Phys 531 Assignment 5

Solutions

1. a) Reflected field is

$$U_{out} = rU_{in} + U_{in}\left[t^2 e^{i\phi} + t^2 r e^{i2\phi} + t^2 r^2 e^{i3\phi} + \dots\right]$$

Field after one bounce:

get t from initial transmission

$\times e^{i\phi}$ from bounce

$\phi = -2kd + \phi_M$

another t from transmission out of gap

$$U_{out} = U_{in}\left[r + t^2 e^{i\phi}\left(1 + re^{i\phi} + r^2 e^{2i\phi} + \dots\right)\right]$$

$$= U_{in}\left[r + t^2 e^{i\phi}\frac{1}{1-re^{i\phi}}\right]$$

$$= U_{in}\left[\frac{r - r^2 e^{i\phi} + t^2 e^{i\phi}}{1-re^{i\phi}}\right]$$

$$\boxed{U_{out} = U_{in}\left[\frac{r-(r^2-t^2)e^{i\phi}}{1-re^{i\phi}}\right]}$$

b) $$I_{out} = I_{in}\left|\frac{r-(r^2-t^2)e^{i\phi}}{1-re^{i\phi}}\right|^2$$

So if $I_{out} = I_{in}$

$$|1-re^{i\phi}|^2 = |r-(r^2-t^2)e^{i\phi}|^2 \quad \text{for all } \phi$$

$$1+|r|^2 - re^{i\phi} - r^* e^{-i\phi} = |r|^2 + |r^2-t^2|^2 - r^*(r^2-t^2)e^{i\phi} - r(r^{*2}-t^{*2})e^{-i\phi}$$

If this is true for all ϕ, must have:

$$1+|r|^2 = |r|^2 + |r^2-t^2|^2$$

and

$$re^{i\phi} = r^*(r^2-t^2)e^{i\phi}$$

and

$$r^* e^{-i\phi} = r(r^{*2}-t^{*2})e^{-i\phi}$$

See pg 2a for proof

From $e^{i\phi}$ equation:

$$r^2 - t^2 = \frac{r}{r^*}$$

if $r = |r| e^{i\beta}$, then $\frac{r}{r^*} = \frac{|r|e^{i\beta}}{|r|e^{-i\beta}} = e^{2i\beta}$

so $|r|^2 e^{2i\beta} - t^2 = e^{2i\beta}$

$$t^2 = e^{2i\beta}(|r|^2 - 1) = -e^{2i\beta}|t|^2$$

since $|r|^2 + |t|^2 = 1$

So, $\boxed{t = \pm i|t|e^{i\beta} = e^{i(\beta \pm \frac{\pi}{2})}|t|}$

Suppose we have a formula

$$A + Be^{i\phi} + Ce^{-i\phi} = 0$$

which holds for all ϕ.

Then we must have $\int_0^{2\pi} e^{im\phi}(A + Be^{i\phi} + Ce^{-i\phi})\,d\phi = 0$

for any m

if $m = 0$, get $2\pi A + B\int_0^{2\pi} e^{i\phi}\,d\phi + C\int_0^{2\pi} e^{-i\phi}\,d\phi = 0$

But $\int_0^{2\pi} e^{\pm i\phi}\,d\phi = \frac{1}{\pm i}\, e^{\pm i\phi}\Big|_0^{2\pi} = 0$

So $A = 0$

if $m = -1$, get $A\int_0^{2\pi} e^{-i\phi}\,d\phi + 2\pi B + C\int_0^{2\pi} e^{-2i\phi}\,d\phi = 0$

$\Rightarrow B = 0$

if $m = +1$, get $A\int_0^{2\pi} e^{i\phi}\,d\phi - B\int_0^{2\pi} e^{2i\phi}\,d\phi + 2\pi C = 0$

$\Rightarrow C = 0$

So, must have $A = B = C = 0$

2. a) Light frequency $\nu = \frac{c}{\lambda} = \frac{3\times10^8 \frac{m}{s}}{800nm} = 3.75\times10^{14}$ Hz

So, number of cycles is

$$N = 50fs \times \nu = 5\times10^{-14} \times 3.75\times10^{14} = 18.75$$

$$\boxed{\approx 19 \text{ cycles}}$$

b) Know that $\Delta\nu \approx \frac{1}{T} = \frac{1}{50\times10^{-15}s}$

$$= 2\times10^{13} \text{ Hz}$$

So $\Delta\lambda \approx \Delta\nu \left|\frac{d\lambda}{d\nu}\right|$

$$= \Delta\nu \frac{c}{\nu^2} = \frac{\Delta\nu}{\nu}\lambda$$

$$= \frac{2\times10^{13}Hz}{3.75\times10^{14}Hz} \cdot 800nm$$

$$= \frac{800nm}{18.75} = 43nm$$

So, λ ranges from about $\boxed{780nm \text{ to } 820nm}$

c) Pulses come out at 100 MHz rate

$$= 10^8 \text{ pulses}/s$$

In one second, get 0.1 J of energy

So each pulse has energy $\frac{0.1 J}{10^8} = 10^{-9} J$

Power in each pulse is $\frac{\text{energy}}{\text{time}} = \frac{10^{-9}J}{50\times10^{-15}s} = \boxed{2\times10^4 W}$

3a)
$$F_\nu = \int_0^T \sin 2\pi\nu_0 t \; e^{-2\pi i\nu t}\,dt$$

$$= \frac{1}{2i}\left[\int_0^T e^{2\pi i\nu_0 t} e^{-2\pi i\nu t}\,dt - \int_0^T e^{-2\pi i\nu_0 t} e^{-2\pi i\nu t}\,dt\right]$$

$$= \frac{1}{2i}\left[\frac{1}{2\pi i(\nu_0-\nu)}\left(e^{2\pi i(\nu_0-\nu)T} - 1\right) + \frac{1}{2\pi i(\nu_0+\nu)}\left(e^{-2\pi i(\nu_0+\nu)T} - 1\right)\right]$$

If $\nu_0 T = N$, then $e^{2\pi i N} = 1$, so

$$F_\nu = \frac{1}{2i}\frac{1}{2\pi i}\left[\frac{1}{(\nu_0-\nu)}\left(e^{-2\pi i\nu T} - 1\right) + \frac{1}{\nu_0+\nu}\left(e^{-2\pi i\nu T} - 1\right)\right]$$

$$= \frac{1}{2\pi i}\left(\frac{1}{\nu_0-\nu} + \frac{1}{\nu_0+\nu}\right) e^{-\pi i\nu T}\left(-\sin\pi\nu T\right)$$

$$= \frac{i}{2\pi}\frac{(\nu_0+\nu)+(\nu_0-\nu)}{\nu_0^2-\nu^2} e^{-\pi i\nu T}\sin\pi\nu T$$

$$\boxed{F_\nu = \frac{i}{\pi}\frac{\nu_0}{\nu_0^2-\nu^2} e^{i\pi\nu T}\sin\pi\nu T}$$

for $\nu = \nu_0 + \delta$, get term $\dfrac{\sin(\pi\nu_0 T + \pi\delta T)}{-2\nu_0\delta}$

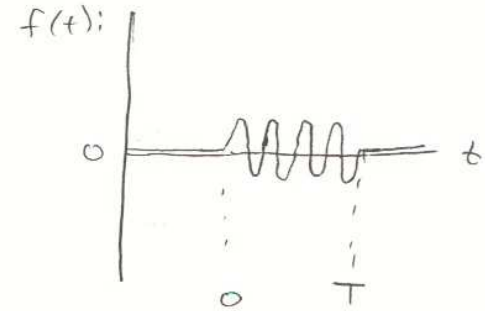
Since $\nu_0 T = N$,
$$= \pm\frac{\sin\pi\delta T}{2\nu_0\delta}$$

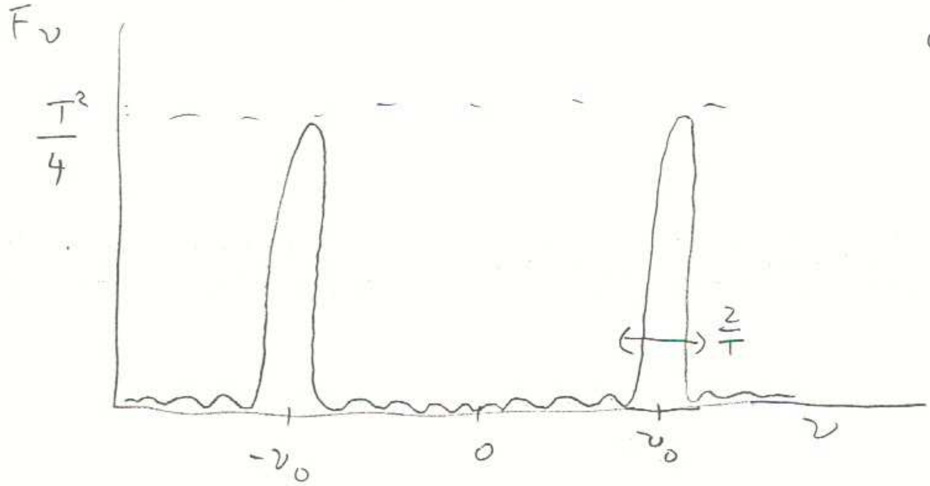
So for $\delta \to 0$, set $\quad \frac{\pi \delta T}{2 \nu_0 \delta} = \frac{\pi}{2}\frac{T}{\nu_0} = \frac{\pi}{2}\frac{T^2}{N}$

So peak value $\quad F_\nu(max) = \pm \frac{i}{\pi}\, \nu_0\, e^{i\pi\nu_0 T}\, \frac{\pi}{2}\frac{T^2}{N}$

$$= \pm \frac{i}{2}\frac{\nu_0 T^2}{N} = \pm i \frac{T}{2}$$

So $\quad |F_\nu(max)|^2 = \frac{T^2}{4}$





b) $\quad F_\nu = \int_{-\infty}^{\infty} e^{-\frac{|t|}{T}} e^{-2\pi i \nu t}\, dt$

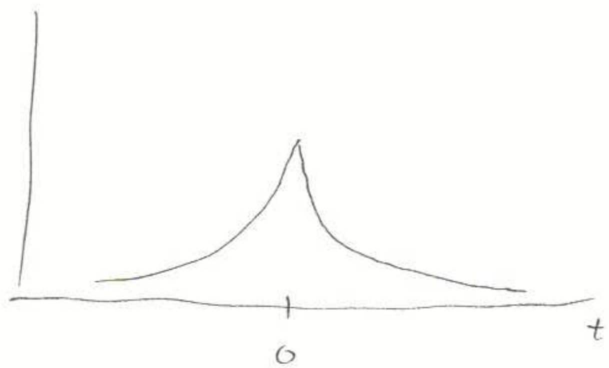
$$= \int_0^{\infty} e^{-(2\pi i \nu + \frac{1}{T})t}\, dt + \int_{-\infty}^{0} e^{-(2\pi i \nu - \frac{1}{T})t}\, dt$$

$$= \frac{1}{-(2\pi i \nu + \frac{1}{T})}\, e^{-(2\pi i\nu + \frac{1}{T})}\Big|_0^{\infty} + \frac{1}{-(2\pi i \nu - \frac{1}{T})}\, e^{-(2\pi i \nu - \frac{1}{T})}\Big|_{-\infty}^{0}$$

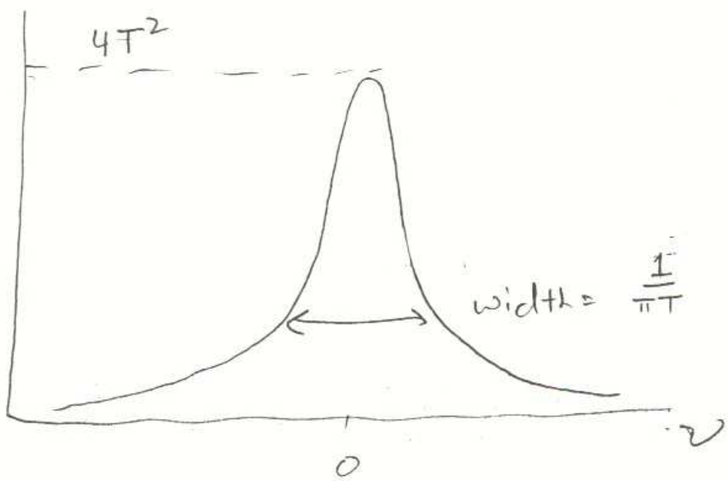
$$F_\nu = \frac{1}{2\pi i\nu + \frac{1}{T}} - \frac{1}{2\pi i\nu - \frac{1}{T}}$$

$$= T\,\frac{(2\pi i\nu T - 1) - (2\pi i\nu T + 1)}{(2\pi i\nu T)^2 - 1} = \boxed{\frac{2T}{4\pi^2\nu^2T^2 + 1}}$$

$f(t)$:



$|F_\nu|^2$



4. a)

Can write

$$u(\vec{r},t) = \frac{A}{2}\left\{ e^{i(2\pi\nu_1 t - k_1 z)} + e^{-i(2\pi\nu_1 t - k_1 z)} + e^{i(2\pi\nu_2 t - k_2 z)} + e^{-i(2\pi\nu_2 t - k_2 z)} \right\}$$

To get complex representation, drop terms with $e^{-i2\pi\nu t}$, and double terms with $e^{+i2\pi\nu t}$:

$$\boxed{U(\vec{r},t) = A\left[e^{i(2\pi\nu_1 t - k_1 z)} + e^{i(2\pi\nu_2 t - k_2 z)} \right]}$$

b) $I(\vec{r},t) = |U|^2$

$$= |A|^2 \left[e^{i(2\pi\nu_1 t - k_1 z)} + e^{i(2\pi\nu_2 t - k_2 z)} \right]\left[e^{-i(2\pi\nu_1 t - k_1 z)} + e^{-i(2\pi\nu_2 t - k_2 z)} \right]$$

$$= |A|^2 \left\{ 1 + 1 + e^{i[2\pi(\nu_1-\nu_2)t - (k_1-k_2)z]} + e^{-i[2\pi(\nu_1-\nu_2)t - (k_1-k_2)z]} \right\}$$

$$\boxed{I = |A|^2 \left\{ 2 + 2\cos\left[2\pi(\nu_1-\nu_2)t - (k_1-k_2)z\right] \right\}}$$

So, I itself looks like wave moving at velocity

$$V = \frac{2\pi(\nu_1 - \nu_2)}{(k_1 - k_2)}$$

But $k_1 = \frac{2\pi\nu_1}{c}$ $\qquad k_2 = \frac{2\pi\nu_2}{c}$

$$V = \frac{2\pi(\nu_1 - \nu_2)}{\frac{1}{c}2\pi(\nu_1 - \nu_2)} = c$$

Intensity pattern moves at speed $\boxed{c}$.