Phys 531 Assigment 6 Solutions

1. $A(\vec{r}) = \frac{A_0}{z+iz_0} e^{-ik\frac{x^2+y^2}{2(z+iz_0)}}$

a) Show a solution of paraxial wave equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - 2ik\frac{\partial A}{\partial z} = 0$$

$$\frac{\partial A}{\partial x} = \frac{A_0}{z+iz_0}\left(-ik\frac{x}{z+iz_0}\right) e^{-ik\frac{x^2+y^2}{2(z+iz_0)}}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{A_0}{z+iz_0}\left[\left(\frac{-ik}{z+iz_0}\right) e^{-ik\frac{x^2+y^2}{2(z+iz_0)}} - \frac{k^2x^2}{(z+iz_0)^2} e^{-ik\frac{x^2+y^2}{2(z+iz_0)}}\right]$$

$$= A(\vec{r})\left[-\frac{ik}{z+iz_0} - \frac{k^2x^2}{(z+iz_0)^2}\right]$$

and

$$\frac{\partial^2 A}{\partial y^2} = A(\vec{r})\left[-\frac{ik}{z+iz_0} - \frac{k^2y^2}{(z+iz_0)^2}\right]$$

$$\frac{\partial A}{\partial z} = -\frac{A_0}{(z+iz_0)^2} e^{-ik\frac{x^2+y^2}{2(z+iz_0)}} + \frac{A_0}{(z+iz_0)}\frac{ik(x^2+y^2)}{2(z+iz_0)^2} e^{-ik\frac{x^2+y^2}{2(z+iz_0)}}$$

$$= A(r)\left[-\frac{1}{z+iz_0} + \frac{ik(x^2+y^2)}{2(z+iz_0)^2}\right]$$

So

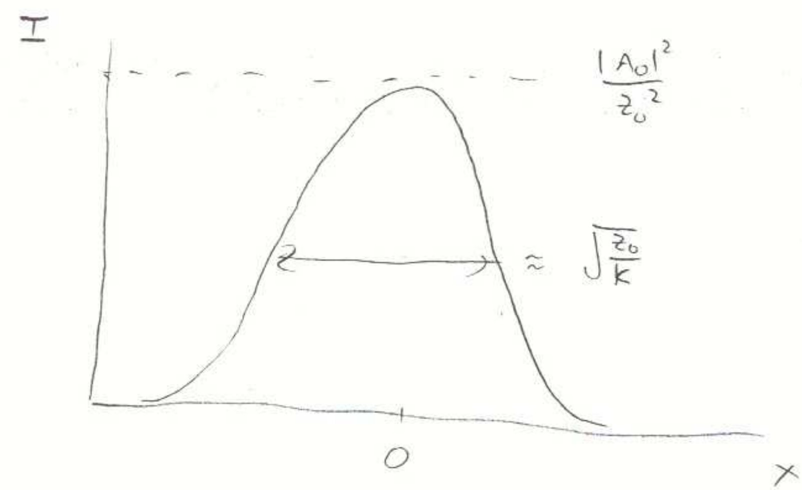
$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - 2ik\frac{\partial A}{\partial z}$$

$$= A(\vec{r})\left[-\frac{2ik}{z+iz_0} - \frac{k^2(x^2+y^2)}{(z+iz_0)^2} + \frac{2ik}{z+iz_0} + \frac{k^2(x^2+y^2)}{(z+iz_0)^2}\right]$$

$$= 0 \quad \checkmark$$

b) $$I(x,y,0) = \left|\frac{A_0}{iz_0} e^{-ik\frac{x^2+y^2}{2iz_0}}\right|^2$$

$$= \frac{|A_0|^2}{z_0^2} e^{-k\frac{x^2+y^2}{z_0}}$$ Gaussian function



2. Use harmonic analysis $\left(k = \frac{2\pi}{\lambda} \right)$

a) $f(x,y) = 1$

Then $U(x,y,z) = e^{-ikz}$

and $g(x,y,d) = e^{-ikd} = \boxed{e^{-2\pi i \frac{d}{\lambda}}}$

b) $f(x,y) = e^{-\frac{i\pi}{\lambda}(x+y)} = e^{-ik\frac{x+y}{2}}$

So $k_x = \frac{k}{2} \quad k_y = \frac{k}{2}$

then $k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{k^2 - \frac{k^2}{4} - \frac{k^2}{4}} = \frac{k}{\sqrt{2}}$

So $U(x,y,z) = e^{-ik\left(\frac{x}{2} + \frac{y}{2} + \frac{z}{\sqrt{2}}\right)}$

$$g(x,y) = e^{-ik\frac{d}{\sqrt{2}}} e^{-ik\frac{x+y}{2}}$$

$$= \boxed{e^{-i\pi\sqrt{2}\frac{d}{\lambda}} e^{-i\frac{\pi}{\lambda}(x+y)}}$$

c) $f(x,y) = \cos\frac{\pi x}{2\lambda} = \frac{1}{2}\left(e^{ik\frac{x}{4}} + e^{-ik\frac{x}{4}}\right)$

So $k_x = \pm \frac{k}{4}$

Then $k_z = \sqrt{k^2 - \frac{k^2}{16}} = k\frac{\sqrt{15}}{4}$

So $U(x,y,z) = \frac{1}{2}\left[e^{-ik\left(\frac{\sqrt{15}}{4}z - \frac{1}{4}x\right)} + e^{-ik\left(\frac{\sqrt{15}}{4}z + \frac{1}{4}x\right)} \right]$

$$g(x,y) = \frac{1}{2} e^{-ikd\frac{\sqrt{15}}{4}} \left(e^{ik\frac{x}{4}} + e^{-ik\frac{x}{4}} \right)$$

$$= \boxed{e^{-i\frac{\pi}{2}\sqrt{15}\frac{d}{\lambda}} \cos\frac{\pi}{2}\frac{x}{\lambda}}$$

d) $f(x,y) = \cos^2 \frac{\pi y}{2\lambda}$

$$= \frac{1}{2}\left(1 + \cos\frac{\pi y}{\lambda}\right) = \frac{1}{2} + \frac{1}{4} e^{i\frac{k}{2}y} + \frac{1}{4} e^{-i\frac{k}{2}y}$$

So $u(x,y,z) = \frac{1}{2} e^{-ikz} + \frac{1}{2} e^{-ik_z z} \cos\frac{\pi y}{\lambda}$

$$k_z = \sqrt{k^2 - \frac{k^2}{4}} = k\frac{\sqrt{3}}{2}$$

So $g(x,y) = \frac{1}{2}\left[e^{-ikd} + e^{-ik\frac{\sqrt{3}}{2}d} \cos\frac{\pi y}{\lambda}\right]$

$$= \boxed{\frac{1}{2}\left[e^{-i\frac{2\pi d}{\lambda}} + e^{-i\frac{\pi\sqrt{3}\,d}{\lambda}} \cos\frac{\pi y}{\lambda}\right]}$$

3. If max spatial frequency is 200 lines/mm, then

$$\Theta_{max} = \sin^{-1} 633\text{nm} \cdot \frac{200}{\text{mm}} = \sin^{-1} 0.127$$

$$\boxed{\Theta_{max} = 7.3^\circ}$$

4. $f(x,y) = A e^{-\frac{x^2+y^2}{w_0^2}}$

a) $F(\nu_x,\nu_y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; A e^{-\frac{x^2+y^2}{w_0^2}} e^{+2\pi i(\nu_x x + \nu_y y)}$

$$= A\left[\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{w_0^2}} e^{2\pi i \nu_x x}\right]\left[\int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{w_0^2}} e^{2\pi i \nu_y y}\right]$$

$$= A\left(\sqrt{\pi}\, w_0 e^{-\pi^2 \nu_x^2 w_0^2}\right)\left(\sqrt{\pi}\, w_0 e^{-\pi^2 \nu_y^2 w_0^2}\right)$$

$$F(\nu_x,\nu_y) = \pi w_0^2 A e^{-\pi^2 w_0^2(\nu_x^2+\nu_y^2)}$$

Then

$$U(x,y,z) = A \iint_{-\infty}^{\infty} d\nu_x d\nu_y \; \pi w_0^2 \, e^{-\pi^2 w_0^2(\nu_x^2+\nu_y^2)} e^{-i(2\pi\nu_x x + 2\pi \nu_y y)} e^{-ik_z z}$$

$$k_z = 2\pi\left(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2\right)^{1/2}$$

If $w_0 \gg \lambda$, then $\nu_x, \nu_y \ll \frac{1}{\lambda}$

$$k_z \approx \frac{2\pi}{\lambda}\left(1 - \frac{1}{2}\lambda^2\nu_x^2 - \frac{1}{2}\lambda^2\nu_y^2\right)$$

$$= k_0 - \pi\lambda\nu_x^2 - \pi\lambda\nu_y^2$$

$$U(x,y,z) = A\pi w_0^2 e^{-ik_0 z} \iint d\nu_x d\nu_y \; e^{-\pi^2 w_0^2 \nu_x^2 + i\pi\lambda z \nu_x^2}$$

$$\times\, e^{-\pi^2 w_0^2 \nu_y^2 + i\pi\lambda z \nu_y^2}$$

$$\times\, e^{-i(2\pi\nu_x x + 2\pi\nu_y y)}$$

$$U(x,y,z) = A\pi w_0^2 e^{-ik_0 z} \int d\nu_x \, e^{-(\pi^2 w_0^2 - i\pi\lambda z)\nu_x^2} e^{-2\pi \nu_x x}$$

$$\times \int d\nu_y \, e^{-(\pi^2 w_0^2 - i\pi\lambda z)\nu_x^2} e^{-2\pi \nu_y y}$$

Like before, but with $w_0^2 \to w_0^2 - i\frac{\lambda}{\pi}z$

$$= A\pi w_0^2 e^{-ik_0 z}\left(\frac{1}{\sqrt{\pi}\sqrt{w_0^2 - i\frac{\lambda}{\pi}z}} e^{-\frac{x^2}{w_0^2 - i\frac{\lambda}{\pi}z}}\right)\left(\frac{1}{\sqrt{\pi}\sqrt{w_0^2 - i\frac{\lambda}{\pi}z}} e^{-\frac{y^2}{w_0^2 - i\frac{\lambda}{\pi}z}}\right)$$

$$= A e^{-ik_0 z} \frac{1}{1 - i\frac{\lambda z}{\pi w_0^2}} e^{-\frac{x^2+y^2}{w_0^2 - i\frac{\lambda}{\pi}z}}$$

if $w_0^2 = \frac{\lambda z_0}{\pi}$

$$U(x,y,z) = A e^{-ik_0 z} \frac{1}{1 - i\frac{z}{z_0}} e^{-\frac{\pi}{\lambda}\frac{x^2+y^2}{z_0 - iz}}$$

$$= \boxed{A e^{-ik_0 z} \frac{iz_0}{z + iz_0} e^{-i\frac{k}{2}\frac{x^2+y^2}{z+iz_0}}}$$

Same as Gaussian wave from problem (1)
(with $A = A' i z_0$)

b) Space domain

$$U(x,y,z) = h_0 \iint_{-\infty}^{\infty} f(x',y')\, e^{-i\pi \frac{(x-x')^2+(y-y')^2}{\lambda z}}\, dx'dy'$$

$$h_0 = \frac{i}{\lambda z} e^{-ikz}$$

$$U = \frac{i}{\lambda z} e^{-ikz} \iint A e^{-\frac{x'^2+y'^2}{w_0^2}} e^{-i\pi \frac{(x-x')^2+(y-y')^2}{\lambda z}} dx'dy'$$

$$= \frac{i}{\lambda z} e^{-ikz} A \int_{-\infty}^{\infty} e^{-\frac{x'^2}{w_0^2} - i\pi \frac{x^2-2x'x+x'^2}{\lambda z}} dx'$$

$$\times \int_{-\infty}^{\infty} e^{-\frac{y'^2}{w_0} - i\pi \frac{y^2-2y'y+y'^2}{\lambda z}} dy'$$

Look at x' integral:

$$e^{-i\pi\frac{x^2}{\lambda z}} \int_{-\infty}^{\infty} e^{-\left[x'^2\left(\frac{1}{w_0^2}+\frac{i\pi}{\lambda z}\right) - \frac{2i\pi}{\lambda z}x'x\right]} dx'$$

Use $\frac{1}{w_0^2} + \frac{i\pi}{\lambda z} = \frac{\pi}{\lambda}\left(\frac{\lambda}{\pi w_0^2} + \frac{i}{z}\right)$

$$= \frac{\pi}{\lambda}\left(\frac{1}{z_0} + \frac{i}{z}\right)$$

$$= e^{-i\pi\frac{x^2}{\lambda z}} \int_{-\infty}^{\infty} e^{-\frac{\pi}{\lambda}\left[x'^2\left(\frac{1}{z_0}+\frac{i}{z}\right) - \frac{2xx'}{z}\right]} dx'$$

Complete the square

$$x'^2\left(\frac{1}{z_0}+\frac{i}{z}\right)-\frac{2xx'}{z}=\left(\frac{1}{z_0}+\frac{i}{z}\right)\left[x'^2-2xx'\frac{1}{z}\left(\frac{1}{z_0}+\frac{i}{z}\right)^{-1}\right]$$

$$=\left(\frac{1}{z_0}+\frac{i}{z}\right)\left[x'^2-2xx'\left(\frac{z}{z_0}+i\right)^{-1}\right]$$

$$=\left(\frac{1}{z_0}+\frac{i}{z}\right)\left[x'^2-2xx'\frac{z_0}{z+iz_0}+x^2\left(\frac{z_0}{z+iz_0}\right)^2\right]$$

$$-\left(\frac{1}{z_0}+\frac{i}{z}\right)x^2\frac{z_0^2}{(z+iz_0)^2}$$

$$=\left(\frac{1}{z_0}+\frac{i}{z}\right)\left(x'-x\frac{z_0}{z+iz_0}\right)^2-\frac{z+iz_0}{z_0 z}x^2\frac{z_0^2}{(z+iz_0)^2}$$

So x integral becomes

$$e^{-i\pi\frac{x^2}{\lambda z}}\,e^{-\frac{\pi}{\lambda}\frac{z_0}{z+iz_0}x^2}\int_{-\infty}^{\infty}e^{-\frac{\pi}{\lambda}\left(\frac{1}{z_0}+\frac{i}{z}\right)\left(x'-x\frac{z_0}{z+iz_0}\right)^2}dx'$$

$$=e^{\frac{\pi}{\lambda}x^2\left[\frac{-i}{z}-\frac{z_0}{z+iz_0}\right]}\sqrt{\pi}\,\frac{1}{\sqrt{\frac{\pi}{\lambda}\left(\frac{1}{z_0}+\frac{i}{z}\right)}}$$

Total expression for U is

$$U(x,y,z)=\frac{i}{\lambda z}e^{-ikz}A\,\frac{\lambda}{\frac{1}{z_0}+\frac{i}{z}}\,e^{\frac{\pi}{\lambda}(x^2+y^2)\left[-\frac{i}{z}-\frac{z_0}{z+iz_0}\right]}$$

$$=\frac{iz_0}{z+iz_0}Ae^{-ikz}\,e^{\frac{\pi}{\lambda}(x^2+y^2)\left[\frac{-iz+z_0-z_0}{z(z+iz_0)}\right]}$$

$$=\frac{iz_0}{z-iz_0}Ae^{-ikz}\,e^{-i\frac{\pi}{\lambda}(x^2+y^2)\frac{1}{z+iz_0}}=\boxed{\frac{iz_0}{z-iz_0}Ae^{-ikz}e^{-i\frac{k}{2}\frac{x^2+y^2}{z+iz_0}}}$$

5. Harmonic oscillator

$$\frac{d^2g}{dt^2} + 2\pi\sigma\frac{dg}{dt} + (2\pi\nu_0)^2 g = f(t)$$

a) Transfer function:

take $f(t) = Ae^{i2\pi\nu t}$

Expect $g = Be^{i2\pi\nu t}$

Then

$$-(2\pi\nu)^2 B + i(2\pi)^2\nu\sigma B + (2\pi\nu_0)^2 B = A$$

So $$B(\nu) = \frac{1}{(2\pi)^2}\frac{A}{\nu_0^2 - \nu^2 + i\nu\sigma}$$

Transfer function $$\mathcal{H}(\nu) = \frac{B}{A} = \boxed{\frac{1}{(2\pi)^2}\frac{1}{\nu_0^2 - \nu^2 + i\nu\sigma}}$$

b) Impulse response

$$h(t) = \int_{-\infty}^{\infty} e^{2\pi i\nu t}\,\mathcal{H}(\nu)\,d\nu$$

From HW 5, inverse transform of $\frac{2T}{1+4\pi^2\nu^2T^2}$

is $e^{-|t|/T}$

Try to make $\mathcal{H}(\nu)$ look like this:

Complete square on

$$\nu^2 - i\nu\sigma - \nu_0^2$$

$$= \left(\nu - i\frac{\sigma}{2}\right)^2 + \frac{\sigma^2}{4} - \nu_0^2$$

So $$h(t) = \frac{-1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{2\pi i \nu t}}{\left(\nu - i\frac{\sigma}{2}\right)^2 + \frac{\sigma^2}{4} - \nu_0^2}\, d\nu$$

$$\nu' = \nu - i\frac{\sigma}{2}$$

so $$\nu = \nu' + i\frac{\sigma}{2}$$

$$h(t) = \frac{-1}{(2\pi)^2} e^{2\pi i t \left(i\frac{\sigma}{2}\right)} \int_{-\infty}^{\infty} \frac{e^{2\pi i \nu' t}}{\nu'^2 + \frac{\sigma^2}{4} - \nu_0^2}\, d\nu$$

Define $$\frac{1}{4\pi^2 T^2} = \frac{\sigma^2}{4} - \nu_0^2$$

$$T = \pm \frac{1}{2\pi} \frac{1}{\sqrt{\frac{\sigma^2}{4} - \nu_0^2}} = \frac{\pm 1}{2\pi i} \frac{1}{\sqrt{\nu_0^2 - \frac{\sigma^2}{4}}}$$

$$h(t) = -\frac{1}{(2\pi)^2} e^{-\pi\sigma t} \int_{-\infty}^{\infty} \frac{4\pi^2 T^2 e^{2\pi i \nu' t}}{1 + 4\pi^2 T^2 \nu'^2}\, d\nu'$$

$$= -\frac{T}{2} e^{-\pi\sigma t} \int_{-\infty}^{\infty} \frac{2T}{1 + 4\pi^2 T^2 \nu'^2}\, e^{2\pi i \nu' t}$$

$$= -\frac{T}{2} e^{-\pi\sigma t} e^{-|t|/T}$$

* Want $e^{+2\pi i t \nu}$, so take $T = \frac{-1}{2\pi i} \frac{1}{\sqrt{\nu_0^2 - \frac{\sigma^2}{4}}}$

$$\boxed{g(t) = \frac{1}{4\pi i} \frac{1}{\sqrt{\nu_0^2 - \frac{\sigma^2}{4}}} e^{-\pi\sigma t} e^{2\pi i |t| \sqrt{\nu_0^2 - \frac{\sigma^2}{4}}}}$$