Retarders and the Jones Calculus

Last time, discussed how polarizers work:

- transmit one polarization
- blocks orthogonal polarization

Described ways to implement
Discussed birefringence

Today:
Discuss retarders
Develop computational tools

Outline:

- Review polarizers
- Retarders
- Matrix technique
- Partially polarized light

Finish polarization

Next time: Interferometers

Polarizers

Ideal polarizer has transmission axis $\mathbf{a}$

transmits light with $\hat{\jmath} = \mathbf{a}$

blocks light with $\hat{\jmath} \perp \mathbf{a}$

If $\mathbf{a}$ complex, transmit circ or ellip polarization

no true "axis"

If $\hat{\jmath}$ neither $\parallel \mathbf{a}$ nor $\perp \mathbf{a}$

use $T = |\hat{\jmath}^* \cdot \mathbf{a}|^2$

Output light is polarized along $\mathbf{a}$

Retarders (Hecht 8.7)

Use polarizers to make linear polarized light

What about circular or elliptical?

Use *retarder*

Most common retarder = wave plate

Based on birefringence

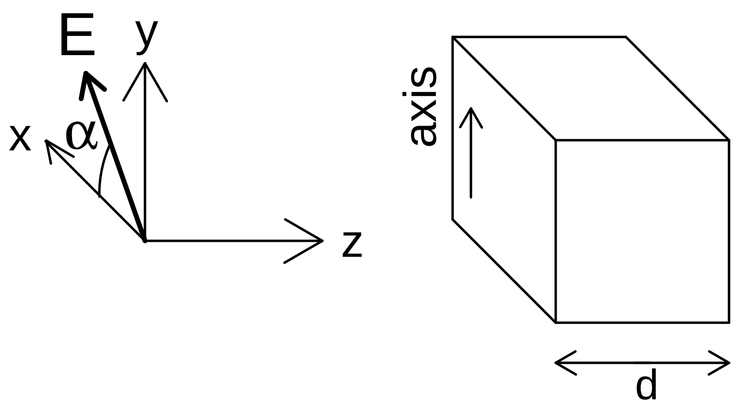
Light polarized along optic axis: index n_e

Light polarized $\perp$ axis: index n_o

Suppose linear polarized light, angle α

Incident on uniaxial crystal with axis vertical

Crystal thickness d



Coordinates as shown: $z = 0$ at front of crystal

Incident wave has $\hat{\jmath} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}}$

$$\mathbf{E}_{\text{inc}} = E_0 \hat{\jmath} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

In crystal $k \to nk_0 = n\omega/c$

different for E_x and E_y components

So

$$\mathbf{E}_{\text{crystal}} = E_0 \left(\cos\alpha\, e^{in_o k_0 z}\,\hat{\mathbf{x}} + \sin\alpha\, e^{in_e k_0 z}\,\hat{\mathbf{y}} \right) e^{-i\omega t}$$

At output $z = d$:

$$\mathbf{E}_d = E_0 \left(\cos\alpha\, e^{in_o k_0 d}\,\hat{\mathbf{x}} + \sin\alpha\, e^{in_e k_0 d}\,\hat{\mathbf{y}} \right) e^{-i\omega t}$$

$$= E_0 e^{in_o k_0 d} \left[\cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\, e^{i(n_e - n_o)k_0 d}\,\hat{\mathbf{y}} \right] e^{-i\omega t}$$

Define $\hat{\jmath}_{\text{out}} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,e^{i(n_e-n_o)k_0 d}\,\hat{\mathbf{y}}$

Then after crystal, have

$$
\begin{aligned}
E(z,t) &= E_0 e^{in_o k_0 d}\,\hat{\jmath}_{\text{out}}\,e^{i[k(z-d)-\omega t]} \\
&= E_0 e^{i(n_o-1)k_0 d}\,\hat{\jmath}_{\text{out}}\,e^{i(kz-\omega t)} \\
&= E_0'\,\hat{\jmath}_{\text{out}}\,e^{i(kz-\omega t)}
\end{aligned}
$$

Get plane wave out with

$$\hat{\jmath} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,e^{i\varepsilon}\,\hat{\mathbf{y}}$$

and $\boxed{\varepsilon = (n_e - n_o)k_0 d} \equiv$ *retardance*

Set d to achieve desired ε value

By adjusting α and ε,
make arbitrary polarization state

Example: $kd(n_e - n_o) = \pi/2$
Then $d(n_e - n_o) = \lambda/4$: call *quarter-wave plate*

Get $\hat{\jmath}_{\text{out}} = \cos\alpha\,\hat{\mathbf{x}} + e^{i\pi/2}\sin\alpha\,\hat{\mathbf{y}}$

$$= \cos\alpha\,\hat{\mathbf{x}} + i\sin\alpha\,\hat{\mathbf{y}}$$

For $\alpha = \pm 45^\circ$, $\hat{\jmath}_{\text{out}} = \dfrac{\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}}{\sqrt{2}}$

Make LHC and RHC polarizations

Main use of quarter-wave plate: convert linear to circular polarization

Other common configuration: $\varepsilon = \pi$

Then $d(n_e - n_o) = \lambda/2$: *half-wave plate*

Have $e^{i\pi} = -1$ so

$$\hat{\jmath}_{\text{out}} = \cos\alpha\,\hat{\mathbf{x}} - \sin\alpha\,\hat{\mathbf{y}}$$

Changes polarization angle $\alpha \to -\alpha$

If $\alpha_{\text{in}} = 45^\circ$, then $\alpha_{\text{out}} = -45^\circ$

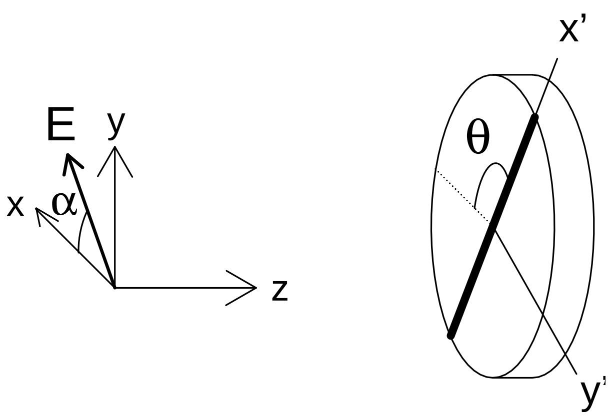
Rotated by 90°: orthogonal to input

Place between crossed polarizers,
get 100% transmission

Generally, use half-wave plate at arb angle θ
how to calculate effect?

$\Rightarrow$ need to express input in crystal basis

Define x' = axis of waveplate



Then

$$\hat{\mathbf{x}}' = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{y}}' = \cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{x}}$$

and $\hat{\jmath} = \cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}}$

Calculate $j_{x'} = \hat{\mathbf{x}}' \cdot \hat{\jmath}$

$$= \cos\alpha\cos\theta - \sin\alpha\sin\theta = \cos(\alpha - \theta)$$

and $j_{y'} = \hat{\mathbf{y}}' \cdot \hat{\jmath}$

$$= -\cos\alpha\sin\theta + \sin\alpha\cos\theta = \sin(\alpha - \theta)$$

So: $\hat{\jmath} = \cos\alpha'\,\hat{\mathbf{x}}' + \sin\alpha'\,\hat{\mathbf{y}}'$

for $\alpha' = \alpha - \theta$

(Or just use x'-y' coords from beginning)

Effect of half wave plate is $\alpha' \to -\alpha'$

So $\alpha_{\text{out}} - \theta = -(\alpha_{\text{in}} - \theta)$

$$\alpha_{\text{out}} = 2\theta - \alpha_{\text{in}}$$

Change θ by $\Delta\theta$, output polarization rotates by $2\Delta\theta$

Main use of half-wave plate: rotate linear polarization by arbitrary angle

Question: What happens when $\theta = \alpha_{\text{in}}$? What is physically happening in this situation?

One problem: waveplates very thin

Quarter wave plate: $d = \lambda/4(n_e - n_o) \sim \lambda$

Often make $\varepsilon = \pi/4 + 2\pi m$ for integer m:
get same effect

Called *multiple order waveplate*
typical $m \approx 10$

Most common waveplate material: quartz
cost $200 for 1-cm diameter plate

Cheaper: plastic ($5 for 5 cm square)
- ε less accurate
- distorts laser beams

Terminology:

- Fast axis: axis of wave plate with lower n
- Slow axis: axis of wave plate with higher n

Doesn't really matter which is optical axis
depends on whether $n_o > n_e$ ("negative" crystal)
or $n_o > n_e$ ("positive" crystal)

Other ways to make retarders (Hecht pg 356-357):
- Fresnel rhomb: use phase shift from TIR
- Babinet-Soliel compensator:
waveplate with variable thickness

Jones Calculus (Hecht 8.13.2-3)

Polarizers and retarders simple in proper basis

Polarizer: pass one component, block other

Retarder: add phase shift to one component

Always good to choose basis carefully

But with multiple elements, no one basis is simple

Half-wave plate example illustrated how to deal:

change basis for each element

Easy way to implement: use matrix notation

Basic idea: $\hat{\jmath}$ is a vector

write $\hat{\jmath} = \begin{bmatrix} j_x \\ j_y \end{bmatrix}$

Polarizers and retarders are linear elements

write as matrices $\hat{\jmath}_{\mathrm{out}} = \mathcal{U}\hat{\jmath}_{\mathrm{in}}$

$\mathcal{U} = 2 \times 2$ matrix

Hecht writes matrix as $\mathcal{A}$

I'll use $\mathcal{U}$

Examples:

Matrix for polarizer with $\mathbf{a} = \hat{\mathbf{x}}$ is

$$\mathcal{U} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Arbitrary incident polarization:

$$\hat{\jmath}_{\text{in}} = \begin{bmatrix} \cos\alpha \\ e^{i\varepsilon}\sin\alpha \end{bmatrix}$$

So $$\hat{\jmath}_{\text{out}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ e^{i\varepsilon}\sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ 0 \end{bmatrix}$$

See $\hat{\jmath}_{\text{out}} \parallel \mathbf{a}$ and $|\hat{\jmath}_{\text{out}}|^2 = |\hat{\jmath}^*_{\text{in}} \cdot \mathbf{a}|^2$
as required

Not really any different than previous calcs
Just written differently:
Matrix is taking $\hat{\jmath} \cdot \mathbf{a}$ for you

Question: What would $\mathcal{U}$ be for polarizer with transmission axis along $\hat{\mathbf{y}}$?

One issue: $\hat{\jmath}$ is supposed to be unit vector

Really have $t\,\hat{\jmath}_{\text{out}} = \mathcal{U}\hat{\jmath}_{\text{in}} = \cos\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

where t = amplitude transmission coeff

$E_{0\text{out}} = tE_{0\text{in}}$

Then intensity transmittance $T = |t|^2$

Easier to write $\hat{\jmath}_{\text{out}} = \mathcal{U}\hat{\jmath}_{\text{in}}$

use $T = |\hat{\jmath}_{\text{out}}|^2$

Just remember to make $|\hat{\jmath}_{\text{in}}|^2 = 1$

More examples:

Wave plate with optic axis along y

Saw $\hat{\jmath} \rightarrow j_x e^{in_o k_0 d}\,\hat{\mathbf{x}} + j_y e^{in_e k_0 d}\,\hat{\mathbf{y}}$

So $\mathcal{U} = \begin{bmatrix} e^{in_o k_0 d} & 0 \\ 0 & e^{in_e k_0 d} \end{bmatrix} = e^{in_o k_0 d} \begin{bmatrix} 1 & 0 \\ 0 & e^{i(n_e - n_o)k_0 d} \end{bmatrix}$

Say quarter wave plate, $n_e > n_o$

- then slow axis along y

$$\mathcal{U} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

ignoring overall phase

Or, quarter wave plate with fast axis along y:

$$\mathcal{U} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

$\equiv$ slow axis along x

(Doesn't matter which is optic axis)

Half wave plate, either axis along y:

$$\mathcal{U} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

since $e^{i\pi} = e^{-i\pi} = -1$

Generally called *Jones matrices*

Easy when element axes aligned to x or y

What if axis at angle θ?

Define two coordinates:

x-y = "lab" coordinates

x'-y' = "element" coordinates

For instance, x' = transmission axis of polarizer

Express $\hat{\jmath}$ in x'-y' coords

If x' axis at angle θ to x, then

$$\hat{\mathbf{x}}' = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{y}}' = \cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{x}}$$

as before

So

$j_{x'} = \hat{\mathbf{x}}' \cdot \hat{\jmath} = \cos\theta j_x + \sin\theta j_y$

$j_{y'} = \hat{\mathbf{y}}' \cdot \hat{\jmath} = -\sin\theta j_x + \cos\theta j_y$

Write

$$\begin{bmatrix} j_{x'} \\ j_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} j_x \\ j_y \end{bmatrix}$$

Define rotation matrix

$$\mathcal{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Then $\hat{\jmath}' = \mathcal{R}(\theta)\hat{\jmath}$

In primed coordinates, $\mathcal{U}$ is easy

For polarizer

$$\mathcal{U}' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\hat{\jmath}'_{\text{out}} = \mathcal{U}'\,\hat{\jmath}'_{\text{in}} = \mathcal{U}'\,\mathcal{R}(\theta)\,\hat{\jmath}$

Still need to get $\hat{\jmath}_{\text{out}}$ in lab coords

Generally $\hat{\jmath} = R(\theta)^{-1}\hat{\jmath}'$

$\mathcal{R}(\theta)^{-1}$ = matrix inverse of $\mathcal{R}(\theta)$

Find $\mathcal{R}(\theta)^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \mathcal{R}(-\theta)$

So $\hat{\jmath}_{\text{out}} = \mathcal{R}(-\theta)\,\mathcal{U}'\,\mathcal{R}(\theta)\hat{\jmath}_{\text{in}}$

Or, without specifying coordinates:

$$\boxed{\mathcal{U}(\theta) = \mathcal{R}(-\theta)\,\mathcal{U}_0\,\mathcal{R}(\theta)}$$

for $\mathcal{U}_0 \equiv \mathcal{U}(\theta = 0)$

So polarizer with axis at angle θ:

$$\mathcal{U}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

If $\hat{\jmath}_{\text{in}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{\mathbf{x}}$

get $\hat{\jmath}_{\text{out}} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

Get $T = |\hat{\jmath}_{\text{out}}|^2 = \cos^2\theta$ as expected

Other examples:

QWP fast axis at angle θ (measured from x axis)

$$\mathcal{U}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix}$$

HWP at angle θ

$$\mathcal{U}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Get matrices for all elements in lab coordinates

For multiple elements, multiply matrices

Example: Suppose light linearly polarized along y passes through a quarter-wave plate with slow axis at $\theta_1 = 30^\circ$ from x, then a half-wave plate with axis at $\theta_2 = 60^\circ$, and finally a polarizer with axis at $\theta_3 = 45^\circ$. What is the total transmission?

Solution:
Get matrix for each element.

$$\mathcal{U}_1 = \begin{bmatrix} 0.75+0.25i & 0.433-0.433i \\ 0.433-0.433i & 0.25+0.75i \end{bmatrix}$$

$$\mathcal{U}_2 = \begin{bmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix} \qquad \mathcal{U}_3 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$\mathcal{U}_{\text{tot}} = \mathcal{U}_3\mathcal{U}_2\mathcal{U}_1 = \begin{bmatrix} 0.433-0.25i & 0.25+0.433i \\ 0.433-0.25i & 0.25+0.433i \end{bmatrix}$$

Given $\hat{\jmath}_{\text{in}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $\hat{\jmath}_{\text{out}} = \begin{bmatrix} 0.25+0.433i \\ 0.25+0.433i \end{bmatrix}$

So $T = |\hat{\jmath}_{\text{out}}|^2 = 2 \times |0.25+0.433|^2 = \boxed{0.5}$

Question: What is the polarization state of the output?

Often output state is "disguised" by extra phase factors

Example:
Light polarized along $\hat{\mathbf{y}}$ into QWP at 45°

Expect to get circular polarized light out

Have $\mathcal{U} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

so $\hat{\jmath}_{\text{out}} = \frac{1}{2}\begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$

Doesn't look like circular polarization!

Note $1 + i = \sqrt{2}e^{i\pi/4}$

$1 - i = \sqrt{2}e^{-i\pi/4}$

So $$\hat{\jmath}_{\text{out}} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\pi/4} \\ e^{-i\pi/4} \end{bmatrix} = \frac{e^{i\pi/4}}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{-i\pi/2} \end{bmatrix}$$

$$= \frac{e^{i\pi/4}}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

= right circular polarization

Good approach:

always factor out phase to make j_x real

Gives standard form $\hat{\jmath} = \cos\beta\hat{\mathbf{x}} + e^{i\varepsilon}\sin\beta\hat{\mathbf{y}}$

Can generalize and work in any basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$

Write $\hat{\jmath} = j_1\hat{\mathbf{e}}_1 + j_2\hat{\mathbf{e}}_2 = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}_{12}$

then $$\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}_{12} = \begin{bmatrix} \hat{\mathbf{e}}_1^* \cdot \hat{\mathbf{x}} & \hat{\mathbf{e}}_1^* \cdot \hat{\mathbf{y}} \\ \hat{\mathbf{e}}_2^* \cdot \hat{\mathbf{x}} & \hat{\mathbf{e}}_2^* \cdot \hat{\mathbf{y}} \end{bmatrix}\begin{bmatrix} j_x \\ j_y \end{bmatrix}_{xy}$$

$$\equiv \mathcal{S}_{(xy\to 12)}\begin{bmatrix} j_x \\ j_y \end{bmatrix}_{xy}$$

Then in 1-2 basis, have

$$\mathcal{U}_{12} = \mathcal{S}_{(xy\to 12)}\,\mathcal{U}_{xy}\,\mathcal{S}^\dagger_{(xy\to 12)}$$

where $\mathcal{S}^\dagger$ = conjugate transposition of $\mathcal{S}$

Partially Polarized Light

Sometimes light is not perfectly polarized

also not unpolarized

Example: pass natural light through poor polarizer

No Jones vector in this case:

$\hat{\jmath}$ is fluctuating

Instead define *coherency matrix*

$$G = \begin{bmatrix} \langle j_x j_x^* \rangle & \langle j_x j_y^* \rangle \\ \langle j_y j_x^* \rangle & \langle j_y j_y^* \rangle \end{bmatrix} = \left\langle \begin{bmatrix} j_x \\ j_y \end{bmatrix} \begin{bmatrix} j_x^* & j_y^* \end{bmatrix} \right\rangle$$

For polarized light, G and $\hat{\jmath}$ have same information

Unpolarized light =

mixture of x and y polarized light

Then $$G_{\text{unpolarized}} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Get G in other cases by propagation

If system has Jones matrix $\mathcal{U}$, then have

$$G_{\text{out}} = \mathcal{U}\, G_{\text{in}} \mathcal{U}^\dagger$$

Example:

Suppose glass surface has $t_\| = 0.95$, $t_\perp = 0.85$
from Fresnel relations

Then $\mathcal{U} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix}$ in $\|$-$\perp$ basis

Illuminate with natural light:

$$G_{\text{out}} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix}$$
$$= \begin{bmatrix} 0.45 & 0 \\ 0 & 0.36 \end{bmatrix}$$

Total irradiance $= |j_x|^2 + |j_y|^2 =$ Trace G
$=$ sum of diagonal elements

Here write

$$G_{\text{out}} = 0.81 \begin{bmatrix} 0.55 & 0 \\ 0 & 0.45 \end{bmatrix}$$

Total transmission is 0.81

Transmitted light is partially polarized:
55% $\|$ and 45% $\perp$

Could then pass through a QWP at 45°

$$\mathcal{U} = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

$$\text{Get } G_{\text{out}} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 0.55 & 0 \\ 0 & 0.45 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.5 & 0.05i \\ -0.05i & 0.5 \end{bmatrix}$$

Describes partial circular polarization

Coherency matrix is generally nice way to describe polarization

- Takes care of phase factors
- Has correct symmetry property:
  180° rotation doesn't change state

Older way to describe partially polarized light:
Stokes parameters (Hecht 8.13.1)

Coherency matrix is better
Reference Klein and Furtak §9.3

Summary:

- Retarders based on birefringence
  - Quarter-wave plate: make circ polarization
  - Half-wave plate: rotate linear polarization

- Use Jones matrices for multiple elements

- Partially polarized light

  $\rightarrow$ coherency matrix
  - Also uses Jones matrices