Phys 531 Final Exam Solutions

1. Any ray matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ has $AD - BC = \frac{n_{in}}{n_{out}}$

For symmetric system, $n_{in} = n_{out}$

so $AD - BC = 1$

This rules out (b): $AD - BC = 4$

and (d): $AD - BC = -1$

Easiest approach: think of a few symmetric systems:

Single thin lens $M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$

Free propagation distance d: $M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

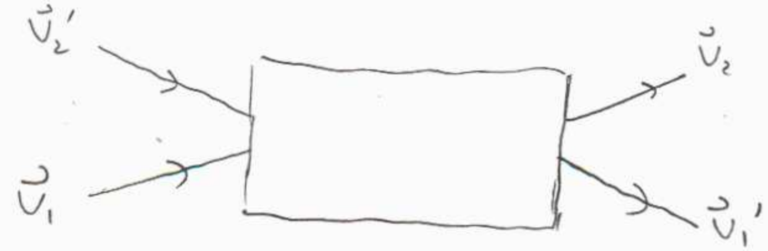
Both of these have $A = D$, so guess $\boxed{(a)}$ with confidence

To make sure:

If input ray $\vec{v}_1$ leads to output $\vec{v}_2$,

then input $\vec{v}_2{}' = \begin{bmatrix} y_2 \\ -\theta_2 \end{bmatrix}$ leads to output $\vec{v}_1{}' = \begin{bmatrix} y_1 \\ -\theta_1 \end{bmatrix}$

for a symmetric system:



Try $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then for (a), $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{v}_2{}' = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

and $M_a \vec{v}_2{}' = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \vec{v}_1{}'$

But for (c), $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2{}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and $M_c \vec{v}_2{}' = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \vec{v}_1{}'$

So answer must be $\boxed{(a)}$

<u>General Solution</u>:

$$v_1 = \begin{bmatrix} y \\ \theta \end{bmatrix} \qquad v_2 = \begin{bmatrix} Ay + B\theta \\ Cy + D\theta \end{bmatrix} \qquad v_2{}' = \begin{bmatrix} Ay + B\theta \\ -Cy - D\theta \end{bmatrix}$$

$$M v_2' = \begin{bmatrix} A^2 y + AB\theta - BCy - BD\theta \\ ACy + BC\theta - DCy - D^2\theta \end{bmatrix} = \begin{bmatrix} y \\ -\theta \end{bmatrix}$$

Need
$$A^2 - BC = 1$$
$$\left.\begin{array}{l} AB - BD = 0 \\ AC - DC = 0 \end{array}\right] \Rightarrow \boxed{A = D}$$
$$D^2 - BC = 1$$

Then $AD - BC = A^2 - BC = D^2 - BC = 1$ anyway.

2. Plane wave is defined by

$$U(\vec{r}) = A e^{-i\vec{k}\cdot\vec{r}}$$

$$u(\vec{r}) = Re\{U(\vec{r})e^{i\omega t}\}$$

$$= |A| \cos(\omega t - \vec{k}\cdot\vec{r} + \phi)$$

So:

(a) : Wave, but not plane wave

(b) : Plane wave with $\phi = -\frac{\pi}{2}$

(c) : Plane wave with $A = i$, $\vec{k} = 0$

(d) : Not a plane wave

(e) : Looks like plane wave with $\vec{k} = \left(-\frac{\pi}{\lambda}, -\frac{\pi}{\lambda}, -\frac{\sqrt{2}\pi}{\lambda}\right)$

check $|\vec{k}| = \frac{\pi}{\lambda}\sqrt{1+1+2} = \frac{2\pi}{\lambda}$

so it is a good solution.

3. Say red slit generates diffracted wave $U_R(t)$

blue slit $U_B(t)$

Then interference term is $\langle U_R^* U_B \rangle$

But if U_R oscillates near ω_R

and U_B oscillates near ω_B

with $\omega_B - \omega_R$ large,

then time average

$$\langle U_R^* U_B \rangle \sim \langle e^{-i(\omega_R - \omega_B)t} \rangle \to 0$$

So no interference observed anywhere, $\boxed{(a)}$

4. Wave propagates as $e^{i(\omega t - \tilde{k} z)}$

$$= e^{-\frac{\alpha z}{2}} e^{i(\omega t - n k_0 z)}$$

$$n = \mathrm{Re}\, \tilde{n}$$

Phase velocity is $c = \frac{\omega}{n k_0} = \frac{c_0}{n}$

$$= \frac{c_0}{1.5} = 0.667\, c_0$$

$\boxed{b}$

5. Label pictures (a) (b)

(c) (d)

(sorry I forgot labels on exam!)

Can rule out (a) + (b) because center of Fraunhofer pattern is always bright (recall problem 6 from midterm.)

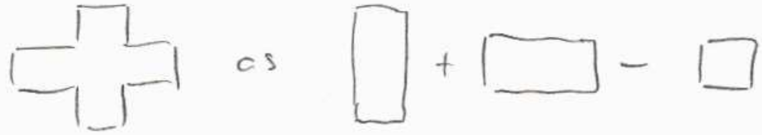
To judge between (c) + (d), note fastest spatial frequency in horizontal direction is $\frac{1}{a}$, while in vertical direction its $\frac{2}{a}$

Since diffraction pattern $\sim F(\nu_x = \frac{x}{\lambda d}, \nu_y = \frac{y}{\lambda d})$ expect large y features to have twice the length scale of large x features.

Look at central big rectangle: in (c), sides have 2:1 ratio. In (d), sides have 3:1 ratio.

So, choose $\boxed{c}$

How I made pictures:

Can write 

Since everything is linear, get

$$U(x,y) = \text{rect}(a, 3a) + \text{rect}(3a, \tfrac{1}{2}a) - \text{rect}(a, \tfrac{1}{2})$$

where $\text{rect}(D_x, D_y)$ is pattern from rectangular slit

$$\text{rect}(D_x, D_y) = D_x D_y \operatorname{sinc} \frac{x D_x}{\lambda d} \operatorname{sinc} \frac{y D_y}{\lambda d}$$

6. Here TE is normal to page, and optic axis is in page, so TE is ordinary, TM is extraordinary.

If $n_e > n_o$, then $n_{TE} < n_{TM}$

So TE light refracts less:

$$\sin\theta_2 = \frac{n_1}{n_2} \sin\theta_1$$

the closer n_1 & n_2 are, the closer θ_1 & θ_2 are.

So, choose $\boxed{(b)}$

Multiple Choice Scores:

	% correct:
1.	15%
2.	3.7%
3.	15%
4.	70%
5.	41%
6.	74%

Average score: 11.3/30

7. $U(z=0) = A\left[\frac{1}{4}\left(e^{\frac{i\pi}{\lambda}(x+y)} + e^{\frac{i\pi}{\lambda}(x-y)} + e^{-\frac{i\pi}{\lambda}(x-y)} + e^{-\frac{i\pi}{\lambda}(x+y)}\right) - 1\right]$

So $\nu_x, \nu_y = \pm \frac{1}{2\lambda}$ for first terms

$= 0$ for last

Then $\nu_z = \sqrt{\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2} = \frac{1}{\sqrt{2}\,\lambda}$ for first terms

$\nu_z = \frac{1}{\lambda}$ for last term

So $U(x,y,z) = A\left[e^{-\frac{i2\pi z}{\sqrt{2}\lambda}} \cos\frac{\pi x}{\lambda} \cos\frac{\pi y}{\lambda} - e^{-i\frac{2\pi z}{\lambda}}\right]$

$U(0,0,z) = A\left[e^{-i\frac{2\pi z}{\sqrt{2}\lambda}} - e^{-i\frac{2\pi z}{\lambda}}\right] = 0$

Need $\frac{2\pi z}{\sqrt{2}\lambda} = 2\pi n + \frac{2\pi z}{\lambda}$

for integer n

$z\left(\frac{1}{\sqrt{2}} - 1\right) = n\lambda$

$z = n\frac{\lambda}{\frac{1}{\sqrt{2}} - 1}$ or $= \boxed{n\frac{\lambda}{1 - \frac{1}{\sqrt{2}}}}$

8. From Fresnel equations,

$$r_x \to -1 \quad \text{if} \quad n_2 \to \infty$$
$$r_y \to +1$$

From Figure 6.2-1 see that in either case get $\vec{E}_{refl} = -\vec{E}_{in}$ at normal incidence

So $\tilde{\vec{E}}_{TOT} = \tilde{\vec{E}}_{in} + \tilde{\vec{E}}_{ref}$

$$= A\hat{x}\, e^{-ikz} - A\hat{x}\, e^{+ikz}$$

$$\boxed{\tilde{\vec{E}}_{TOT} = -2iA\,\hat{x}\,\sin kz}$$

Then $\tilde{\vec{H}} = \tilde{\vec{H}}_{in} + \tilde{\vec{H}}_{ref}$

$$\tilde{\vec{H}}_{in} = \frac{1}{\mathfrak{Z}_0}\hat{k}_{in} \times \tilde{\vec{E}}_{in} = \frac{1}{\mathfrak{Z}_0}(\hat{z} \times \hat{x})\, A e^{-ikz}$$
$$= \frac{1}{\mathfrak{Z}_0}\hat{y}\, A e^{-ikz}$$

while $\tilde{\vec{H}}_{ref} = \frac{1}{\mathfrak{Z}_0}\vec{k}_{ref} \times \vec{E}_{refl} = \frac{1}{\mathfrak{Z}_0}(-\hat{z} \times -\hat{x})\, A e^{+ikz}$

$$= \frac{1}{\mathfrak{Z}_0}\hat{y}\, A e^{+ikz}$$

So $\tilde{\vec{H}}_{TOT} = \frac{A}{\mathfrak{Z}_0}\hat{y}\left(e^{-ikz} + e^{+ikz}\right) = \boxed{\frac{2A}{\mathfrak{Z}_0}\hat{y}\cos kz}$

Above for $z < 0$. For $z > 0$, have $E, H \sim e^{-\frac{\alpha z}{2}} \to 0$ within conductor

9. Say initial polarization is $\vec{J}_{in} = \begin{bmatrix} J_x \\ J_y \end{bmatrix}$

write as
$$\vec{J}_{in} = \frac{J_x}{2}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{J_x}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{J_y}{2i}\begin{bmatrix} 1 \\ i \end{bmatrix} - \frac{J_y}{2i}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\left[J_x\left(\vec{J}_R + \vec{J}_L\right) + \frac{1}{i} J_y \left(\vec{J}_R - \vec{J}_L\right)\right]$$

Pass through medium:
$$\vec{J}_{out} = \frac{1}{\sqrt{2}}\left[J_x\left(t_R \vec{J}_R + t_L \vec{J}_L\right) + \frac{1}{i} J_y\left(t_R \vec{J}_R - t_L \vec{J}_L\right)\right]$$
$$= \frac{1}{2}\left[J_x\left(t_R\begin{bmatrix} 1 \\ i \end{bmatrix} + t_L\begin{bmatrix} 1 \\ -i \end{bmatrix}\right) + \frac{1}{i} J_y\left(t_R\begin{bmatrix} 1 \\ i \end{bmatrix} - t_L\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)\right]$$
$$= \frac{1}{2}\begin{bmatrix} J_x(t_R+t_L) - i J_y(t_R - t_L) \\ i J_x(t_R - t_L) + J_y(t_R + t_L) \end{bmatrix}$$

So $$\boxed{\vec{J}_{out} = \frac{1}{2}\begin{bmatrix} t_R + t_L & -i(t_R - t_L) \\ i(t_R - t_L) & t_R + t_L \end{bmatrix} \vec{J}_{in}}$$

Could also write $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and solve $M\begin{bmatrix} 1 \\ i \end{bmatrix} = t_R \begin{bmatrix} 1 \\ i \end{bmatrix}$
$M\begin{bmatrix} 1 \\ -i \end{bmatrix} = t_L \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Or, write $M = U^{-1}\begin{bmatrix} t_R & 0 \\ 0 & t_L \end{bmatrix} U$ for $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$

10.

Evaluate $G(\tau) = \langle u^*(t)\, u(t+\tau) \rangle$

$$u(t) = \frac{1}{\sqrt{2}}\left[u_0(t) + u_0(t-a)\right]$$

$$G(\tau) = \frac{1}{2}\left\langle \left[u_0^*(t) + u_0^*(t-a)\right]\left[u_0(t+\tau) + u_0(t-a+\tau)\right]\right\rangle$$

$$= \frac{1}{2}\langle u_0^*(t)u_0(t+\tau) + u_0^*(t-a)u_0(t+\tau) + u_0^*(t)u_0(t-a+\tau) + u_0^*(t-a)u_0(t-a+\tau)\rangle$$

$$= \frac{1}{2}\left[G_0(\tau) + G_0(\tau+a) + G_0(\tau-a) + G_0(\tau)\right]$$

where $G_0(\tau) = \langle u_0^*(t)\, u_0(t+\tau)\rangle$

Then $S(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i \nu \tau} G(\tau)\, d\tau$

$$= S_0(\nu) + \frac{1}{2}\int_{-\infty}^{\infty} e^{-2\pi i \nu \tau}\left[G_0(\tau+a) + G_0(\tau-a)\right] d\tau$$

$$= S_0(\nu) + \frac{1}{2}\int_{-\infty}^{\infty}\left[e^{-2\pi i \nu(\tau-a)} + e^{-2\pi i \nu(\tau+a)}\right] G_0(\tau)\, d\tau$$

$$= S_0(\nu) + \frac{1}{2}\left(e^{2\pi i \nu a} + e^{-2\pi i \nu a}\right) S_0(\nu)$$

$$\boxed{S(\nu) = S_0(\nu)\left[1 + \cos(2\pi \nu a)\right]}$$