Phys 531 Assignment 3 Solutions

1. a)

$$z = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} \quad \text{So} \quad \boxed{\begin{aligned} \mathrm{Re}\, z &= \frac{a}{a^2+b^2} \\ \mathrm{Im}\, z &= \frac{-b}{a^2+b^2} \end{aligned}}$$

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} = \frac{\sqrt{a^2+b^2}}{a^2+b^2} \quad \text{so} \quad \boxed{|z| = \frac{1}{\sqrt{a^2+b^2}}}$$

$$\tan\phi = \frac{\mathrm{Im}\, z}{\mathrm{Re}\, z} = -\frac{b}{a}$$

$$\text{so} \quad \boxed{\phi = -\tan^{-1}\frac{b}{a}}$$

b) $$z = \frac{1+i}{2+i} = \frac{(1+i)(2-i)}{2^2+1} = \frac{2+2i-i+1}{5} = \frac{3+i}{5}$$

$$\mathrm{Re}\, z = \frac{3}{5}$$

$$\mathrm{Im}\, z = \frac{1}{5}$$

$$|z| = \frac{1}{5}\sqrt{9+1} = \sqrt{\frac{2}{5}}$$

$$\phi = \tan^{-1}\frac{1/5}{3/5} = \tan^{-1}\frac{1}{3} = 18.43^\circ$$

c) $$z \equiv (3-2i)e^{i3\pi/4} = (3-2i)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= (3-2i)\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(-3+2i+3i+2)$$

$$= \frac{1}{\sqrt{2}}(-1+5i)$$

$$\mathrm{Re}\, z = -\frac{1}{\sqrt{2}} \qquad |z| = \frac{1}{\sqrt{2}}\sqrt{1+25} = \sqrt{13}$$

$$\mathrm{Im}\, z = \frac{5}{\sqrt{2}} \qquad \phi = \tan^{-1}\frac{5}{-1} = -78.7^\circ$$

d) $z = e^{i\frac{\pi}{3}} \cdot e^{i\frac{\pi}{4}}$

$= e^{i\pi(\frac{1}{3}+\frac{1}{4})} = e^{i\pi\frac{7}{12}}$

So $|z| = \sqrt{z^* z} = 1$

$\phi = \frac{7\pi}{12}$

$\text{Re } z = \cos\frac{7\pi}{12} = -0.259$

$\text{Im } z = \sin\frac{7\pi}{12} = 0.966$

e) $z = (1+i)^{3/2}$

Note $1+i = \sqrt{2} \times [\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}] = \sqrt{2}\, e^{i\pi/4}$

So $z = (2^{1/2} e^{i\pi/4})^{3/2} = 2^{3/4} e^{i3\pi/8}$

So $|z| = 2^{3/4}$

$\phi = \frac{3\pi}{8}$

$\text{Re } z = 2^{3/4}\cos\frac{3\pi}{8} = 0.644$

$\text{Im } z = 2^{3/4}\sin\frac{3\pi}{8} = 1.554$

f) $z = \cos(a+ib) = \frac{1}{2}[e^{i(a+ib)} + e^{-i(a+ib)}]$

$= \frac{1}{2}[e^{ia-b} + e^{-ia+b}]$

$= \frac{1}{2}[e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a)]$

$= \frac{1}{2}[\cos a(e^{b}+e^{-b}) - i\sin a(e^{b}-e^{-b})]$

$= \cos a \cosh b - i\sin a \sinh b$

f) So

$Re\, z = \cos a \cosh b$

$Im\, z = \sin a \sinh b$

$|z| = \sqrt{\cos^2 a \cosh^2 b + \sin^2 a \sinh^2 b}$ (OK answer)

$= \sqrt{\cos^2 a\,(1+\sinh^2 b) + (1-\cos^2 a)\sinh^2 b}$

$|z| = \sqrt{\cos^2 a + \sinh^2 b}$

2. Spherical wave is $u(\vec{r}) = A \dfrac{e^{ikr}}{r}$

Intensity is $|u|^2 = \dfrac{|A|^2}{r^2}$

Relate $|A|^2$ to power:

Know $P = \int I\, dA$

Here light emitted into sphere, surface area $4\pi r^2$

So $P = \int I\, dA = \dfrac{|A|^2}{r^2} \cdot 4\pi r^2 = 4\pi |A|^2$

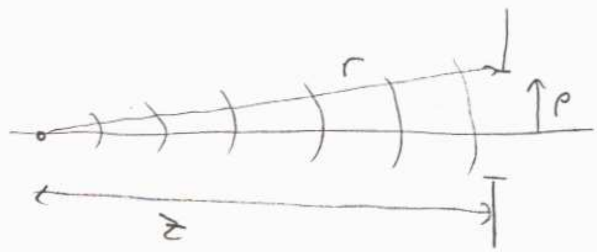
and $|A|^2 = \dfrac{P}{4\pi}$

Finally, $\boxed{I = \dfrac{P}{4\pi r^2}}$

Makes sense, could just say $I = \dfrac{P}{Area(r)} = \dfrac{P}{4\pi r^2}$

3. Spherical wave is $\frac{e^{-ikr}}{r}$

Parabolic wave is $\frac{e^{-ikz}}{z} e^{-ik\frac{\rho^2}{2z}}$ $\qquad \rho^2 = x^2 + y^2$



At edge of circle with radius a, $r = \sqrt{a^2+z^2}$

$\rho = a$

So phase difference between parabolic and spherical wave is

$$\delta\phi = kr - \left(kz + k\frac{\rho^2}{2z}\right)$$

$$= k\left(\sqrt{a^2+z^2} - z - \frac{a^2}{2z}\right)$$

$$= kz\left[\sqrt{\left(\tfrac{a}{z}\right)^2+1} - 1 - \frac{1}{2}\left(\frac{a}{z}\right)^2\right]$$

want $|\delta\phi| < 0.1$

Could expand square root:

$$\sqrt{1+\left(\tfrac{a}{z}\right)^2} = 1 + \frac{1}{2}\left(\frac{a}{z}\right)^2 - \frac{1}{8}\left(\frac{a}{z}\right)^4$$

$$|\delta\phi| = \left|kz\left(-\frac{1}{8}\right)\frac{a^4}{z^4}\right| = \frac{2\pi}{\lambda}\,\frac{1}{8}\,\frac{a^4}{z^3} = 0.1$$

$$a^4 = \frac{4}{\pi}\lambda z^3 \times 0.1$$

gives $\quad a = 0.017\,\text{m}$

Or solve exactly, if you're masochistic:

set $x = \frac{a}{z}$, $q = \frac{\phi_0}{kz} = \frac{\lambda}{z}\frac{0.1}{2\pi} = 10^{-8}$

then $\sqrt{1+x^2} - 1 - \frac{x^2}{2} = -q$ (since we know $\sqrt{1+x^2} < 1 + \frac{x^2}{2}$)

$$1+x^2 = \left(-q + 1 + \frac{x^2}{2}\right)^2$$

$$= q^2 + 1 + \frac{x^4}{4} - 2q - qx^2 + x^2$$

$$0 = \frac{x^4}{4} - qx^2 + q^2 - 2q$$

$$x^2 = \frac{1}{2(\frac{1}{4})}\left[q \pm \sqrt{q^2 - (q^2 - 2q)}\right]$$

$$= 2\left[q \pm \sqrt{2q}\right]$$

need $x > 0$, and have $q \ll 1$, so

$$x = \sqrt{2(q + \sqrt{2q})}$$

$$a = z\sqrt{2(10^{-8} + \sqrt{2}\times 10^{-4})} = 0.017\,\text{m}$$

Maximum angle is $\Theta_m = \frac{a}{z} = 17\,\text{mrad} = 1^\circ$

Fresnel number is $N_F = \frac{a^2}{\lambda z} = 456$

So $\frac{N_F \Theta_m^2}{4} = 0.033 \ll \underline{1}$

4. For each layer q, acquire phase $e^{-in_q k_o d_q}$

So total amplitude is

$$A = A_{in} e^{-in_1 k_o d_1} e^{-in_2 k_o d_2} \cdot \ldots \cdot e^{-in_N k_o d_N}$$

$$= A_{in} e^{-ik_o \sum_q n_q d_q}$$

So $$\boxed{t = e^{-ik_o \sum_q n_q d_q}}$$

If no plates, would need to travel distance

$$\boxed{d = \sum_q n_q d_q}$$

since $t = e^{-ik_o d}$ for free space

See that $d =$ optical path length

5.
$$H = \frac{1}{2a} \int_{y_0-a}^{y_0+a} e^{-iky} dy$$

Define $v = y - y_0$

$$H = \frac{1}{2a} \int_{-a}^{a} e^{-ik(y_0+v)} dv = \frac{1}{2a} e^{-iky_0} \int_{-a}^{a} e^{-ikv} dv$$

$$= \frac{1}{2a} e^{-iky_0} \frac{1}{-ik} \left(e^{-ika} - e^{ika} \right)$$

$$= \frac{1}{ka} e^{-iky_0} \sin ka$$

$$\boxed{H = e^{-iky_0} \operatorname{sinc} ka}$$