Gaussian Beams

Last time, finished Fourier optics

Lots of interesting applications

Next topic: interference and coherence

emphasis on applications

Today, discuss laser beams

Typically use lasers for interference applications

Outline:

- Recall solution
- Properties
- Beams and optical systems

None of this in Hecht

See Saleh and Teich, chapter 3

Next time: interferometers

The Gaussian Beam

Actually derived already:

Lecture 15, slides 32–38

Start with Gaussian field at $z = 0$

$$E(x, y, 0) = E_0 e^{-(x^2+y^2)/w_0^2}$$

Use Fresnel approximation to get field at all z:

$$E(x, y, z) = E_0 \frac{w_0^2}{Q^2} e^{ikz} e^{(x^2+y^2)/Q^2}$$

for $Q^2 = w_0^2 + i\dfrac{2z}{k}$

Conform to conventional notation,

change definition of Q:

Define $q = z - i\dfrac{kw_0^2}{2} \qquad \left(= -i\dfrac{k}{2} \times Q^2\right)$

Then

$$E(x, y, z) = -iE_0 \left(\frac{\pi w_0^2}{\lambda q}\right) e^{ikz} e^{ik(x^2+y^2)/2q}$$

Mathematically equivalent

Spend today exploring solution

Helpful to define

$$\boxed{z_0 = \frac{kw_0^2}{2} = \frac{\pi w_0^2}{\lambda}}$$

$\equiv$ Rayleigh length

Will see importance shortly

Convenient:
Have $\boxed{q = z - iz_0}$ and

$$\boxed{E(x, y, z) = -iE_0 \frac{z_0}{q} e^{ikz} e^{ik(x^2+y^2)/2q}}$$

Properties

- Irradiance

$$I = \frac{1}{2\eta_0}|E|^2$$

$$= I_0 \frac{z_0^2}{|q|^2} \left[e^{ik\rho^2/2q} \times e^{-ik\rho^2/2q^*} \right]$$

for $I_0 = |E_0|^2/(2\eta_0)$ and $\rho^2 = x^2 + y^2$

Total exponent is

$$\frac{ik\rho^2}{2(z - iz_0)} - \frac{ik\rho^2}{2(z + iz_0)} = -\frac{k\rho^2 z_0}{z^2 + z_0^2}$$

Write

$$I(\rho, z) = I_0 \frac{z_0^2}{z^2 + z_0^2} e^{-2\rho^2/w^2}$$

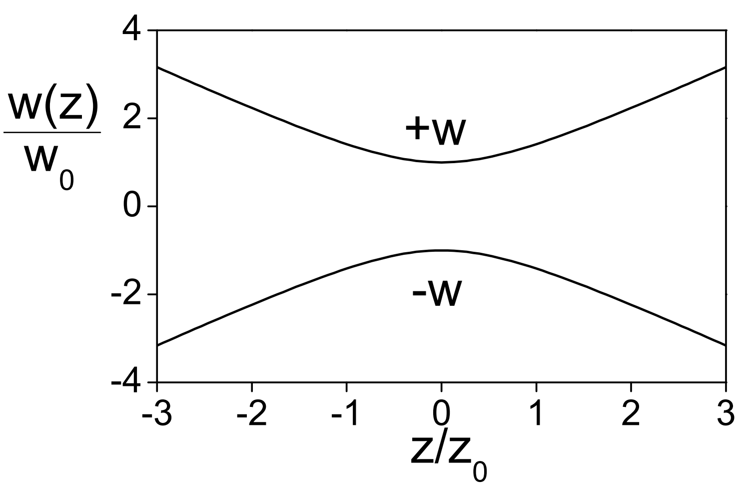
where $w = w(z) = \sqrt{\frac{2}{kz_0}(z^2 + z_0^2)}$

is the *beam width* at position z

Using $z_0 = kw_0^2/2$, have

$$\boxed{w(z) = w_0\sqrt{1 + \frac{z^2}{z_0^2}}}$$

As beam propagates, width expands



Shows profile of beam as it travels

Minimum width = w_0 at $z = 0$

Beam has focus at $z = 0$
  - called *beam waist*

Call w_0 = waist radius

For $|z| \ll z_0$, $w(z) \approx w_0$

At $z = \pm z_0$, width increased by $\sqrt{2}$
  Rayleigh length ≈ depth of focus
  Gives length over which beam stays focused

Question: If I double the beam waist, what happens to the Rayleigh length?

For $z \gg z_0$, beam diverges
  (Due to diffraction)

At large z

$$w(z) \approx \frac{w_0 z}{z_0} = \frac{\lambda}{\pi w_0} z$$

Divergence angle $\theta = \lambda/(\pi w_0)$
  = minimum possible divergence for spot size w_0

Laser beams spread more slowly than other sources
  but they still spread

Example:

Suppose $\lambda = 633$ nm, $w_0 = 2$ mm

Then $z_0 = 20$ m and $\theta = 100$ μrad

Laser beam near focus is collimated

= not diverging

Strange but true:

only place laser is collimated is at focus

Usually say "collimated" if z_0 large

"focused" if z_0 small

- Power

Total power in beam is

$$P = \iint I(x,y)\,dx\,dy$$

$$= 2\pi \int_0^\infty I(\rho,z)\rho\,d\rho$$

Note

$$I(\rho,z) = I_0 \frac{z_0^2}{z^2+z_0^2}\,e^{-2\rho^2/w^2}$$

$$= I_0 \frac{w_0^2}{w^2}\,e^{-2\rho^2/w^2}$$

So $P = \frac{2\pi I_0 w_0^2}{w^2} \int_0^\infty \rho e^{-2\rho^2/w^2}\, d\rho$

Change variables $u = 2\rho^2/w^2$

$$du = \frac{4\rho}{w^2} d\rho$$

Then $P = \frac{\pi}{2} I_0 w_0^2 \int_0^\infty e^{-u}\, du = \frac{\pi}{2} I_0 w_0^2$

Here I_0 = max irradiance at center of focus

Usually measure P, want I_0:

$$\boxed{I_0 = \frac{2P}{\pi w_0^2}}$$

Generally

$$I_{\text{max}}(z) = \frac{2P}{\pi w^2}$$

Effective area of beam = $\pi w^2/2$
- radius $w/\sqrt{2}$

Want larger radius if passing through aperture:
86% of power in $\rho < w$
98% of power in $\rho < 2w$

Rule of thumb: make aperture diameter = πw
gives 95% transmission

• Phase

Have

$$E(x,y,z) = E_0 \left(\frac{-iz_0}{q}\right) e^{ikz}\, e^{ik(x^2+y^2)/2q}$$

with $q = z - iz_0$

Write $\dfrac{1}{q} = \dfrac{1}{z - iz_0} = \dfrac{z + iz_0}{z^2 + z_0^2}$

Imaginary part:

$$\frac{z_0}{z^2 + z_0^2} = \frac{w_0^2}{z_0 w^2} = \frac{\lambda}{\pi w^2} = \frac{2}{kw^2}$$

using definitions of w and z_0

Real part:

Define $\dfrac{z}{z^2 + z_0^2} = \dfrac{1}{R}$

So $\dfrac{1}{q} = \dfrac{1}{R} + \dfrac{2i}{kw^2}$

Use in exponent:

$$e^{ik\rho^2/2q} = e^{ik\rho^2/2R} e^{-\rho^2/w^2}$$

Recognize phase and amplitude terms

Also write prefactor in polar form

$$\frac{-iz_0}{z - iz_0} = \frac{1}{1 + i(z/z_0)}$$
$$= \frac{1}{|1 + i(z/z_0)|} e^{-i\zeta}$$

with

$$\frac{1}{|1 + i(z/z_0)|} = \frac{z_0}{\sqrt{z^2 + z_0^2}} = \frac{w_0}{w}$$

and

$$\tan\zeta = \frac{z}{z_0}$$

So express

$$E = E_0 \frac{w_0}{w} \, e^{i\phi} \, e^{-\rho^2/w^2}$$

with phase

$$\phi(z) = -\zeta(z) + k\left(z + \frac{\rho^2}{2R(z)}\right)$$

Rewrite

$$\phi(z) = -\zeta + k(z - R) + k\left(R + \frac{\rho^2}{2R}\right)$$

Recognize $R + \rho^2/(2R)$ as expansion of sphere wave

Radius of curvature R

For $|z| \gg z_0$, $R \to z$

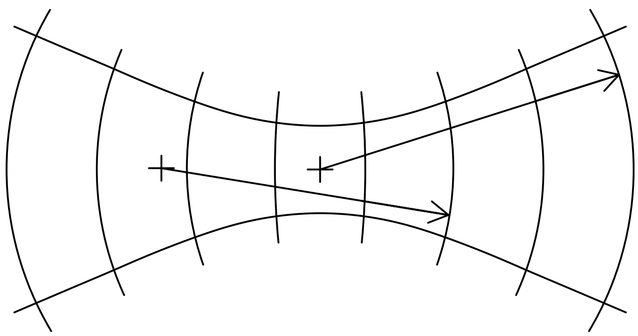
- sphere wave centered at focus

For $|z| \ll z_0$, $R \to \infty$

- plane wave

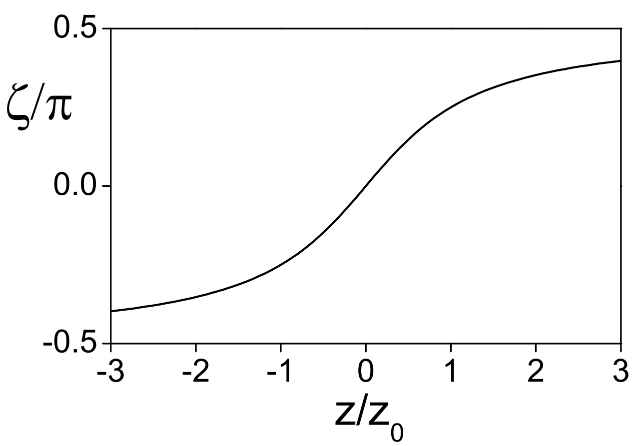
= collimated, as before

Draw wave fronts:



On axis $\rho = 0$ have $\phi = kz - \zeta(z)$

Like plane wave with extra phase $\zeta = \tan^{-1}(z/z_0)$
called "Guoy" phase



180° phase shift through focus
sometimes important

- Complex Radius

At position z, have $q = z - iz_0$

In general, Re z = distance to focus
Im z = Rayleigh length of focus ($\times -1$)

From z_0 get $w_0 = \sqrt{\lambda z_0/\pi}$

So q specifies beam parameters at focus

Question: Suppose that a Gaussian beam is propagating in the $+z$ direction. At some position, you determine that $q =$ 0.3 m - i0.05 m. Where is the focus of the beam relative to your position?

But

$$\frac{1}{q} = \frac{1}{R} + \frac{i\lambda}{\pi w^2}$$

R = radius of curvature at z
w = beam width at z

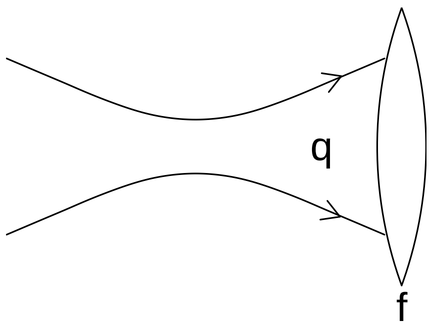
So $1/q$ specifies beam parameters at z

Inverting q:
Transforms between "local" properties at z
and "focal" properties at $z = 0$

q is useful: called *complex beam radius*

Beams and Lenses

What happens when we put Gaussian beam through lens?



Say incident beam has complex radius q

Thin lens, focal length f

Effect of lens:

Change radius of curvature R

Spherical wave centered at $z = -s_o$

$\rightarrow$ sphere wave centered at $z = s_i$

with $\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$

Here $s_o = R_{\text{in}}$

$s_i = -R_{\text{out}}$

So

$$\frac{1}{R_{\text{out}}} = \frac{1}{R_{\text{in}}} - \frac{1}{f}$$

Beam width doesn't change

So $$\frac{1}{q_{\text{out}}} = \frac{1}{R_{\text{out}}} + i\frac{\lambda}{\pi w^2}$$

$$= \frac{1}{R_{\text{out}}} - \frac{1}{f} + i\frac{\lambda}{\pi w^2}$$

$$\boxed{\frac{1}{q_{\text{out}}} = \frac{1}{q_{\text{in}}} - \frac{1}{f}}$$

This is transformation law for Gaussian beam

Example: Suppose a Gaussian beam with $\lambda = 633$ nm has a focus with spot size $w_0 = 75$ μm. A lens with $f = 25$ mm is placed a distance 50 mm after the focus. At what position is the light refocused after the lens, and what beam waist is obtained?

Solution:
Incident beam has $q = z - iz_0$ for $z = 50$ mm and $z_0 = \pi w_0^2/\lambda = 28$ mm. So

$$\frac{1}{q_{\text{in}}} = \frac{1}{50 - 28i} = 0.0152 + i0.0085 \text{ mm}^{-1}$$

Then

$$\frac{1}{q_{\text{out}}} = \frac{1}{q_{\text{in}}} - \frac{1}{f} = -0.0248 + i0.0085 \text{ mm}^{-1}$$

Invert to get $q_{\text{out}} = -36 - i12.4$ mm.

So light is refocused $\boxed{36 \text{ mm}}$ after lens.

Beam waist $\boxed{w_0 = \sqrt{\lambda z_0/\pi} = 50\ \mu\text{m}}$.

Note ray optics predicts focus at

$$\frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o} = \frac{1}{25} - \frac{1}{50} = \frac{1}{50} \text{ mm}^{-1}$$

which is incorrect

Beams and Ray Matrices

General optical system described by ray matrix

$$\mathcal{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Transforms ray vector

$$\mathbf{v} = \begin{bmatrix} n\alpha \\ y \end{bmatrix}$$

with $\mathbf{v}_{\text{out}} = \mathcal{M}\mathbf{v}_{\text{in}}$

Useful for thick lenses, multilens systems

Relate to Gaussian beams

Recall $\mathcal{M}$ composed of two elements:

- Refraction matrix

$$\mathcal{R} = \begin{bmatrix} 1 & -\mathcal{D} \\ 0 & 1 \end{bmatrix}$$

where $\mathcal{D}$ = refractive power of element

For thin lens $\mathcal{D} = 1/f$

- Transfer matrix

$$\mathcal{T} = \begin{bmatrix} 1 & 0 \\ d/n & 1 \end{bmatrix}$$

d = propagation distance, n = index

Get effect of elements on Gaussian beam

Refraction: for lens have

$$\frac{1}{q_{\text{out}}} = \frac{1}{q_{\text{in}}} - \frac{1}{f}$$

Generally get

$$\frac{1}{q_{\text{out}}} = \frac{1}{q_{\text{in}}} - \mathcal{D} = \frac{1}{q_{\text{in}}} + B$$

where B = element of $\mathcal{R}$ matrix

So

$$q_{\text{out}} = \frac{1}{1/q_{\text{in}} + B} = \frac{q_{\text{in}}}{1 + Bq_{\text{in}}}$$

Free propagation:

Have $q = z - iz_0$

Free propagation just changes z:

$$q_{\text{out}} = q_{\text{in}} + d$$

Modified in medium: wavelength is different

Define λ, k = values in vacuum
λ', k' = values in medium

So $\lambda' = \lambda/n$ and $k' = nk$

Then in medium have $z_0' = \pi w_0^2/\lambda' = nz_0$
and $q' = z - iz_0'$

Field evolves as

$$E(z) = -iE_0 \frac{z_0'}{q'} e^{ik'z} e^{ik'\rho^2/2q'}$$

$$= -iE_0 \frac{nz_0}{z - inz_0} e^{ik'z} \exp\left[\frac{ink\rho^2}{2(z - inz_0)}\right]$$

$$= -iE_0 \frac{z_0}{z/n - iz_0} e^{ik'z} \exp\left[\frac{ik\rho^2}{2(z/n - iz_0)}\right]$$

$$= -iE_0 \frac{z_0}{q} e^{ik'z} e^{ik\rho^2/2q}$$

for $q = z/n - iz_0$

Ignoring overall phase, distance d in medium has

$$q_{\text{out}} = q_{\text{in}} + \frac{d}{n}$$

In terms of ray matrix $\mathcal{T}$

$$q_{\text{out}} = q_{\text{in}} + C$$

Get effect of both $\mathcal{R}$ and $\mathcal{T}$ with

$$\boxed{q_{\text{out}} = \frac{C + Dq_{\text{in}}}{A + Bq_{\text{in}}}}$$

Formula works for multiple systems too

Suppose $\mathcal{M}_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ and $\mathcal{M}_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$

Then for arbitrary q_0 let

$$q_1 = \frac{C_1 + D_1 q_0}{A_1 + B_1 q_0}$$

(= output of element 1)

$$q_2 = \frac{C_2 + D_2 q_1}{A_2 + B_2 q_1}$$

(= output of element 2)

Substitute for q_1, find

$$q_2 = \frac{C_T + D_T q_0}{A_T + B_T q_0}$$

for (A_T, B_T, C_T, D_T) satisfying

$$\begin{bmatrix} A_T & B_T \\ C_T & D_T \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$

So q_0 related to q_2 by system matrix

$$\mathcal{M}_T = \mathcal{M}_2 \mathcal{M}_1$$

Works for any number of elements

For arbitrary system with $\mathcal{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

have $q_{\text{out}} = \dfrac{C + D q_{\text{in}}}{A + B q_{\text{in}}}$

Easy to find Gaussian beam output of any paraxial system

Important:
Various conventions for $\mathcal{M}$ and $\mathbf{v}$

Laser books usually have

$$q_{\text{out}} = \frac{Aq_{\text{in}} + B}{Cq_{\text{in}} + D}$$

Elements of $\mathcal{M}$ rearranged

Also λ = wavelength in medium
not wavelength in vacuum

Summary

- Laser beams $\approx$ Gaussian beam solution
- Collimated at focus, diverge at ∞
- Rayleigh length z_0 = depth of focus
  - z_0 small if w_0 small
- Use ray matrices for propagation