Phys 531 Final Exam Solutions

1. a) From notes, $I_0 = \frac{2P}{\pi w_0^2} = \frac{0.2\,W}{\pi \times 10^{-6}\,m^2} = \boxed{6.3 \times 10^4 \frac{W}{m^2}}$

(2 pts)

b) Generally have

$$I = \frac{E^2}{2 Z_0}$$

$$E = \sqrt{2 Z_0 I} = \sqrt{2 \times 377\,\Omega \times 6.3 \times 10^4 \frac{W}{m^2}}$$

1 pt

$$= \boxed{6.93 \times 10^3 \frac{V}{m}}$$

c) $B = \frac{E}{c} = \boxed{2.3 \times 10^{-5}\ T}$

1 pt

d) $n = \frac{u}{\hbar\omega}$ $u =$ energy density $= \frac{I}{c}$

$$\hbar\omega = (1.054 \times 10^{-34}\ J \cdot s)\left(\frac{2\pi c}{\lambda}\right)$$

$\omega = 2.98 \times 10^{15} \frac{rad}{s}$

$\nu = 4.7 \times 10^{14}\ Hz$

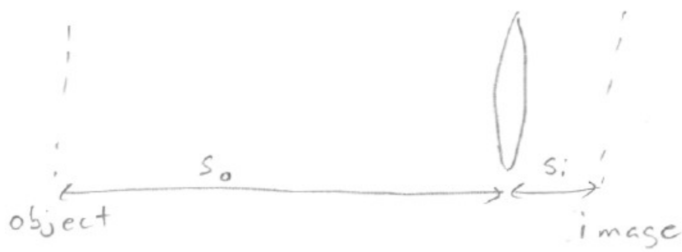
$$= 3.14 \times 10^{-19}\ J$$ 1 pt

$$u = \frac{6.3 \times 10^4 \frac{W}{m^2}}{c} = 2.1 \times 10^{-4} \frac{J}{m^3}$$

$$\boxed{n = 6.7 \times 10^{14} \frac{photons}{m^3}}$$

2. Want magnification $= \frac{7\text{ mm}}{5\text{ cm}} = 0.14 = \left|\frac{s_i}{s_o}\right|$ (2 pts)

Looks like



With $s_o = \frac{s_i}{0.14} = 7.14\, s_i$

Since $s_o \gg s_i$, have $s_i \approx f$

$s_o \approx 7f$

Need $s_o > 15\text{ cm} \Rightarrow f > \frac{15}{7}\text{ cm} \approx 2\text{ cm}$

So 12 mm lens won't work, but $\boxed{26\text{ mm}}$ lens should.

With that lens, want

$$\frac{1}{s_i} + \frac{1}{s_o} = \frac{1}{f}$$

$$\frac{1}{s_i} + \frac{0.14}{s_i} = \frac{1}{f} \qquad s_i = 1.14\, f$$

$$= 29.6\text{ mm}$$

$$s_o = 7.14\, s_i = 21.1\text{ cm}$$

So lens is $\boxed{21.1\text{ cm}}$ from board,

and sensor is $\boxed{2.96\text{ cm}}$ from lens

3. From tables:

$$\delta(x - x_0) \to e^{-ik_x x_0}$$

$$e^{i\alpha x} \to 2\pi\,\delta(k_x - \alpha)$$

and

$$\begin{cases} 1 & (|y| < b) \\ 0 & \text{else} \end{cases} \to 2b \operatorname{sinc} k_y b$$

So $f(x, y) \to$

$$\boxed{F(k_x, k_y) = 2b\left[A e^{-ik_x x_0} + 2\pi B\,\delta(k_x - \alpha)\right] \operatorname{sinc} k_y b}$$

4. If $z=0$ at front of plate, then at

$z = t_0 + a$, have

$$E = \underbrace{E_0 \, e^{inkt}}_{\text{transmission through glass}} \; \underbrace{e^{ik(t_0+a-t)}}_{\text{transmission through air}}$$

$$= E_0 \, e^{ink(t_0 + a\sin\beta x)} \, e^{ik(t_0 + a - t_0 - a\sin\beta x)}$$

$$= E_0 \, e^{inkt_0} \, e^{ika} \, e^{i(n-1)ka\sin\beta x}$$

$$= E_0' \, e^{i(n-1)ka\sin\beta x}$$

$$E_0' = E_0 e^{ik(nt_0+a)}$$

But note $ka \ll 1$

$$E \approx E_0' \left[1 + i(n-1)ka\sin\beta x\right]$$

$$= E_0' \left\{1 + \frac{(n-1)ka}{2}\left[e^{i\beta x} + e^{-i\beta x}\right]\right\}$$

Sum of harmonic functions

Each evolves with phase $e^{i\varkappa z}$ $\quad \varkappa = \sqrt{k^2 - k_x^2}$

So

$$E(d) = E_0'\left[e^{ikd} + \frac{(n-1)ka}{2}\left(e^{i\beta x} + e^{-i\beta x}\right)e^{i\sqrt{k^2-\beta^2}\,d}\right]$$

$$\boxed{E(d) = E_0'\left[e^{ikd} + i(n-1)ka\sin\beta x \; e^{i\sqrt{k^2-\beta^2}\,d}\right]}$$

5. Know that circular aperture of radius a produces field

$$E = -\frac{i}{\lambda d} e^{i\phi} \pi a^2 \frac{2 J_1\left(\frac{k\rho a}{d}\right)}{k\rho a/d} E_0$$

$$\phi = kd + k\frac{\rho^2}{2d}$$

Can write field from ring as $E_2 - E_1$

E_2 = field from aperture radius R_2

E_1 = field from aperture radius R_1

Since E_1 will be "missing"

$$E_{TOT} = -\frac{i}{\lambda d} e^{i\phi} \frac{2\pi d}{k\rho}\left[R_2 J_1\left(\frac{k\rho R_2}{d}\right) - R_1 J_1\left(\frac{k\rho R_1}{d}\right)\right] E_0$$

Interference pattern

$$\boxed{|E_{TOT}|^2 = \frac{1}{\rho^2}\left[R_2 J_1\left(\frac{k\rho R_2}{d}\right) - R_1 J_1\left(\frac{k\rho R_1}{d}\right)\right]^2 |E_0|^2}$$

6. Matrix for half wave plate

$$M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$\hat{J}_{in} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = LCP$$

So $$\hat{J}_{out} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos 2\theta + i \sin 2\theta \\ \sin 2\theta - i \cos 2\theta \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i2\theta} \\ -i e^{i2\theta} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} e^{i2\theta} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Overall phase is irrelevent, so

$$\hat{J}_{out} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \boxed{RCP}$$

7.a) At normal incidence, Fresnel equations become $r = \frac{n_i - n_t}{n_i + n_t}$

$$r_{1a} = \frac{n_1 - n_a}{n_1 + n_a} \qquad r_{a2} = \frac{n_a - n_2}{n_a + n_2}$$

Supposed to be equal, so

$$\frac{n_1 - n_a}{n_1 + n_a} = \frac{n_a - n_2}{n_a + n_2} \rightarrow +2$$

$$(n_1 - n_a)(n_2 + n_a) = (n_a - n_2)(n_a + n_1)$$

$$n_1 n_2 - n_a n_2 + n_1 n_a - n_a^2 = n_a^2 - n_2 n_a + n_a n_1 - n_1 n_2$$

$$n_1 n_2 - n_a^2 = n_a^2 - n_1 n_2$$

$$n_a^2 = n_1 n_2$$

$$\boxed{n_a = \sqrt{n_1 n_2}}$$

b) Reflection from $a \to 2$ interface travels extra distance $2h$

To get π phase shift, need $2h = \lambda\left(\frac{1}{2} + m\right)$

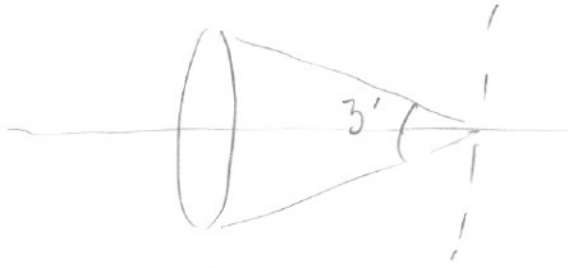
m = integer

$$h = \frac{\lambda}{2}\left(\frac{1}{2} + m\right)$$

But need λ in medium $= \frac{\lambda_0}{n_a}$

$$\boxed{h = \frac{\lambda}{2n_a}\left(\frac{1}{2} + m\right)} \quad \text{integer } m$$

8. From image plane, looks like light coming from source with angle $\vartheta' = \frac{D}{f}$:



So, have $\rho_c \approx \frac{\lambda}{\vartheta'} = \boxed{\frac{\lambda f}{D}}$

Or: each point on source $\Rightarrow$ diffraction pattern with $\Delta x \approx \frac{\lambda f}{D}$

Within this Δx, can't distinguish which source point light is from
$\Rightarrow$ field is coherent

Or from van Cittert-Zernike, $\rho_c \approx$ width of diffraction pattern $= \frac{\lambda f}{D}$