$$= -\tan\left(\frac{\pi}{2} - \Theta_1\right)$$

$$= \tan\left(\Theta_1 - \frac{\pi}{2}\right)$$

So, $\boxed{\Theta_2 = \Theta_1 - \frac{\pi}{2},}$ 90° rotated

Finally, sense of rotation given by ϕ:

If $0 < \phi < \pi$, clockwise rotation

$-\pi < \phi < 0$, counterclockwise

Since $\phi_2 = \phi_1 \pm \pi$, rotation sense is $\boxed{\text{opposite}}$

[See alternate solution pg 9]

3. Half wave retarder: $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (fast axis along x)

Input polarization $\vec{J} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ (θ = angle between plane of polarization and x)

So, $\vec{J}_{out} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta' \\ \sin\theta' \end{bmatrix}$ for $\theta' = -\theta$

So - output is rotated by angle 2θ

Here rotation angle depends on input polarization.
For true rotator, all inputs rotated same amount.

4. $\boxed{T_a = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}}$

$$T_b = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T_b}$$

$$T_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} = T_c}$$

So $T_{tot} = T_c T_b T_a$

$$= \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

Compare: matrix for $90°$ rotator is $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

5. $\vec{J}_{in} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad T_1 = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$ (6.1-18)

$$T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So $\vec{J}_{out} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{J}_{out} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta\cos\theta \end{bmatrix}$$

Then $\dfrac{P_{out}}{P_{in}} = \dfrac{|J_{out}|^2}{|J_{in}|^2} = \boxed{\sin^2\theta\cos^2\theta}$

$$= \frac{1}{4}\sin^2 2\theta$$

Max for $\boxed{\theta = 45^\circ,}$ transmission $= \boxed{\frac{1}{4}}$

Alternate solution to # 2 (nicest version from Ben Teolis)

Orthogonality implies that $\frac{a_{1x}}{a_{1y}} e^{-i\phi_1} = -\frac{a_{2y}}{a_{2x}} e^{-i\phi_2}$

which means that $\vec{J}_2$ can be expressed as

$$\vec{J}_2 = \begin{bmatrix} a_{1y} \\ -a_{1x} e^{-i\phi_1} \end{bmatrix} = \begin{bmatrix} a_{1y} e^{+i\phi_1} \\ -a_{1x} \end{bmatrix}$$

So $\vec{J}_2 = R(90°)\, C\, \vec{J}_1$

$$R(90°) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$ rotation operator

and C is conjugate operator, defined by

$$C \begin{bmatrix} J_x \\ J_y \end{bmatrix} = \begin{bmatrix} J_x^* \\ J_y^* \end{bmatrix}$$

Effect of C is to reverse rotation direction,

Effect of $R(90°)$ is to rotate ellipse by 90°

So, ellipse for $\vec{J}_2$ is $\perp$ to ellipse for $\vec{J}_1$,

and rotation sense is opposite.