Lens System Analysis

Last time, finished survey of optical elements
ray diagrams, mirrors, stops, prisms

Also, analyzed simple lens system:
two thin lenses separated by d

Analyzed by iterating thin lens equation
image of first lens = object of second

Today, develop a better way

Outline:

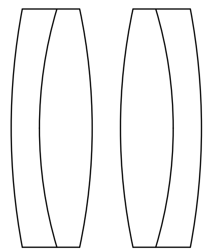
- Ray matrix method
- Thick lens formalism

Next time: study some real systems

Ray Matrices (Hecht 6.2.1)

Good lens systems typically have 4-8 elements

Individual elements not always thin

Plössl eyepiece: 

Common in telescopes

To analyze:

(a) Find image from first surface

= object of second

(b) Find image from second surface

= object of third

...

Iterate until finished

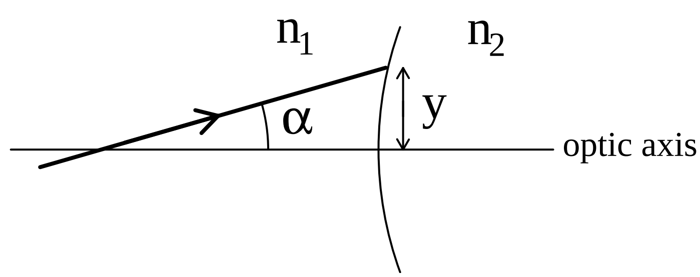
Analytical solution useless for > two surfaces

Computer solution OK

- cumbersome to program
- little insight

Better way:

Consider single ray at some interface



Described by two parameters:
angle α height y

Can write as vector $\begin{bmatrix} \alpha \\ y \end{bmatrix}$

Note: $\begin{bmatrix} \alpha \\ y \end{bmatrix}$ not a "normal" vector

- units are different
- (α, y) not coordinates of anything

Can use just like a vector anyway

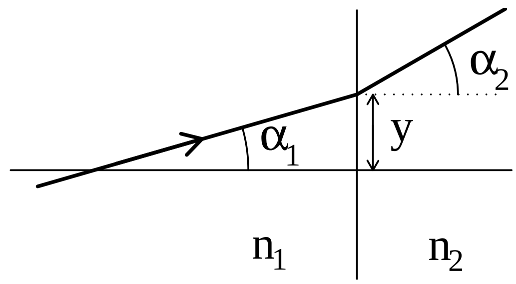
Better definition: $$\boxed{\mathbf{v} = \begin{bmatrix} n\alpha \\ y \end{bmatrix}}$$

Optics books: $\begin{bmatrix} n\alpha \\ y \end{bmatrix}$ Laser books: $\begin{bmatrix} y \\ \alpha \end{bmatrix}$

Strategy:

Determine how $\mathbf{v}$ propagates through system

First, plane interface:



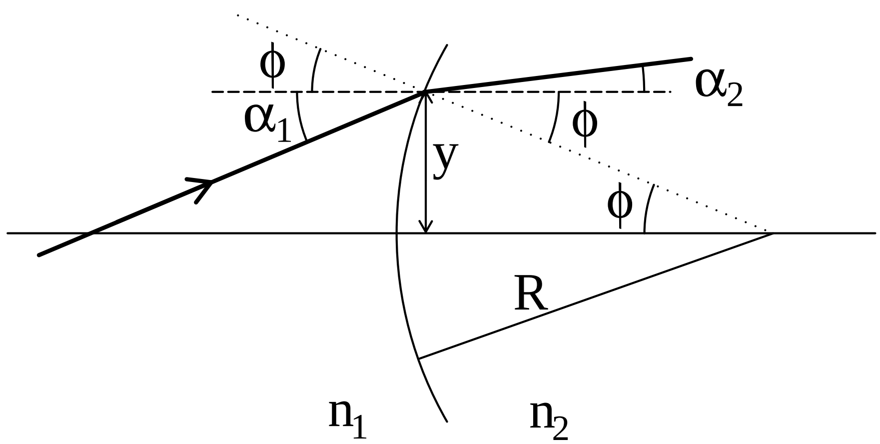
Paraxial limit: $\alpha \ll 1$

Snell's Law says $n_1\alpha_1 = n_2\alpha_2$

At boundary, y doesn't change

With optics convention, $\mathbf{v}$ is constant!

Consider curved interface:



Snell's Law: $n_1(\alpha_1 + \phi) = n_2(\alpha_2 + \phi)$

with $\phi = \dfrac{y}{R}$

So $n_2\alpha_2 = n_1\alpha_1 + (n_1 - n_2)\phi$

$$= n_1\alpha_1 + \frac{n_1 - n_2}{R} y$$

Define $\mathcal{D} = \frac{n_2 - n_1}{R} =$ power of surface

Like power = $1/f$ for thin lens

unit = diopter

Then $n_2\alpha_2 = n_1\alpha_1 - \mathcal{D}y$

Again, y same on either side

Linear relationship: can write in matrix form:

$$\begin{bmatrix} n_1\alpha_1 \\ y \end{bmatrix} = \begin{bmatrix} 1 & -\mathcal{D} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n_2\alpha_2 \\ y \end{bmatrix}$$

or

$$\mathbf{v}_1 = \mathcal{R}\mathbf{v}_2$$

$\mathcal{R}$ = refraction matrix

Note, if curvature $R \to \infty$, matrix $\mathcal{R} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{v}_1 = \mathbf{v}_2$, as before

For thin lens in air, use $\mathcal{D} = 1/f$

$$\mathcal{R}_{\text{lens}} = \begin{bmatrix} 1 & -1/f \\ 0 & 1 \end{bmatrix}$$

Question: What is the matrix for a spherical mirror with radius R?

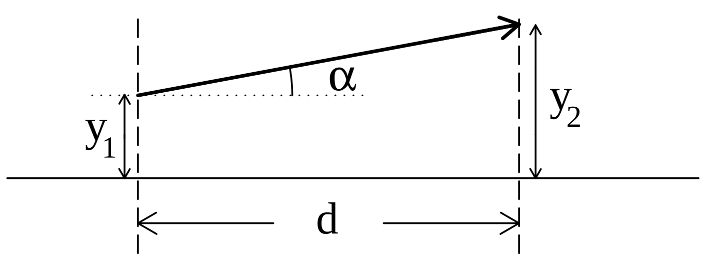
So, effect of lens = matrix multiplication

Handle many lenses by multiplying matrices

Computationally easy!

But also need ray propagation between surfaces

Free propagation:



Paraxial limit: $y_2 = y_1 + \alpha d$ and $n\alpha =$ constant

$$\begin{bmatrix} n_2\alpha_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d/n & 1 \end{bmatrix} \begin{bmatrix} n_1\alpha_1 \\ y_1 \end{bmatrix}$$

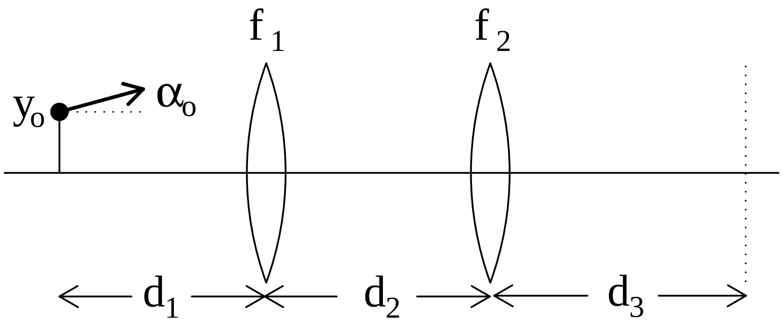
or $\mathbf{v}_2 = \mathcal{T}\mathbf{v}_1$

Transfer matrix $\mathcal{T}$

In general, $\mathcal{R}, \mathcal{T}$ called *ray matrices*

Multiply to get matrix for complete system

Example:



Consider ray starting at $\mathbf{v}_o = \begin{bmatrix} \alpha_o \\ y_o \end{bmatrix}$

Just before first lens: $\mathbf{v} = \mathcal{T}(d_1)\mathbf{v}_o$

Just after first lens: $\mathbf{v} = \mathcal{R}(f_1)\mathcal{T}(d_1)\mathbf{v}_o$

...

At final plane: $\mathbf{v}_f = \mathcal{T}(d_3)\mathcal{R}(f_2)\mathcal{T}(d_2)\mathcal{R}(f_1)\mathcal{T}(d_1)\mathbf{v}_o$

Note reversed order!

Define $\mathbf{v}_f = \mathcal{M}\mathbf{v}_o$

$\mathcal{M}$ = system matrix

relates rays at output to rays at input

For simple system, can get $\mathcal{M}$ analytically

For any system, easy to get $\mathcal{M}$ numerically
like tracing all possible rays at once

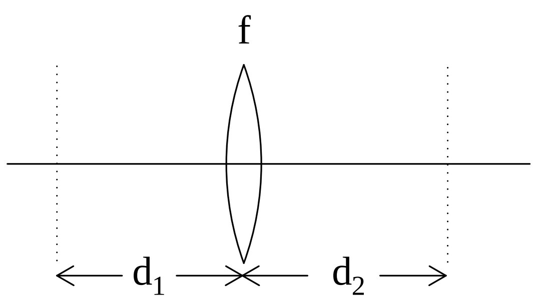
Generally write

$$\mathcal{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad \left(\text{Hecht: } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$$

Coefficients (A, B, C, D) completely define system
(in paraxial limit)

Ray matrices sometimes called “ABCD matrices”

Example:



Say $d_1 = 10$ cm, $f = 25$ cm, $d_2 = 20$ cm

$$\text{Then } \mathcal{M} = \begin{bmatrix} 1 & 0 \\ 20 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.04 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 20 & 1 \end{bmatrix} \begin{bmatrix} 0.06 & -0.04 \\ 10 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.04 \text{ cm}^{-1} \\ 22 \text{ cm} & 0.2 \end{bmatrix}$$

Analytically, get

$$\mathcal{M} = \begin{bmatrix} 1 - \frac{d_1}{f} & -\frac{1}{f} \\ d_1 + d_2 - \frac{d_1 d_2}{f} & 1 - \frac{d_2}{f} \end{bmatrix}$$

Supposed to know everything about system

ask: given $s_o = d_1$, where is image formed?

Imaging system: y_{out} depends on y_{in}, not α_{in}

Compare $\begin{bmatrix} \alpha \\ y \end{bmatrix}_{\text{out}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix}_{\text{in}}$

Need $C = 0$

In example $C = d_1 + d_2 - \frac{d_1 d_2}{f}$

So need $d_1 + d_2 = \dfrac{d_1 d_2}{f}$

or $\dfrac{1}{d_1} + \dfrac{1}{d_2} = \dfrac{1}{f}$

Works!

Can apply to any system

Procedure:
(1) Calculate vertex-to-vertex matrix $\mathcal{M}_v$
= just before first surface to just after last

Say $\mathcal{M}_v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(2) Given object distance s_o, calculate

$$\mathcal{M} = \mathcal{T}(s_i)\mathcal{M}_v\mathcal{T}(s_o)$$

for arbitary s_i:

$$\mathcal{M} = \begin{bmatrix} a + bs_o & b \\ as_i + bs_os_i + c + ds_o & d + bs_i \end{bmatrix}$$

(3) Solve for $C = 0$:

$$as_i + bs_os_i + c + ds_o = 0$$

gives

$$\boxed{s_i = -\frac{c + ds_o}{a + bs_o}}$$

Also magnification $m = y_i/y_o$
but if $C = 0$, then $y_i = Dy_o$
$\Rightarrow m = D = \boxed{d + bs_i}$

This is really powerful!

Explore imaging further...

But first, a general property:

Any ray matrix has determinant = 1

$$\det \mathcal{M} \equiv \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 1$$

Proof:

- Know $\det \mathcal{R} = 1$ and $\det \mathcal{T} = 1$
- Matrix property:
  $\det(\mathcal{M}_1 \mathcal{M}_2) = \det(\mathcal{M}_1)\det(\mathcal{M}_2)$
- Ray matrix $\mathcal{M}$ = product of $\mathcal{T}$'s and $\mathcal{R}$'s
- So $\det \mathcal{M}$ = product of 1's = 1

Previous example:

$$\mathcal{M} = \begin{bmatrix} 0.6 & -0.04 \text{ cm}^{-1} \\ 22 \text{ cm} & 0.2 \end{bmatrix}$$

So $AD - BC = 0.6 \cdot 0.2 + 0.04 \cdot 22 = 1$

Identity useful in derivations

Practial application:

error check for calculations

. . . back to imaging

Ray matrix defines system
Like focal length defines thin lens

For thin lens, ray diagrams still helpful:
visual representation of lens effect

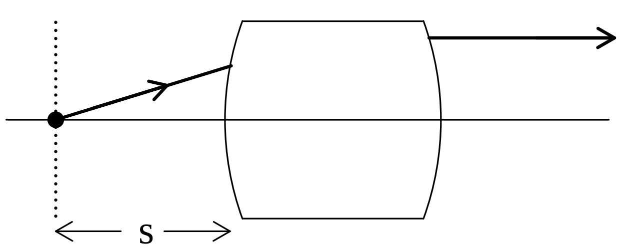
Develop ray diagram formalism for general system

Thick Lens Picture (Hecht 6.1)

Suppose vertex-vertex matrix $\mathcal{M}_v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Where are focal points?

Front focal point = object point imaged to ∞



Want output angle $\alpha_2 = 0$
for input position $y_1 = 0$

Have

$$\mathbf{v}_2 = \mathcal{M}_v \mathcal{T}(s) \mathbf{v}_1$$

or

$$\begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} a + sb & b \\ c + sd & d \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}$$

So require $a + sb = 0$

Define $s =$ *front focal length* (ffl)

$$\text{ffl } = -\frac{a}{b}$$

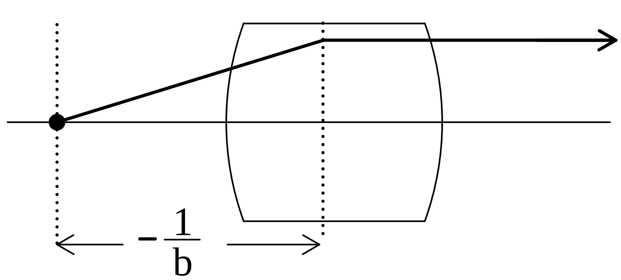
= distance of front focal point from front vertex

For rays emitted from focal point, have

$$y_2 = (c + sd)\alpha_1 = \left(c - \frac{ad}{b}\right)\alpha_1$$

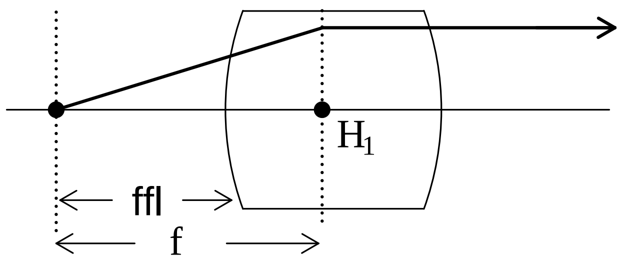
$$= \left(\frac{bc - ad}{b}\right)\alpha_1 = -\frac{1}{b}\alpha_1$$

Acts like thin lens located $-1/b$ from focal point



Define focal length of system $= -1/b$

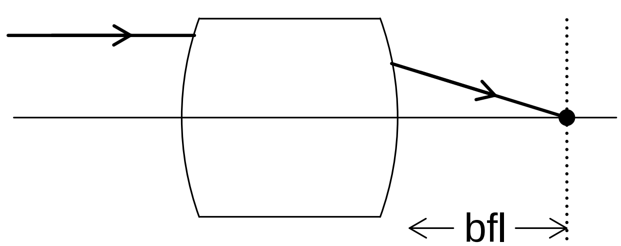
Call location of effective lens =
front principle plane
Intersection with axis = (front) principle point H_1



Distance from vertex to principle point

$$= f - \text{ffl} = \frac{a-1}{b}$$

Similar: define back focal point
= focus of horizontal input rays



Back focal length = distance from back vertex
to back focal point

To find back focal point:

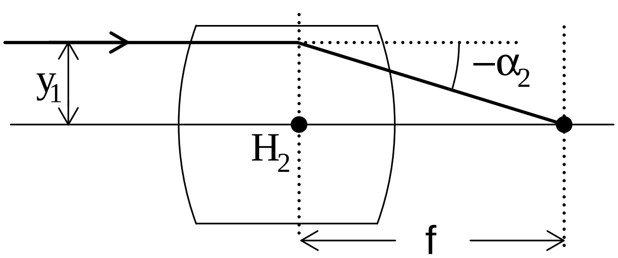
$$\begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ as + c & bs + d \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} b \\ bs + d \end{bmatrix} y_1$$

So bfl $= s = -\dfrac{d}{b}$

Intersection of rays defines back principle plane



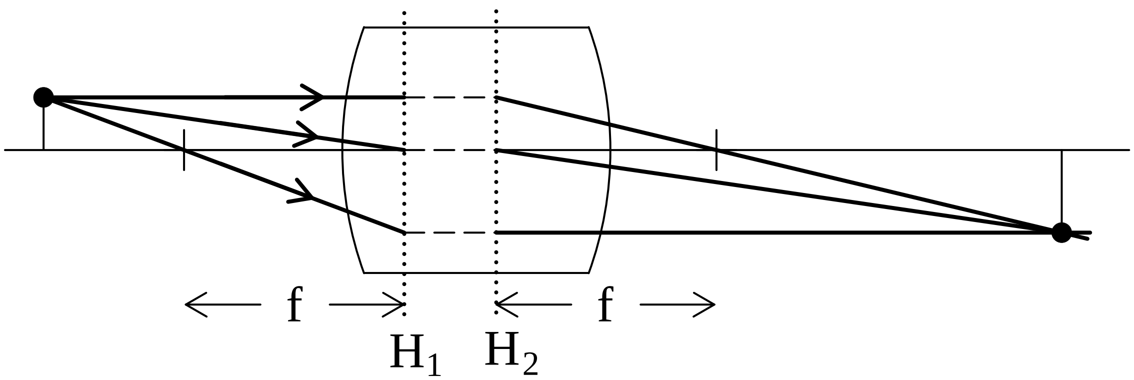
Have $y_1 = -f\alpha_2$

Again $f = -\dfrac{1}{b}$

Distance from back principle point to back vertex

$$= f - \text{bfl} = \frac{d-1}{b}$$

Gives thick lens picture



Draw diagram just like thin lens
but rays skip across between principle planes

Even get that ray aimed at H_1 exits from H_2
without deviation

Can specify system with either (f, ffl, bfl) or $\mathcal{M}_v$

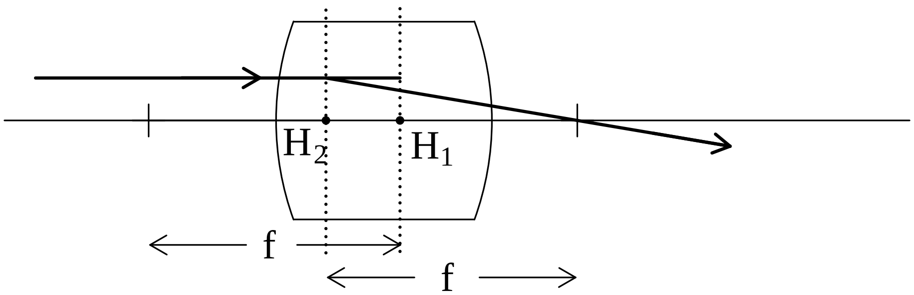
- matrix good for calculations
- focal lengths good for picture

Question: There are only three focal parameters, but four elements of $\mathcal{M}_v$. How many parameters does it take to specify system?

Note really one more parameter:
vertex-to-vertex distance

Note, still have sign convention:
if $f < 0$, then "front" focal point is behind lens

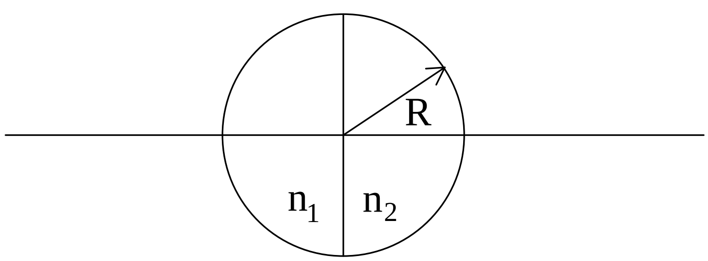
Often have order of H_1, H_2 reversed
in picture, rays skip backwards



Question: If the focal length of a system is 5 cm and the back focal length is -2 cm, where is the back principle point?

Finish with example

System: glass sphere radius $R = 1$ cm
two hemispheres: front $n_1 = 1.5$, back $n_2 = 1.7$



Want to characterize lens

First find $\mathcal{M}_v$:

$$\mathcal{M}_v = \mathcal{R}(-R, n_2 \to \text{air})\, \mathcal{T}(R, n_2)\, \mathcal{T}(R, n_1)\, \mathcal{R}(R, \text{air} \to n_1)$$

with

$$\mathcal{R}(R, n_1 \to n_2) = \begin{bmatrix} 1 & \frac{n_1 - n_2}{R} \\ 0 & 1 \end{bmatrix}$$

and

$$\mathcal{T}(d, n) = \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix}$$

Remember, can ignore plane boundaries!

Put in numbers:

$$\mathcal{M}_v = \begin{bmatrix} 1 & -0.7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.588 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.667 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{M}_v = \begin{bmatrix} 0.125 & -0.761 \text{ cm}^{-1} \\ 1.255 \text{ cm} & 0.373 \end{bmatrix}$$

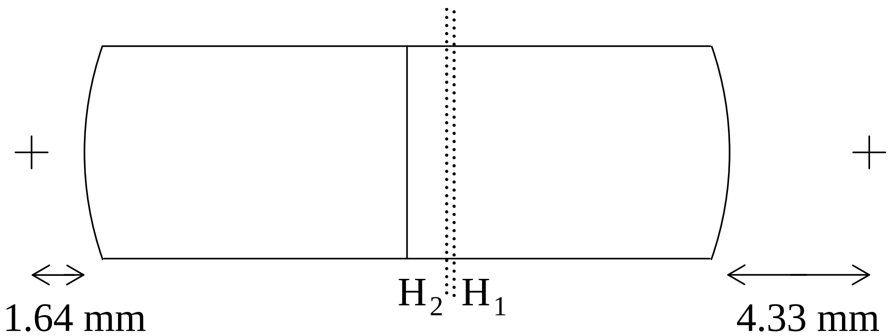
Check: $\det(\mathcal{M}_v) = 1$

Then $f = -1/b = 1.31$ cm

ffl $= -a/b = af = 0.164$ cm

bfl $= -d/b = df = 0.433$ cm

Picture



Principle planes separated by 0.2 mm
system acts almost exactly like thin lens

Summary

- Can analyze paraxial rays with matrix technique
- Arbitrary system decribed by single matrix
- Alternatively, use thick lens picture