Fourier Transform

Last time, looked at how waves add

Spatial variations: interference pattern

Time variations: beat note

Claimed that you could construct arbitrary pulse by adding fields with different ω's

Today, show how: Fourier transform

Outline:

- Motivation
- Definition
- Transform properties
- Spatial transforms

Lots of math today

Next time:

- Apply Fourier methods to wave propagation
- Start working on diffraction

Motivation

Lecture 1:
Claimed any wave = sum of plane waves

For now, show that:
Any function of time $f(t)$
$= $ sum of harmonic functions $e^{-i\omega t}$

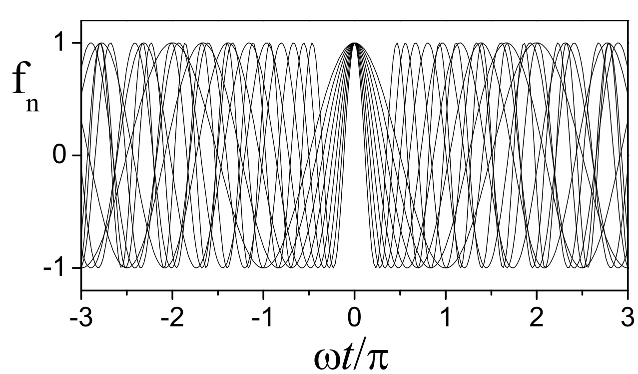
More general: *any* function
More specific: single variable
imagine $f(t) = E(\mathbf{r}, t)$ at fixed $\mathbf{r}$

Talk about full waves again at end

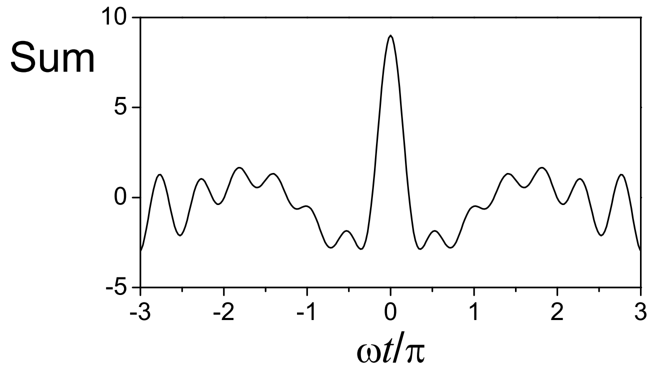
Why should $f(t) =$ sum of $e^{i\omega t}$'s?
$=$ sum of sines and cosines?

Make components add constructively where f large
destructively where f small

Example: add $f_n = \cos[(1.2)^n \omega t]$ for $n = 1$ to 9:



Sum gives peak at $t = 0$:



More cosines → sharper peak, flatter background

If you can make sharp peaks:
any $f(t)$ = sum of peaks at different t's

Fourier Transform (Hecht 7.3, 7.4, 11.1)

Most general sum = integral

Can write $f(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$

$F(\omega)$ = coefficients of sum
$1/2\pi$ = normalizing factor

Fine, but how to determine $F(\omega)$?

Basic Fourier trick:
multiply both sides by $e^{i\beta t}$ and integrate over t

$$\int_{-\infty}^{\infty} e^{i\beta t} f(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\beta t} F(\omega) e^{-i\omega t} d\omega \, dt$$

Change order of integrals on rhs:

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left(\int_{-\infty}^{\infty} e^{i(\beta-\omega)t} dt \right) d\omega$$
$$= \int_{-\infty}^{\infty} F(\omega) \delta(\beta - \omega) d\omega$$

for

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} dt \equiv \text{“delta function”}$$

Consider $\delta(\omega)$ (Hecht 11.2.3)

If $\omega \neq 0$, then $e^{i\omega t}$ oscillates +/-

So $\int e^{i\omega t} dt$ averages to zero:
Expect $\delta(\omega) = 0$ for $\omega \neq 0$

But for $\omega = 0$, $e^{i\omega t} = e^0 = 1$

So $\int_{-\infty}^{\infty} e^{i\omega t} dt \to \int_{-\infty}^{\infty} dt = \infty$

Like adding up infinite number of cosines:
get infinitely high, infinitely narrow peak

Important property:

$$\int_{-\infty}^{\infty} \delta(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega t} dt\, d\omega = 1$$

delta function is normalized

Derived in handout

Go back to

$$\int_{-\infty}^{\infty} e^{i\beta t} f(t) dt = \int_{-\infty}^{\infty} F(\omega)\delta(\beta - \omega) d\omega$$

delta function peaked at $\omega = \beta$, zero elsewhere

At $\omega = \beta$, have $F(\omega) = F(\beta)$

So have

$$\int_{-\infty}^{\infty} e^{i\beta t} f(t) dt = F(\beta) \int_{-\infty}^{\infty} \delta(\beta - \omega) d\omega = F(\beta)$$

Usually write

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt}$$

Call F = *Fourier transform* of f

Then f = *inverse Fourier transform* of F:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega}$$

Other definitions possible:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

or $F(\nu) = \int_{-\infty}^{\infty} f(t) e^{i2\pi\nu t} dt$

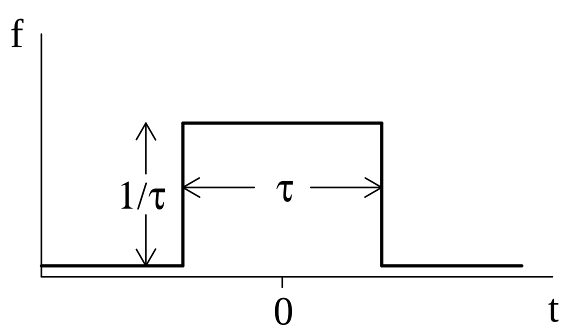
$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{-i2\pi\nu t} d\nu$$

$\nu = \omega/2\pi$ = frequency in Hz

Our version: all ω integrals have $1/2\pi$ factor

Do an example:

Say $f(t) = 1/\tau$ if $-\frac{\tau}{2} < t < \frac{\tau}{2}$
$= 0$ otherwise



Normalized to 1

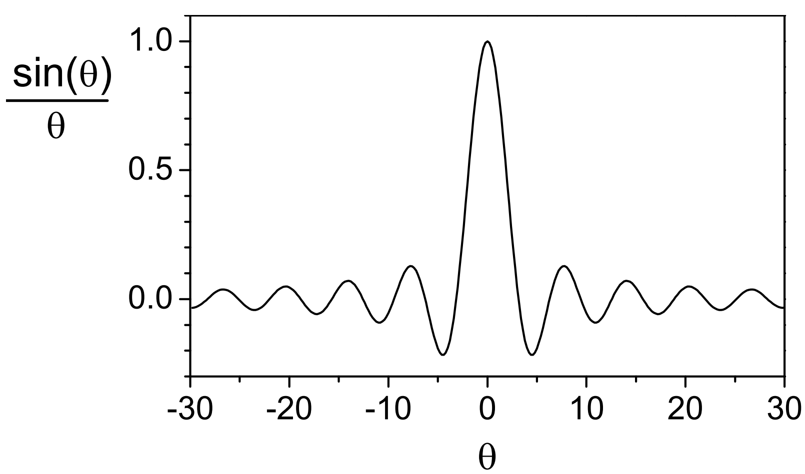
Calculate $F(\omega)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} e^{i\omega t} dt$$

$$= \frac{1}{i\omega\tau} \left(e^{i\frac{\omega\tau}{2}} - e^{-i\frac{\omega\tau}{2}} \right)$$

$$= \frac{2}{\omega\tau} \sin\left(\frac{\omega\tau}{2}\right)$$



Define $\sin(\theta)/\theta \equiv \text{sinc}\,\theta$

Have $\text{sinc}(0) = 1$ (peak value)

$\text{sinc}(n\pi) = 0$ (integer $n \neq 0$)

Peak width $\Delta\theta \approx 2\pi$

So $F(\omega) = \text{sinc}\left(\frac{\omega\tau}{2}\right)$

Peaked at $\omega = 0$

Width $\Delta\omega = \pi/\tau$

General feature:
width $\Delta\omega$ of $F(\omega)$ larger
when width Δt of $f(t)$ is smaller

Can show $\Delta\omega\Delta t \geq 1/2$
(for particular definition of widths)

Need high frequencies if f changes quickly
always expect $\omega_{\text{max}} \approx 1/\delta t$
$\delta t =$ time scale for $f(t)$ to change

For rectangular pulse, $\delta t \to 0$

See $F(\omega)$ decreases slowly $\propto \omega^{-1}$ for $\omega \to \infty$
no definite ω_{max}

Question: If we set $F(\omega) = 0$ for $|\omega|$ greater than some ω_{max}, how would $f(t)$ change?

Properties of Fourier Transform

(Hecht 11.2, handout)

A. Even for real $f(t)$, $F(\omega)$ can be complex

$$F(\omega) = \int f(t) e^{i\omega t} dt$$

$$F^*(\omega) = \int f(t) e^{-i\omega t} dt$$

So $F - F^* = \int f(t) \left(e^{i\omega t} - e^{-i\omega t} \right) dt$

$$= 2i \int f(t) \sin(\omega t) dt$$

$= 0$ only if $f(t) = f(-t)$

Why is F complex?
Because we defined F with complex exponentials

Also explains why we get $\omega < 0$ terms:
in complex space $\omega < 0$ and $\omega > 0$ are different

If f real, then $F(-\omega) = F^*(\omega)$
all information in $\omega > 0$ terms

Fits well with complex representation of fields:
we're just suppressing $\omega < 0$ components

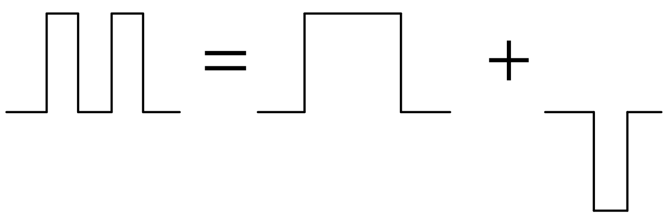
B. Linearity

If $f(t) = af_1(t) + bf_2(t)$ then

$$F(\omega) = aF_1(\omega) + bF_2(\omega)$$

where F_1 = transform of f_1
F_2 = transform of f_2

Very useful:
Often complicated f = sum of simple f's

Example: 

Say pulses width T, height = A
Gap width = T

Remember sinc($\omega\tau/2$) =
transform of pulse width τ, height $1/\tau$

First pulse width $\tau = 3T$, second pulse $\tau = T$
Adjust amplitudes of F accordingly

Then $F(\omega) = 3AT\,\text{sinc}\left(\frac{3\omega T}{2}\right) - AT\,\text{sinc}\left(\frac{\omega T}{2}\right)$

C. Translation Properties

If $f(t) = g(t+\tau)$ then

$$F(\omega) = e^{-i\omega\tau} G(\omega)$$

where G = transform of g

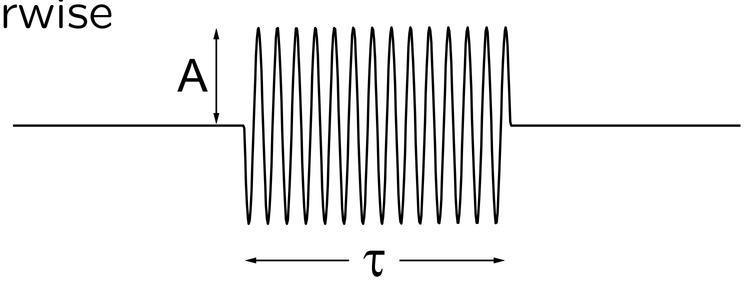
If $f(t) = e^{-i\omega_0 t} g(t)$ then

$$F(\omega) = G(\omega - \omega_0)$$

Also useful for obtaining new transforms

Example: pulsed harmonic signal

$$f = Ae^{-i\omega_0 t} \text{ for } |t| < \tau/2$$
$$= 0 \text{ otherwise}$$



Then $F(\omega) = A\tau \,\text{sinc}\left[\frac{(\omega - \omega_0)\tau}{2}\right]$

Peak in ω space centered at ω_0

Question: What would the transform look like if $f = \cos(\omega_0 t)$ for $|t| < \tau/2$ and $f = 0$ otherwise?

D. Convolution

If $F(\omega) = F_1(\omega)F_2(\omega)$, then

$$f(t) = \int_{-\infty}^{\infty} f_1(T) f_2(t-T) dT$$

where $f_1, f_2 =$ inverse transforms of F_1, F_2

Say that $f =$ *convolution* of f_1 and f_2

Lets you modify $F(\omega)$ and understand result

Example: $f_1(t) = 1/\tau$ if $(|t| < \tau/2)$

Then $F_1(\omega) = \mathrm{sinc}\,(\omega\tau/2)$

Multiply F_1 by F_2

$F_2(\omega) = 1$ if $|\omega| < \omega_{\mathrm{max}}$
$F_2(\omega) = 0$ otherwise

Chops off high frequencies, as before

What is $f_2(t)$?

$$\begin{aligned} f_2(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(\omega) e^{-i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_m}^{\omega_m} e^{-i\omega t} d\omega \\ &= \frac{1}{\pi t} \sin(\omega_m t) = \frac{\omega_m}{\pi} \mathrm{sinc}\, \omega_m t \end{aligned}$$

(Form of f and F interchangable)

So we get

$$f(t) = \int_{-\infty}^{\infty} f_1(T) f_2(t-T) dT$$

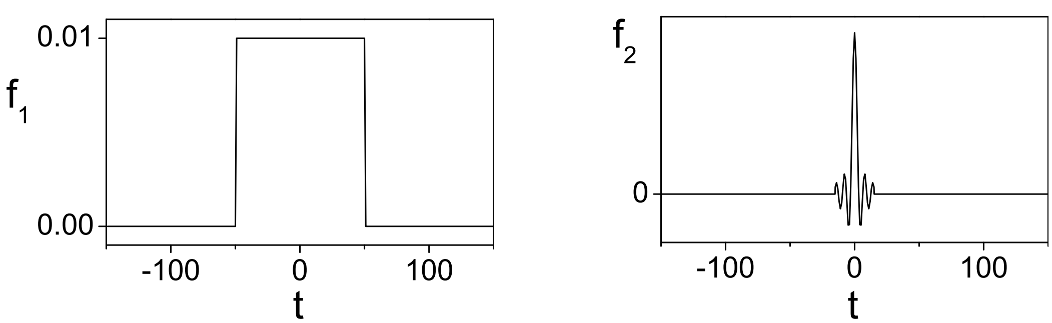
If ω_m is large then f_2 is sharp peak

- only large for $|\omega_m(t-T)| < \pi$

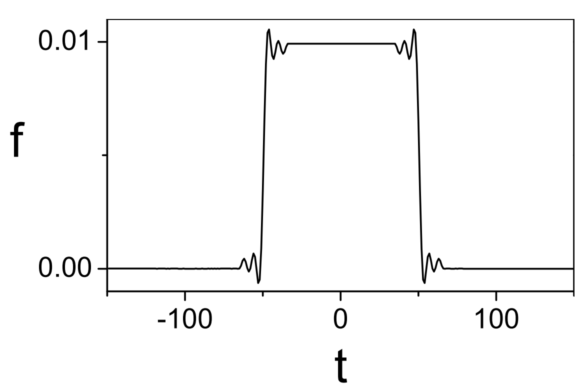
So need T pretty close to t:

$$f(t) \approx f_1(t)$$

But edges of pulse "blurred" like sinc



Gives:



E. Correlation

Suppose $f(t) = \int_{-\infty}^{\infty} f_1^*(T) f_2(t+T) dT$

Say f = *correlation* of f_1 and f_2

Fancy way to compare two functions
- we'll use later

Obtain $F(\omega) = F_1^*(\omega) F_2(\omega)$

Similar to convolution result

F. Parseval's Theorem

If $F(\omega)$ is transform of $f(t)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

For wave, $\int |f(t)|^2 dt \propto$ total energy in wave

Interpret $|F(\omega)|^2 d\omega \propto$ energy in frequency band $d\omega$

Can measure:
Send light pulse through dispersing prism
separates colors = ω components

Intensity of each color $\propto |F(\omega)|^2$

List of transforms

$f(t)$	$F(\omega)$
$\frac{1}{\tau} \quad \left(\lvert t\rvert < \frac{\tau}{2}\right)$	$\text{sinc}\left(\frac{\omega\tau}{2}\right)$
$e^{-i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\delta(t-t_0)$	$e^{i\omega t_0}$
$\frac{1}{\tau\sqrt{\pi}}\, e^{-t^2/\tau^2}$	$e^{-\omega^2\tau^2/4}$

Use with linearity and scaling properties:
gives most of what we need

Spatial transforms

If $f(z)$ is function of spatial coordinate

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikz}dk$$

$$F(k) = \int_{-\infty}^{\infty} f(z)e^{-ikz}dz$$

So (z,k) like (t,ω): everything works the same

Question: My definition of $F(\omega)$ had $e^{i\omega t}$. Why did I change the sign for $F(k)$?

For 3D functions, need 3D transform:

$$f(\mathbf{r}) = \frac{1}{(2\pi)^3} \iiint F(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3k$$

$$F(\mathbf{k}) = \iiint f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r$$

integrals over all space

Same ideas, sometimes integrals are harder

We'll see one example later

For instance:

$|F(\mathbf{k})|^2$ = energy density at wave vector $\mathbf{k}$

What about space and time together?

Write

$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int F(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} d^3k \, d\omega$$

$$F(\mathbf{k}, \omega) = \int f(\mathbf{r}, t) e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} d^3r \, dt$$

Works for any function f

Can write any function as sum of plane waves!

as advertised in Lecture 1

What about waves?

Say electric field $E(\mathbf{r}, t)$

Write transform as $\mathcal{E}(\mathbf{k}, \omega)$

$$E(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int \mathcal{E}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} d^3k\, d\omega$$

But, if E is solution of wave equation, need

$$k^2 = \frac{n^2\omega^2}{c^2}$$

Not all functions are waves

Then ω and $\mathbf{k}$ aren't independent

Really only three variables: use $\mathbf{k}$
then $\omega = \omega(\mathbf{k}) \equiv \omega_k$

Then if

$$\mathcal{E}(\mathbf{k}) = \int E(\mathbf{r}, 0) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r$$

get $E(\mathbf{r}, t) = \dfrac{1}{(2\pi)^3} \displaystyle\int \mathcal{E}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} d^3k$

Gives wave at all times in terms of $E(\mathbf{r}, t = 0)$

Question: What if $E(t = 0)$ is zero everywhere, and at some later time I turn on a source?

Summary:

- Fourier transform lets you express functions as sum of harmonic functions
- Evaluate transform by doing integral
- Covered several important properties
- Can do transforms in space and/or time
- For waves, space and time dependence related