Superposition and Interference

Last time: wrapped up ray optics

- aberrations
- practical considerations

Today: return to wave optics

Start developing tools for real calculations

Outline:

- Interference
- Interference examples
- Interference in time
- Pulse propagation: group and phase v

Next time, see how to construct pulses from frequency components

Next time: Fourier transforms

Interference (Hecht 7.1)

Consider two waves

$$E_1 = A_1 e^{i(kz-\omega t)}$$

$$E_2 = A_2 e^{i(kz-\omega t)}$$

Note: for next six lectures, ignore polarization
If you want, assume all fields polarized along $\hat{\mathbf{x}}$

Also remember really

$$E_1 = |A_1|\cos(kz - \omega t + \phi_1)$$

for $A_1 = |A_1|e^{i\phi_1}$

Total field is sum of individual waves

$$E_{\text{tot}} = E_1 + E_2 = (A_1 + A_2)e^{i(kz-\omega t)}$$

What is the real field?
Need magnitude and phase of $A_{\text{tot}} = A_1 + A_2$

Work out:

$$\begin{aligned}
|A_{\text{tot}}|^2 &= A_{\text{tot}} \cdot A^*_{\text{tot}} \\
&= (|A_1|e^{i\phi_1} + |A_2|e^{i\phi_2})(|A_1|e^{-i\phi_1} + |A_2|e^{-i\phi_2}) \\
&= |A_1|^2 + |A_2|^2 + |A_1||A_2|e^{i(\phi_1-\phi_2)} \\
&\quad + |A_1||A_2|e^{-i(\phi_1-\phi_2)} \\
&= |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos(\phi_1 - \phi_2)
\end{aligned}$$

and

$$\tan\phi_{\text{tot}} = \frac{\text{Im } A_{\text{tot}}}{\text{Re } A_{\text{tot}}}$$
$$= \frac{\text{Im } A_1 + \text{Im } A_2}{\text{Re } A_1 + \text{Re } A_2}$$
$$= \frac{|A_1|\sin\phi_1 + |A_2|\sin\phi_2}{|A_1|\cos\phi_1 + |A_2|\cos\phi_2}$$

So, can fine amplitude and phase of total wave

Usually amplitude is more interesting:

$|A_{\text{tot}}|^2 \propto I_{\text{tot}}$

$$\boxed{|A_{\text{tot}}|^2 = |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos(\phi_1 - \phi_2)}$$

Lets us be quantitative about interference:

"Constructive" interference

when $\phi_1 \approx \phi_2$

$|A_{\text{tot}}|^2 \approx (|A_1| + |A_2|)^2$

"Destructive" interference

when $\phi_1 \approx -\phi_2$

$|A_{\text{tot}}|^2 \approx (|A_1| - |A_2|)^2$

Given any $\delta \equiv \phi_1 - \phi_2$, we have formula

Note if $\delta = \pi/2$ then

$$|A_{\text{tot}}|^2 = |A_1|^2 + |A_2|^2$$

Irradiances add

Irradiances also add if δ fluctuates randomly

$$I_{\text{tot}} = \frac{1}{2\eta_0}\langle |E|^2 \rangle \propto \langle |A_{\text{tot}}|^2 \rangle$$

so interference term $\to 0$ if $\langle \cos\delta \rangle = 0$

Then say waves are *incoherent*

Question: Can two harmonic waves really be incoherent?

Generalizes to N waves:

$$A_{\text{tot}} = \sum_n A_n$$

$$\begin{aligned} |A_{\text{tot}}|^2 &= \sum_n A_n \sum_m A_m^* \\ &= \sum_{n,m} |A_n||A_m| e^{i}(\phi_n - \phi_m) \\ &= \sum_{n=1}^{N} |A_n|^2 + \sum_{n \neq m} |A_n||A_m| e^{i(\phi_n - \phi_m)} \end{aligned}$$

Second term $\to 0$ for incoherent fields

Also get

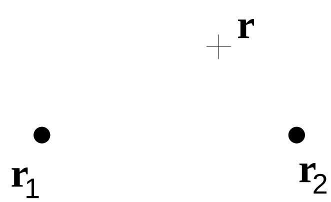
$$\tan\phi_{\text{tot}} = \frac{\sum_n |A_n| \sin\phi_n}{\sum_n |A_n| \cos\phi_n}$$

For incoherent fields, ϕ_{tot} fluctuates randomly

More about incoherent fields at end of course
generally assume coherent now

Look closer at phases

Say waves generated by two point sources



Field from 1 $= A_1 e^{i(k|\mathbf{r}-\mathbf{r}_1|-\omega t)}$

Field from 2 $= A_2 e^{i(k|\mathbf{r}-\mathbf{r}_2|-\omega t)}$

Total field $|E_{\text{tot}}|^2 = |E_1 + E_2|^2 = |A_1|^2 + |A_2|^2$
$+2|A_1||A_2| \cos(\phi_1 - \phi_2 + k|\mathbf{r}-\mathbf{r}_1| - k|\mathbf{r}-\mathbf{r}_2|)$

Interference term depends on position:

get *interference pattern* in space

In medium, $k = nk_o$

Optical path length difference $\Lambda = \mathcal{S}_1 - \mathcal{S}_2$

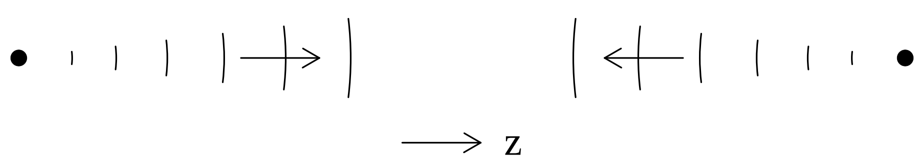
$$\Lambda = n(|\mathbf{r}_1 - \mathbf{r}| - |\mathbf{r}_2 - \mathbf{r}|)$$

Write interference phase $\delta = \phi_1 - \phi_2 + k_0\Lambda(\mathbf{r})$

Like in scattering picture:

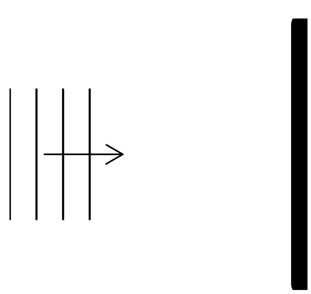
$\mathcal{S}$ determines interference effects

Example: Two sources at ∞ (Hecht 7.1.4)



Then fields $\rightarrow$ plane waves

Equivalent to plane wave reflected by mirror



What is E_{tot}?

$$E_{\text{tot}} = A_1 e^{i(kz-\omega t)} + A_2 e^{i(-kz-\omega t)}$$

Suppose $A_1 = A_2$

Then

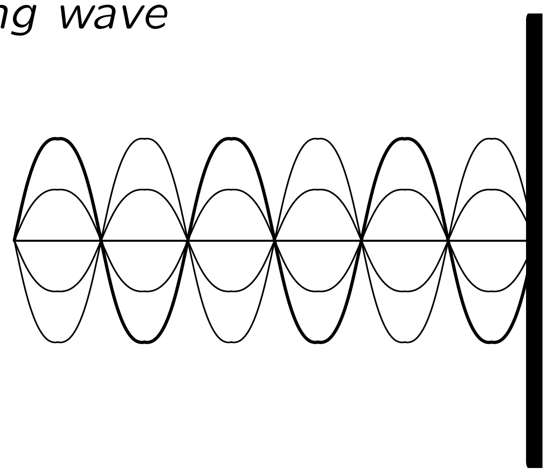
$$\begin{aligned} E_{\text{tot}} &= A_1 e^{-i\omega t}(e^{ikz} + e^{-ikz}) \\ &= 2A_1 e^{-i\omega t} \cos(kz) \end{aligned}$$

So $|E_{\text{tot}}| = 2|A_1||\cos kz|$

and real field $E_{\text{tot}} = 2|A_1| \cos(\phi_1 - \omega t) \cos(kz)$

Not a travelling wave!

Called *standing wave*



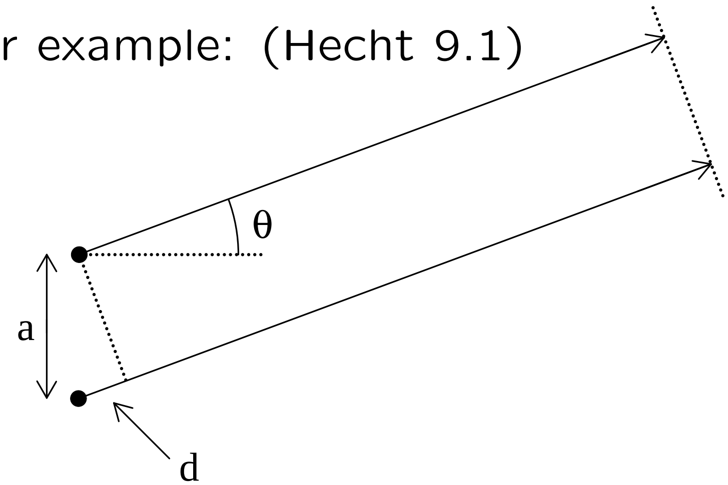
Fixed pattern oscillating in time

Nodes: points where $E_{\text{tot}} = 0$

Antinodes: points where $|E_{\text{tot}}| = \max = 2|A_1|$

Standing wave is one interference pattern

Another example: (Hecht 9.1)



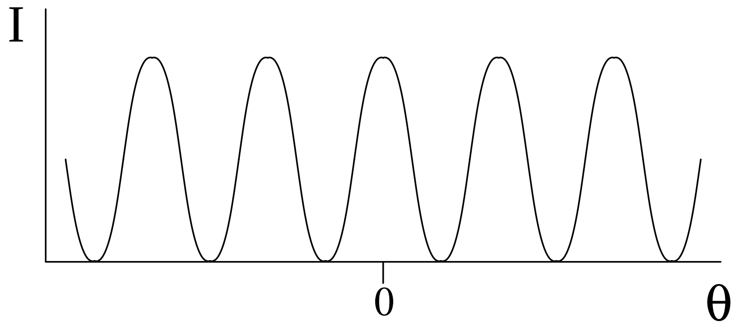
Two point sources separated by a

Observation point $\to \infty$ at angle θ

Here $\Lambda = d = a \sin\theta \approx a\theta$

If $A_1 = A_2$, then

$$
\begin{aligned}
|E_{\text{tot}}|^2 &= |A_1|^2 + |A_2|^2 + 2|A_1||A_2| \cos(\phi_1 - \phi_2 + k_0\Lambda) \\
&= 2|A_1|^2(1 + \cos k_0\Lambda) \\
&= 4|A_1| \cos^2\left(\frac{k_0\Lambda}{2}\right) \\
&\approx 4|A_1| \cos^2\left(\frac{k_0 a\theta}{2}\right)
\end{aligned}
$$



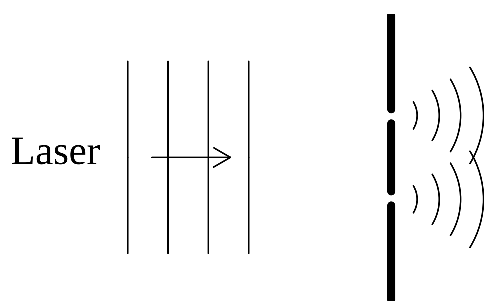
Nodes where $k_0\Lambda/2 = (n + \frac{1}{2})\pi$

$$k_0 a\theta = \left(n + \frac{1}{2}\right) 2\pi$$

or $\theta \approx \dfrac{\lambda}{a}\left(n + \dfrac{1}{2}\right)$ since $k_0 = \dfrac{2\pi}{\lambda}$

Demo: two slit interference

Setup:



Due to diffraction, slits ≈ point sources

For large θ, more complicated:

Can't use $\sin\theta \approx \theta$

Can't treat slit as point source

For now, gives idea

Basic points:

- Calculate field at position $\mathbf{r}$ by algebraically summing incident waves

- Phase of each field depends on
  - source phase
  - optical path length between source and $\mathbf{r}$

Question: If the phase of the two point sources were different, how would the two-slit pattern change?

Question: What if the amplitudes of the sources were different?

Temporal Interference (Hecht 7.2)

Can also get interference in time

Say $E_1 = A_1 e^{i(k_1 z - \omega_1 t)}$

$E_2 = A_2 e^{i(k_2 z - \omega_2 t)}$

with $k_i = \dfrac{n(\omega_i)\omega_i}{c}$ and $\omega_1 \neq \omega_2$

What is total field?

$$E_{\text{tot}} = E_1 + E_2$$

Define $\alpha_i = k_i z - \omega_i t + \phi_i$

$$|E_{\text{tot}}|^2 = \left(|A_1|e^{i\alpha_1} + |A_2|e^{i\alpha_2}\right)\left(|A_1|e^{-i\alpha_1} + |A_2|e^{-i\alpha_2}\right)$$
$$= |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos(\alpha_1 - \alpha_2)$$

Intererence term =
$$\cos\left[(k_1 - k_2)z + (\omega_1 - \omega_2)t + \phi_1 - \phi_2\right]$$

At fixed z, irradiance oscillates at $\omega_1 - \omega_2$
$\equiv$ *beat note*

Identical effect with sound waves – demo

Often useful to think of E_{tot} as single plane wave
with changing amplitude

Say $|A_1| = |A_2|$:

$$E_{\text{tot}} = |A_1|e^{i(k_1 z - \omega_1 t + \phi_1)} + |A_1|e^{i(k_2 z - \omega_2 t + \phi_2)}$$

Define $\bar{\omega} = \dfrac{\omega_1 + \omega_2}{2}$ and $\omega_m = \omega_1 - \omega_2$

Then $\omega_1 = \bar{\omega} + \dfrac{\omega_m}{2}$
$\omega_2 = \bar{\omega} - \dfrac{\omega_m}{2}$

Define similarly for $\bar{k}$, k_m, $\bar{\phi}$, ϕ_m

Then write

$$E_{\text{tot}} = |A_1| e^{i(\bar{k}z - \bar{\omega}t + \bar{\phi})} \times \left[e^{\frac{i}{2}(k_m z - \omega_m t + \phi_m)} + e^{-\frac{i}{2}(k_m z - \omega_m t + \phi_m)} \right]$$
$$= 2|A_1| e^{i(\bar{k}z - \bar{\omega}t + \bar{\phi})} \cos\left(\frac{k_m z - \omega_m t + \phi_m}{2}\right)$$

Looks like plane wave $e^{i(\bar{k}z - \bar{\omega}t)}$ with amplitude

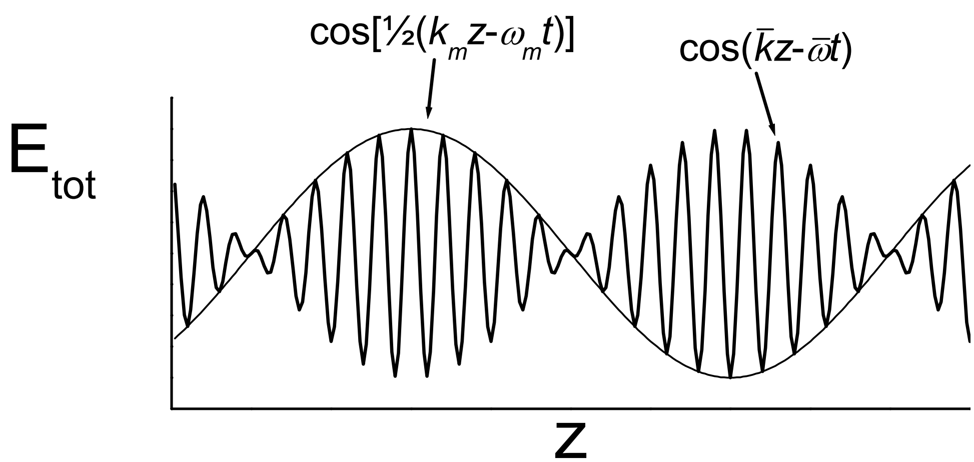
$$A_{\text{tot}}(z,t) = 2|A_1| e^{i\bar{\phi}} \cos\left(\frac{k_m z - \omega_m t + \phi_m}{2}\right)$$

Most useful if $\omega_m \ll \bar{\omega}$

Then A_{tot} slowly varying

Field $\approx$ plane wave at any z, t

Picture (at fixed t):



This is example of a *modulated* wave

like a plane wave,

but amplitude and/or phase slowly varies

Call $e^{i(\bar{k}z-\bar{\omega}t)}$ = carrier wave

Call $A_{\text{tot}}(z,t)$ = envelope function
"contains" carrier wave

Might ask, how fast does wave propagate?
Important: envelope used to transmit information

Define *phase velocity*
= speed of carrier wave
= speed of individual peaks

$$v_{\text{phase}} = \frac{\bar{\omega}}{\bar{k}} = c\,\frac{\omega_1+\omega_2}{n_1\omega_1+n_2\omega_2}$$

If $\omega_1 \approx \omega_2$ then $n_1 \approx n_2$
and $v_{\text{phase}} \to c/n$

Makes sense

Define *group velocity* = speed of envelope

$$v_{\text{group}} = \frac{\omega_m}{k_m} = \frac{\omega_1-\omega_2}{k_1-k_2} = c\,\frac{\omega_1-\omega_2}{n_1\omega_1-n_2\omega_2}$$

If $\omega_m \ll \bar{\omega}$, then get

$$v_{\text{group}} \to \left.\frac{d\omega}{dk}\right|_{\bar{\omega}} \neq v_{\text{phase}}$$

So envelope travels at entirely different speed

Generalizes to any modulated wave:
Complicated envelopes =
more frequency components

$\Rightarrow$ single pulse of light travels at v_{group}

Usually easiest to evaluate

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left(\frac{n\omega}{c}\right) = \frac{1}{c}\left(\frac{dn}{d\omega}\bar{\omega} + n\right)$$

So $v_g = \dfrac{c}{n + \bar{\omega}\frac{dn}{d\omega}}$

In vacuum, $n = 1$ and $dn/d\omega = 0$:
so $v_{\text{group}} = v_{\text{phase}} = c$

Only matters in medium

Recall for normal dispersion, $dn/d\omega > 0$

Then $v_{\text{group}} < \dfrac{c}{n} = v_{\text{phase}}$

Example: In BK7 glass, $n(\lambda = 546 \text{ nm}) = 1.51872$, and $n(588 \text{ nm}) = 1.5168$. What is the difference between the group and phase velocities near these wavelengths?

Solution:
Average $n = 1.5178$, so phase velocity $\approx 0.6588c$.

For v_g, need $\omega\dfrac{dn}{d\omega} = \omega\dfrac{dn}{d\lambda}\dfrac{d\lambda}{d\omega}$

Have $\dfrac{dn}{d\lambda} = \dfrac{1.51872 - 1.5168}{546 - 588} = -4.6 \times 10^{-5}\ \text{nm}^{-1}$

Also $\lambda = \dfrac{2\pi c}{\omega}$, so $\omega\dfrac{d\lambda}{d\omega} = -\omega\dfrac{2\pi c}{\omega^2} = -\lambda$. Use $\bar{\lambda} = 567$ nm

So $\omega\dfrac{dn}{d\omega} = -\lambda\dfrac{dn}{d\lambda} = 576\ \text{nm} \times 4.6 \times 10^{-5}\ \text{nm}^{-1} = 0.0265$

$$v_g = \frac{c}{n + \omega\frac{dn}{d\omega}} = \frac{c}{1.5178 + 0.0265} = 0.6475c$$

Difference $\boxed{v_p - v_g = 0.011c \approx 2\%}$

Usually a small effect

But not always! Near resonance, $|\omega\frac{dn}{d\omega}| \gg 1$

Generally, v_{phase} and v_{group} both arbitrary

Can get $n < 1$ near resonance $\Rightarrow v_p > c$

Get $v_g > c$ for large anomalous dispersion

Didn't worry about $v_p > c$:

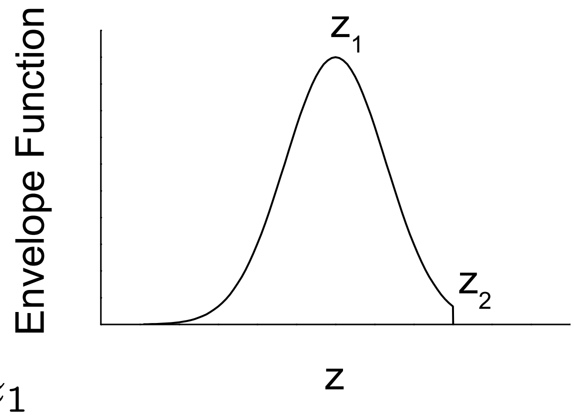
carrier has no information

Should we worry about $v_g > c$?

Remember, in scattering picture fields travel at c
$\rightarrow c$ remains ultimate speed limit

What does $v_g > c$ mean?

Consider pulse of light:



Two features:

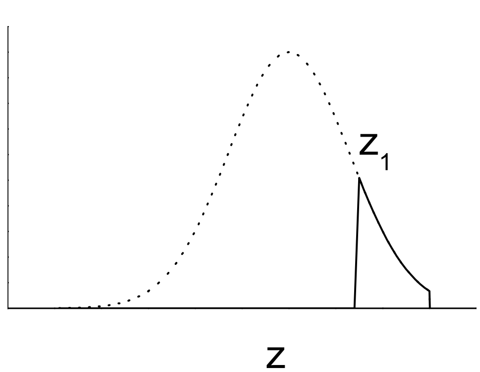
- Peak at some position z_1
- Leading edge at position z_2

Leading edge indicates time you initiated pulse:
physical discontinuity

Group velocity gives speed of peak
Can be larger than c

Simple example:

Suppose medium transmits for time $\ll \Delta t_{\text{pulse}}$
Only front part of pulse transmitted



Peak z_1 has advanced:
$v_g > c$

However, can show that leading edge *always* travels at c

Why? Property of discontinuity

No definite carrier $\bar{\omega}$, and ω_m not small simple analysis fails

More: see Jackson §7.11

Question: Can there really be a discontinuity in the electric field? Aren't Maxwell's equations continuous?

Summary

- Compute total field by summing components
- Result generally depends on position: interference pattern
- If waves have different ω, interference depends on time
- Pulse of light travels at v_{group}
- Information always travels $v \leq c$