Phys 531 Assignment 9 Solutions

1. $I = 250 \frac{W}{m^2}$ $\lambda = 550\,nm$

Have $I = \frac{|E_0|^2}{2\eta_0}$ $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega$

So, $E_0 = \sqrt{2\eta_0 I} = \sqrt{2 \times 377\,\Omega \times 250 \frac{W}{m^2}}$

$= \boxed{434 \frac{V}{m}}$

Go through units in a sensible way:

Know power $= \frac{\text{voltage}^2}{\text{resistance}}$

So $W = \frac{V^2}{\Omega}$

$\Omega = \frac{V^2}{W}$

So $\frac{\Omega W}{m^2} = \frac{V^2}{m^2}$, as required

Then $H_0 = \frac{1}{\eta_0} E_0$, and $B_0 = \mu_0 H_0$

so $B_0 = \frac{\mu_0}{\eta_0} E_0 = \mu_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 = \frac{1}{c_0} E_0$

$B_0 = \frac{434 \frac{V}{m}}{3 \times 10^8 \, m/s} = \boxed{1.45 \times 10^{-6}\, T}$

Units: know Force $= qE + q\vec{v} \times \vec{B}$

So B must have units of $\frac{E}{v} = \frac{\frac{V}{m}}{m/s} = Vs$

as required

Energy density

$$u(t) = \frac{\varepsilon_0}{2} E(t)^2 + \frac{\mu_0}{2} H(t)^2 = \frac{\varepsilon_0}{2} E(t)^2 + \frac{1}{2\mu_0} B(t)^2$$

but $\langle E(t) \rangle^2 = \frac{1}{2} E_0^2$ since real $E(t) = E_0 \cos\omega t$

$\langle B(t) \rangle^2 = \frac{1}{2} B_0^2$

So $$\langle u \rangle = \frac{1}{4}\left(\varepsilon_0 E_0^2 + \frac{1}{\mu_0} B_0^2\right) = \frac{\varepsilon_0}{4}\left(E_0^2 + \frac{1}{\mu_0 \varepsilon_0} B_0^2\right)$$

$$= \frac{\varepsilon_0}{4}\left(E_0^2 + c_0^2 B_0^2\right)$$ use $B_0 = \frac{1}{c_0} E_0$,

$$= \frac{\varepsilon_0}{2} E_0^2$$

$$= \frac{8.8 \times 10^{-12}\ F/m}{2} \left(434 \frac{V}{m}\right)^2$$

$$\boxed{\langle u \rangle = 8.3 \times 10^{-7} \frac{J}{m^3}}$$

Units: energy in capacitor $\approx CV^2 =$

so $F = \frac{J}{V^2}$, as required.

2. $\vec{E}(\vec{r}) = E_0 \sin\beta y\, e^{-i\beta z} \hat{x}$

a) Need $\nabla^2 \vec{E} = -k^2 \vec{E}$

$$\frac{\partial^2}{\partial y^2} \vec{E} = -\beta^2 \vec{E}$$

$$\frac{\partial^2}{\partial z^2} \vec{E} = -\beta^2 \vec{E}$$

$$\frac{\partial^2}{\partial x^2} \vec{E} = 0$$

So $\nabla^2 \vec{E} = -2\beta^2 E$

$2\beta^2 = k^2$

$\beta = \frac{1}{\sqrt{2}} k = \frac{1}{\sqrt{2}} \frac{2\pi}{\lambda}$

So $$\boxed{\beta = \sqrt{2} \frac{\pi}{\lambda_0}}$$

b) Have

$$\vec{\nabla} \times \vec{E} = -i\omega \vec{B} = -i\omega \mu_0 \vec{H}$$
$$= -i c_0 \mu_0 k \vec{H}$$
$$= -i \eta_0 k \vec{H}$$

While
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y} \frac{\partial E_x}{\partial z} - \hat{z} \frac{\partial E_x}{\partial y}$$

$$= \hat{y}\left(-i\beta E_0 \sin\beta y \, e^{-i\beta z}\right) - \hat{z}\left(\beta E_0 \cos\beta y \, e^{-i\beta z}\right)$$

and $\beta = \frac{k}{\sqrt{2}}$

So $$\vec{H}(\vec{r}) = \frac{i}{\eta_0 k} \frac{k}{\sqrt{2}} E_0 e^{-i\beta z}\left(-i\hat{y}\sin\beta y - \hat{z}\cos\beta y\right)$$

$$\boxed{\vec{H}(\vec{r}) = \frac{1}{\sqrt{2}} \frac{E_0}{\eta_0} e^{-i\beta z}\left(\hat{y}\sin\beta y - \hat{z}\, i\cos\beta y\right)}$$

c) Calculate $\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^*$

$$\vec{E} \times \vec{H}^* = \left(E_0 \sin\beta y \, e^{-i\beta z}\right)\left(\frac{1}{\sqrt{2}} \frac{E_0^*}{\eta_0} e^{+i\beta z}\right) \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 0 \\ 0 & \sin\beta y & +i\cos\beta y \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \frac{|E_0|^2}{\eta_0} \sin\beta y \left(-\hat{y}\, i\cos\beta y + \hat{z}\sin\beta y\right)$$

Then
$$\langle \mathcal{S} \rangle = \operatorname{Re} \vec{S} = \frac{1}{\sqrt{2}} \frac{|E_0|^2}{\eta_0} \sin^2\beta y \, \hat{z}$$

$\boxed{\text{Power flows in } \hat{z} \text{ direction}}$

d) Write $\vec{E} = E_0 \frac{e^{i\beta y} - e^{-i\beta y}}{2i} e^{-i\beta z} \hat{x}$

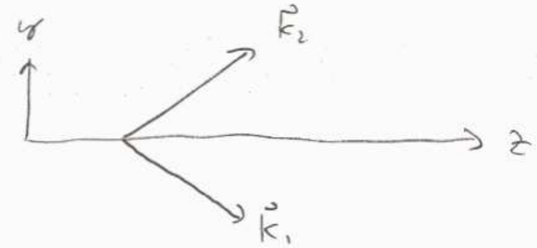
$$= \frac{E_0}{2i}\left[e^{-i\beta(z-y)} - e^{-i\beta(z+y)}\right]\hat{x}$$

$$= \frac{E_0}{2i}\left[e^{-ik\frac{z-y}{\sqrt{2}}} - e^{-ik\frac{z+y}{\sqrt{2}}}\right]\hat{x}$$

Form $e^{-i\vec{k}_1\cdot\vec{r}} + e^{-i\vec{k}_2\cdot\vec{r}}$, sum of two plane waves

$$\vec{k}_1 = \frac{k}{\sqrt{2}}(\hat{z}-\hat{y}) \qquad \vec{k}_2 = \frac{k}{\sqrt{2}}(\hat{z}+\hat{y})$$

So, component waves travelling in y-z plane at $\pm 45^\circ$ to z axis:



3. Maxwell's equations become, for monochromatic waves

$$\nabla\times\vec{H} = i\omega\varepsilon\vec{E} + \vec{J} = (i\omega\varepsilon+\sigma)\vec{E}$$

$$\nabla\times\vec{E} = -i\omega\mu_0\vec{H}$$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{H} = 0$$

(For monochromatic waves, $\frac{\partial}{\partial t} \to i\omega$)

Then $\vec{H} = \frac{-1}{i\omega\mu_0}(\vec{\nabla}\times\vec{E})$

so
$$\nabla\times\left[\frac{-1}{i\omega\mu_0}\vec{\nabla}\times\vec{E}\right] = (i\omega\varepsilon+\sigma)\vec{E}$$

$$\underbrace{\nabla\times(\nabla\times\vec{E})}= -i\omega\mu_0(i\omega\varepsilon+\sigma)\vec{E}$$
$$-\nabla^2\vec{E} = (\omega^2\varepsilon\mu_0 - i\omega\mu_0\sigma)\vec{E}$$

Helmholtz equation
$$\nabla^2\vec{E} + k^2\vec{E} = 0$$
Satisfied if $\boxed{k = \sqrt{\omega^2\varepsilon\mu_0 - i\omega\mu_0\sigma}}$

4. a) In general
$$\vec{\nabla}\cdot\vec{D} = \rho_{free}$$
and
$$\varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho_{total} = \rho_{free} + \rho_{bound}$$
Assume free charge $\rho_{free} = 0$

Also have $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

So, $\vec{\nabla}\cdot\vec{D} = \varepsilon_0\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P}$

or, $0 = \rho_{bound} + \vec{\nabla}\cdot\vec{P}$

Then $\boxed{\rho_{bound} = -\vec{\nabla}\cdot\vec{P}}$

b) If $\vec{P} = \varepsilon_0 \chi \vec{E}$

Then $\vec{\nabla} \cdot \vec{P} = \varepsilon_0 \vec{\nabla} \cdot (\chi \vec{E})$

$$-\rho = \varepsilon_0 \vec{E} \cdot \vec{\nabla}\chi + \varepsilon_0 \chi \vec{\nabla} \cdot \vec{E}$$

$$= \varepsilon_0 \vec{E} \cdot \vec{\nabla}\chi + \chi \rho$$

So $\rho(1+\chi) = -\varepsilon_0 \vec{E} \cdot \vec{\nabla}\chi$

$$\boxed{\rho = -\frac{\varepsilon_0}{1+\chi} \vec{E} \cdot \vec{\nabla}\chi}$$

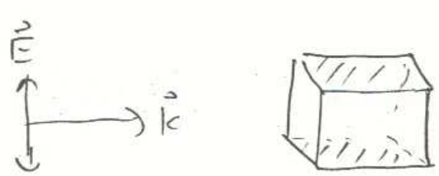
c) Only get ρ where $\vec{\nabla}\chi \neq 0$
= boundaries of cube

At boundary, direction of $\vec{\nabla}\chi$ is along normal to boundary

So, need $\vec{E} \perp$ boundary to have $\vec{E} \cdot \vec{\nabla}\chi \neq 0$

Get charges on faces of cube $\perp$ to $\vec{E}$:

if



Get charges on top & bottom faces

d) If χ = constant, $\vec{\nabla}\chi = 0$

so $\rho = 0$ everywhere

Effect on waves comes from

$$\boxed{\vec{\nabla} \times \vec{H} = i\omega\varepsilon \vec{E}}$$ equation

Note $\varepsilon = \varepsilon_0 (1+\chi)$

so $$\vec{\nabla} \times \vec{H} = i\omega\varepsilon_0 \vec{E} + i\omega\varepsilon_0 \chi \vec{E}$$

$$= i\omega\varepsilon_0 \vec{E} + i\omega \vec{P}$$

General equation: $\vec{\nabla} \times \vec{H} = i\omega\varepsilon_0 \vec{E} + \vec{J}$

So, can identify $\vec{J} = i\omega\vec{P}$

or in time $\vec{J} = \frac{d\vec{P}}{dt}$

So, even though $\rho = 0$ everywhere, have $\vec{J} \neq 0$

gives rise to scattered wave

Current makes sense: in cube example (c),

charge on faces of cube was oscillating

This requires current flowing through cube