Phys 531 Assignment 4 Solutions

1. a) Wave is $U(\vec{r}) = A e^{-i\vec{k}\cdot\vec{r}}$

Here $\vec{k} = k\left(\sin\theta_1 \hat{x} + \cos\theta_1 \hat{z}\right)$

$\approx k\left(\theta_1 \hat{x} + \hat{z}\right)$

So in paraxial limit

$$\boxed{U(\vec{r}) = A e^{-ikz} e^{-ik\theta_1 x}}$$

b) Transmittance is

$$t(x,y) = h_0 e^{-i(n-1)k_0 d(x,y)}$$

Here $d(x,y) = \alpha x$

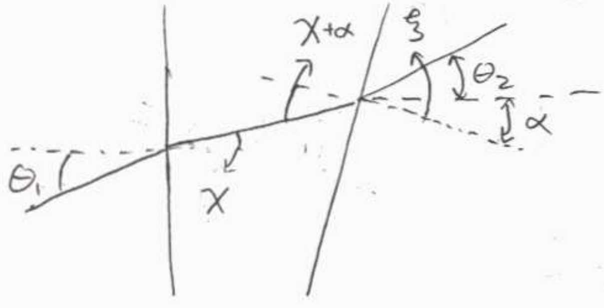
So $t(x,y) = h_0 e^{-i(n-1)k_0 \alpha x}$

c) Transmitted wave is $t\, U_{in}$

$$= A h_0 e^{-ikz} e^{-ikx[\theta_1 + \alpha(n-1)]}$$

form of a plane wave travelling at angle $\boxed{\theta_2 = \theta_1 + \alpha(n-1)}$

d) Ray optics says



$\chi = \frac{\theta}{n}$

$\xi = n(\chi + \alpha) = \theta_1 + n\alpha$

So $\theta_2 = \xi - \alpha$

$$\boxed{\theta_2 = \theta_1 + (n-1)\alpha}$$

Same result

2.

Incident amplitude (at $z=0$):

$$U_{in} = \frac{A}{z_1} e^{-ik\frac{x^2+y^2}{2z_1}}$$

Paraboloidal wave: $U = \frac{A}{z-z_0} e^{-ik\frac{x^2+y^2}{2(z-z_0)}}$

Transmittance $t = h_0 e^{+ik\frac{x^2+y^2}{2f}}$

So transmitted wave is

$$U_{out} = \frac{A}{z_1} h_0 e^{-ik\frac{x^2+y^2}{2}\left(\frac{1}{z_1}-\frac{1}{f}\right)}$$

Wave converging towards z_2 looks like

$$-\frac{A'}{z_2} e^{+ik\frac{x^2+y^2}{2}\frac{1}{z_2}}$$

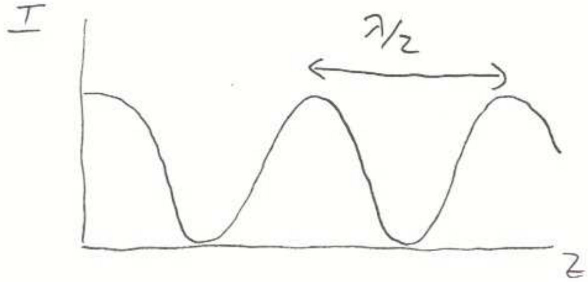
So, have $-\frac{1}{z_1}+\frac{1}{f}=\frac{1}{z_2}$

or $\boxed{\frac{1}{z_1}+\frac{1}{z_2}=\frac{1}{f}}$ and $A' = -\frac{z_2}{z_1} h_0 A$

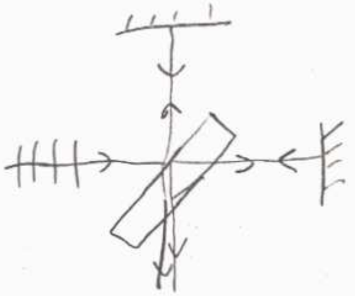
3. Waves are $U_1 = Ae^{ikz}$ and $U_2 = Ae^{-ikz}$

So $U_1+U_2 = A(e^{ikz}+e^{-ikz}) = 2A\cos kz$

$$I = |U_1+U_2|^2 = \boxed{4|A|^2\cos^2 kz}$$



4. Output of Michelson interferometer:

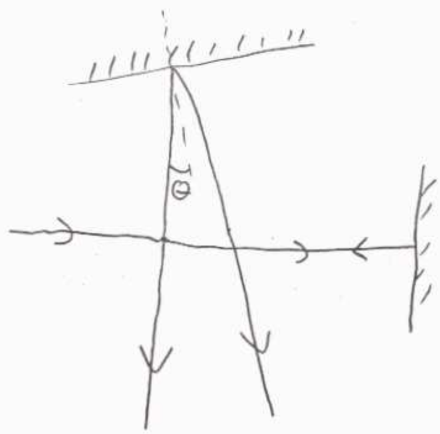


$$U_{out} = U_1 + U_2$$

U_1 and U_2 are fields from different paths

Both U_1 and U_2 are (locally) plane waves

If one mirror is tilted by angle Θ, then plane waves are at angle 2Θ:



So

$$U_{out} = U_1 e^{-ikz} + U_2 e^{-ik(z+2\Theta x)}$$

Take $U_1 = U_2 = A$

$$U_{out} = A e^{-ikz}\left(1 + e^{-i2k\Theta x}\right)$$

$$= A e^{-ikz} e^{-ik\Theta x}\left(e^{ik\Theta x} + e^{-ik\Theta x}\right)$$

$$= 2A e^{-ik(z+\Theta x)} \cos k\Theta x$$

Then $I_{out}(x) = 4|A|^2 \cos^2 k\Theta x$

Interference pattern is sinusoidal: looks like 

4. Continued

As other mirror moves, introduce phase $\phi = -2kd$

d = distance moved

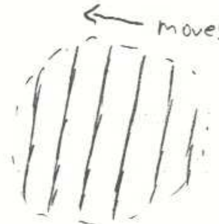
Then $U_{out} = U_1 e^{-ikz} + U_2 e^{-ik(z+2\theta x) - 2ikd}$

So output becomes

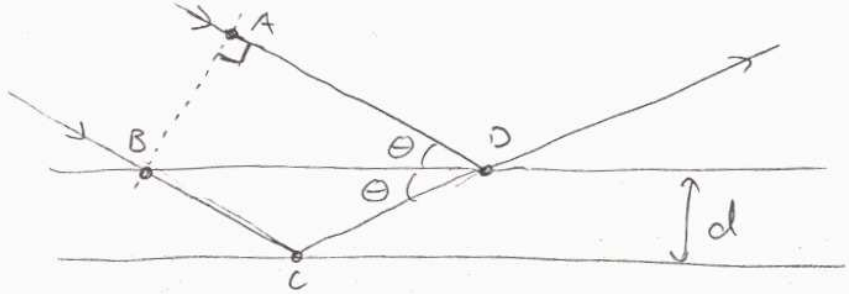
$$I_{out}(x) = 4|A|^2 \cos^2(k\theta x + kd)$$

As d changes, pattern "slides" across output beam

← moves as d increases



5. Two waves:



Dotted line shows input wave front = surface of constant phase

Then phase difference ϕ is given by path length difference $\overline{BCD} - \overline{AD}$

See that $\overline{BC} = \overline{CD} = \dfrac{d}{\sin\theta}$

Also $\overline{BD} = 2\dfrac{d}{\tan\theta} = 2d\dfrac{\cos\theta}{\sin\theta}$

from which $\overline{AD} = \overline{BD}\cos\theta = 2d\dfrac{\cos^2\theta}{\sin\theta}$

So path difference is

$$\overline{BCD} - \overline{AD} = \frac{2d}{\sin\theta} - 2d\frac{\cos^2\theta}{\sin\theta}$$

$$= 2d\,\frac{1-\cos^2\theta}{\sin\theta} = 2d\,\frac{\sin^2\theta}{\sin\theta}$$

$$= 2d\sin\theta$$

Then phase difference is $\boxed{\phi = 2kd\sin\theta}$

From 2.5-10, get maximum reflection when $\phi = 2\pi q$ for integer q

or $2\cdot\frac{2\pi}{\lambda}d\sin\theta = 2\pi q$

So $\sin\theta = \frac{\lambda}{2d}$ for $q = 1$