Phys 531 Assignment 8

Solutions

1. Have $F(\nu_x,\nu_y) = \iint dx\,dy\, f(x,y)\, e^{i2\pi(\nu_x x + \nu_y y)}$

$$f(x,y) = \sum_{n=0}^{N-1} f_1(x-nB, y)$$

where f_1 is transmission function of a single hole

$$F(\nu_x,\nu_y) = \sum_{n=0}^{N-1} \iint dx\,dy\, f_1(x-nB,y)\, e^{i2\pi(\nu_x x+\nu_y y)}$$

Substitute $x' = x - nB$

so $x = x' + nB$

$$= \sum_{n=0}^{N-1} \iint dx'\,dy\, f_1(x',y)\, e^{i2\pi(\nu_x x' + \nu_x nB + \nu_y y)}$$

$$= \sum_{n=0}^{N-1} e^{i2\pi n\nu_x B} \iint dx'\,dy'\, f_1(x',y)\, e^{i2\pi(\nu_x x' + \nu_y y)}$$

$$= \sum_{n=0}^{N-1} e^{i2\pi n\nu_x B}\, F_1(\nu_x,\nu_y)$$

From result for geometrical series

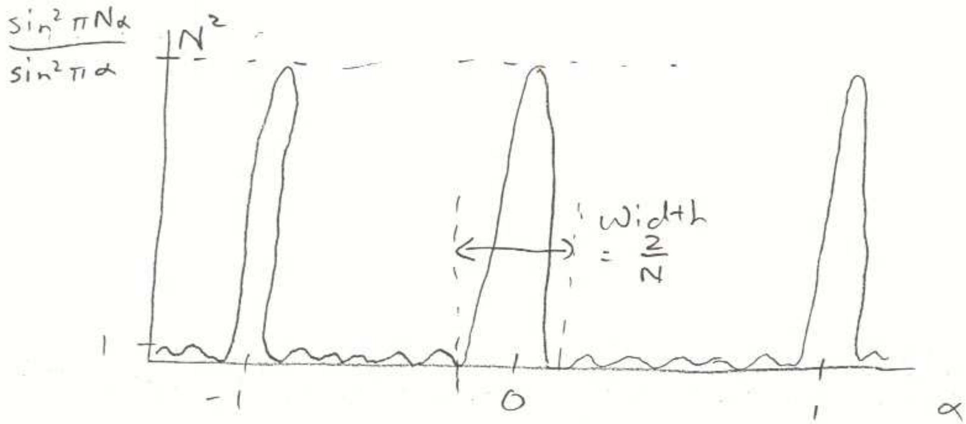
$$F(\nu_x,\nu_y) = F_1(\nu_x,\nu_y)\,\frac{1-e^{i2\pi NB\nu_x}}{1-e^{i2\pi B\nu_x}}$$

$$= e^{i\pi(N-1)B\nu_x}\, F_1(\nu_x,\nu_y)\,\frac{\sin \pi NB\nu_x}{\sin \pi B\nu_x}$$

So
$$\boxed{I(x,y) = \left|F_1\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)\right|^2 \frac{\sin^2 \pi N \frac{Bx}{\lambda d}}{\sin^2 \pi \frac{Bx}{\lambda d}}}$$

See that $\dfrac{\sin^2 \pi N \alpha}{\sin^2 \pi \alpha} \rightarrow N^2$ when $\alpha \rightarrow$ integer.

At other points, $\dfrac{\sin^2 \pi N \alpha}{\sin^2 \pi \alpha} \lesssim 1$



$\alpha =$ integer m

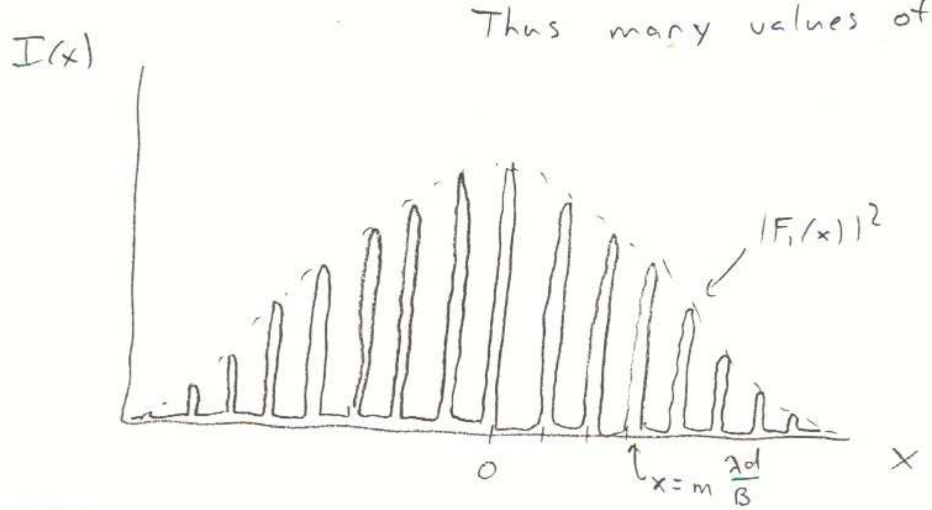
$\Rightarrow \quad x = m \dfrac{\lambda d}{B}$

Multiplied by $F_1\left(\frac{x}{\lambda d}\right)$: width $\Delta \nu_x \gg \frac{1}{B}$

so $\dfrac{\Delta x}{\lambda d} \gg \dfrac{1}{B}$

or $\Delta x \gg \dfrac{\lambda d}{B}$

Thus many values of m are included:



2. Given

$$f(x,y) = \begin{cases} i & x>0 \\ -i & x<0 \end{cases}$$

$$h(x,y) = \text{rect}(x)\,\delta(y)$$

So

$$g(x,y) = \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dx'\, f(x',y')\, \text{rect}(x-x')\,\delta(y-y')$$

$$= \int_{-\infty}^{\infty} dx'\, f(x',y)\, \text{rect}(x-x')$$

$$= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(x',y)\, dx'$$

Three cases possible:

$x \le -\frac{1}{2}$: Then $x' \le 0$ always

$$g(x,y) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} (-i)\, dx' = -i$$

$x \ge \frac{1}{2}$: Then $x' \ge 0$ always

$$g(x,y) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} (i)\, dx' = +i$$

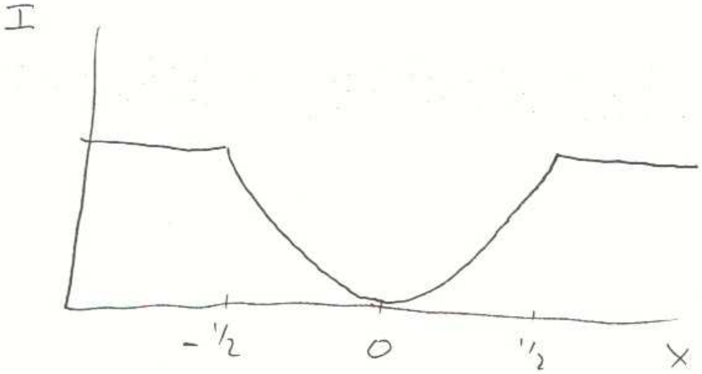
$-\frac{1}{2} < x < \frac{1}{2}$: Then split integral

$$g(x,y) = \int_{x-\frac{1}{2}}^{0} (-i)\, dx' + \int_{0}^{x+\frac{1}{2}} (i)\, dx'$$

$$= (-i)(-x+\tfrac{1}{2}) + i(x+\tfrac{1}{2})$$

$$= 2ix$$

So, $I(x,y) = \begin{cases} 1 & |x| > \frac{1}{2} \\ 4x^2 & |x| \le \frac{1}{2} \end{cases}$



So, even though $|f(x,y)|^2 = 1$ everywhere,
output wave is not uniform.

3. a) Impulse response of lens is

$$h(x,y) = h_0 \frac{2J_1\left(\frac{\pi D\rho}{\lambda d_2}\right)}{\pi D\rho/\lambda d_2}$$ = wave produced by point source

So if object points lie on x-axis, then output wave on x-axis is

$$g(x,y=0) = h_0\left[\frac{2J_1\left(\frac{\pi Dx}{\lambda d_2}\right)}{\pi Dx/\lambda d_2} + \frac{2J_1\left(\frac{\pi D(x-B)}{\lambda d_2}\right)}{\pi D(x-B)/\lambda d_2}\right]$$

$$= h_0\left[\rho(\alpha) + \rho(\alpha-\beta)\right]$$

for $\rho(\alpha) = \dfrac{2J_1(\alpha)}{\alpha}$

$$\alpha = \frac{\pi Dx}{\lambda d_2} \qquad \beta = \frac{\pi DB}{\lambda d_2}$$

So, plot $\rho(\alpha) + \rho(\alpha - \beta)$ vs α for various β. Find β where two peaks merge. See plots on next page

Looking closely at peak, dip appears at $\beta = 4.602$

Or, $$B = \frac{4.602}{\pi} \frac{\lambda d_2}{D}$$

$$\boxed{B = 1.465 \frac{\lambda d_2}{D}}$$

b) If sources are incoherent, then

$$I_{out}(x,y) = |h_o|^2 \left[\frac{2 J_1\left(\frac{\pi D \rho}{\lambda d_2}\right)}{\frac{\pi D \rho}{\lambda d_2}} \right]^2 + |h_o|^2 \left[\frac{2 J_1\left(\frac{\pi D \rho}{\lambda d_2}\right)}{\frac{\pi D \rho}{\lambda d_2}} \right]^2$$

So to get resolution, look at

$$q(\alpha) + q(\alpha + \beta)$$

$q = \rho^2$

α, β as before.

See plots two pages hence.

Find that dip appears at $\beta = 2.992$

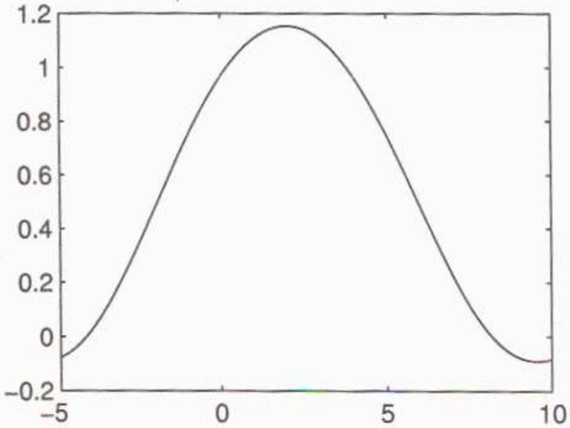
or $$B = \frac{2.992}{\pi} \frac{\lambda d_2}{D}$$

$$\boxed{B = 0.953 \frac{\lambda d_2}{D}}$$

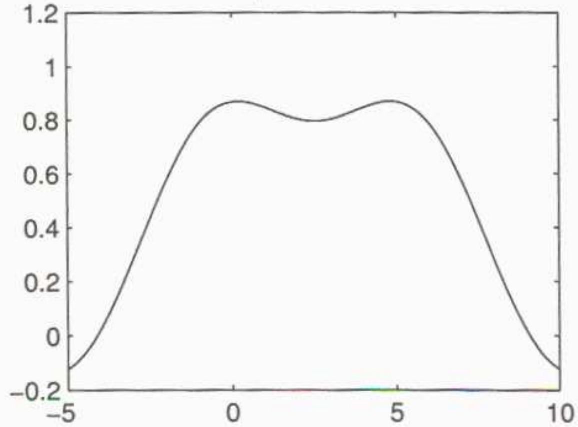
$\rho(\alpha) + \rho(\alpha+\beta)$ vs α

Various β

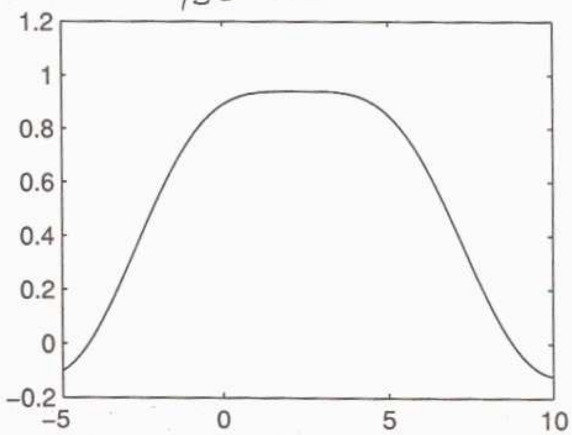
$\beta = 4$



$\beta = 5$



$\beta = 4.602$



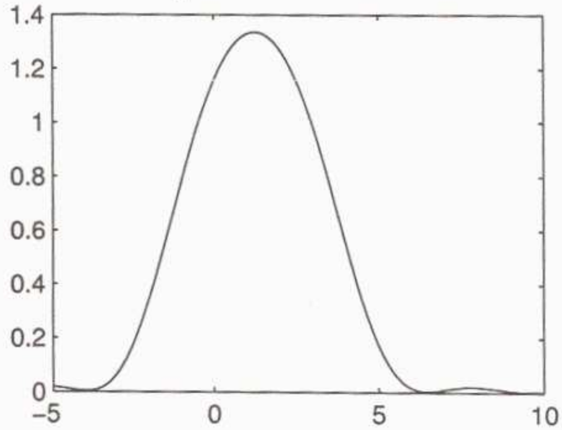
$$\rho(\alpha) = \frac{2J_1(\alpha)}{\alpha}$$

5b

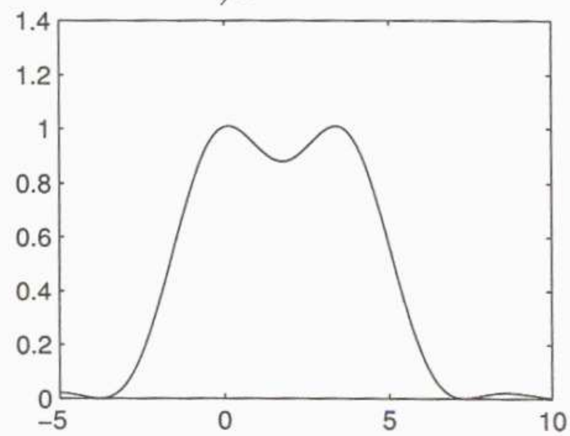
$q(\alpha) + q(\alpha+\beta)$ vs α

Various β

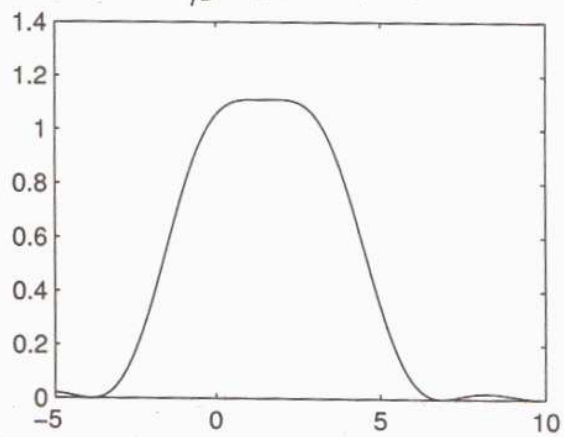
$\beta = 2.5$



$\beta = 3.5$



$\beta = 2.992$



$$q(\alpha) = \left[\frac{2J_1(\alpha)}{\alpha}\right]^2$$