Coherence Theory: Temporal

Last time, discussed interferometers

Michelson

Fabry-Perot

Mostly considered monochromatic light

makes interference easy

Can also use incoherent light

sometimes necessary

Today, discuss incoherent sources

be more quantitative than before

Outline:

- Random waves
- Temporal coherence function
- Applications to interference
- Power spectral density

More Fourier transforms today!

Developing toward Hecht Chapter 12

Material from Ch 7, 9, 11

In Ch 12, Hecht focuses on spatial coherence:

We'll cover in next lecture

Today, apply same ideas to temporal coherence

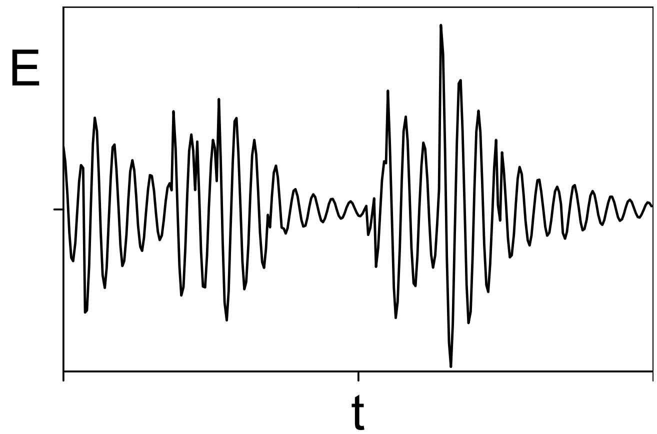
Easier way to start

Random Waves (Hecht 7.4.3)

Most light sources produce wave that fluctuates

Then $E(t)$ varies $\sim$ randomly in time

Example:

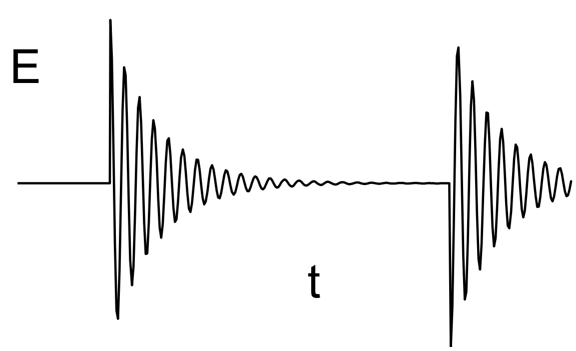


Why?

Source composed of many atoms
$=$ many radiators

Quantum mechanics:
Each atom excited in discrete steps
Radiates briefly, then stops (until excited again)

So atom produces pulses of light:



Sum over many atoms, get random field

Typical atomic decay time = 10 ns

Makes field that is *coherent* over 10 ns timescale

Recall coherent = oscillating with constant phase

Define *coherence time* τ_c
= time over which wave is coherent

Question: What would be the coherence time of a pulsed laser?

Note 10 ns $\approx 10^7$ optical periods
Atomic radiation is rather coherent

Most thermal sources not that good:
Examples: light bulb, candle, sunlight

Atoms constantly collide with neighbors
- interrupt phase of oscillation

Typical coherence time = 2-3 fs
1–2 optical periods

Want to understand effect on interference

Random Waves (Hecht 12.3)

How to treat mathematically?

Don't work with $E(t)$ directly

Assume we have sample of "possible" $E(t)$'s

Work with averages

$\langle .. \rangle$ = average over sample

Imagine running experiment many times,

collecting data $E_1(t)$, $E_2(t)$, $E_3(t)$ . . .

For instance:

$\langle E(t) \rangle$ = average possible values of E at time t

= 0 for truly random wave

But might have $E(t) = \bar{E}(t) + \delta E(t)$

$\bar{E}(t)$ = non-random = *deterministic*

$\delta E(t)$ = random noise

Then $\langle E(t) \rangle = \bar{E}(t)$ = deterministic part

Already know how deterministic part works

For today, assume $\langle E(t) \rangle = 0$

Also assume that averages are independent of time

$\Rightarrow$ Fluctuations in E have constant character

Say that $E(t)$ is *stationary*

For stationary wave,
can record samples $E(t)$ sequentially in time

Sample length T, get N samples in time NT

Then $\langle .. \rangle$ equivalent to time average

$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt$$

What can we measure?

Know irradiance $\neq 0$:

$$I = \frac{1}{2\eta_0} \left\langle |E|^2 \right\rangle$$

For stationary wave, I is constant

Same whether light is coherent or not

Really want to know how $E(t)$ compares to $E(t+\tau)$

Tells how correlations decay in time

Define *temporal coherence function*

$$\boxed{\Gamma(\tau) = \langle E(t+\tau)\, E^*(t)\rangle}$$

Have $\Gamma(0) \equiv 2\eta_0 I$
- has irradiance information

Also has coherence information:
If $\tau \gg \tau_c$, then $E(t+\tau)$ independent of $E(t)$
$\Rightarrow \Gamma(\tau) = 0$

If $\tau \ll \tau_c$, then $E(t+\tau)$ determined by $E(t)$
$\Rightarrow \Gamma(\tau)$ large

$\Gamma(\tau)$ gives precise measure of coherence

Examples:

- Monochromatic wave $E(t) = E_0 e^{-i\omega_0 t}$

$$\Gamma(\tau) = \left\langle E_0 e^{-i\omega_0 (t+\tau)} E_0^* e^{i\omega_0 t} \right\rangle$$

$$= |E_0|^2 e^{-i\omega_0 \tau}$$

Oscillates at ω_0

Magnitude $= |E_0|^2$ for all τ
- perfectly coherent wave

- Atomic radiation, frequency ω_0:

$$\Gamma(\tau) = |E_0|^2 e^{-|\tau|/\tau_c} e^{-i\omega_0 \tau}$$

exponential decay, time constant τ_c

Question: Is it possible to have $|\Gamma(\tau)| > \Gamma(0)$?

Generally, can't calculate $\Gamma(\tau)$ in optics

Need to know about physics of source
→ Usually quantum mechanics

Can measure for given source

Often use normalized version of Γ

$$\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)}$$

so γ independent of irradiance

Called *complex degree of temporal coherence*
(pretty dumb name)

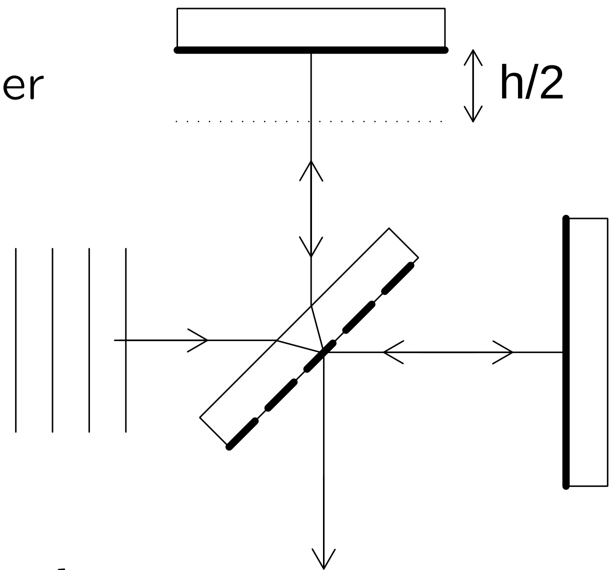
For atomic radiation

$$\gamma(\tau) = e^{-|\tau|/\tau_c} e^{-i\omega_0 \tau}$$

Interference (Hecht 9.2, 9.3)

Use $\Gamma(\tau)$ to analyze interference

Basic example:
Michelson interferometer



Arm length difference = h

Suppose source = random wave $E_0(t)$

Let length of arm 1 = d

Then output $E_1(t) = \beta E_0(t - d/c)$
- transmission factor β
- time delay d/c

Length of arm 2 = $d + h$

And output $E_2(t) = \beta E_0(t - d/c - h/c)$

Output irradiance given by

$$|E(t)|^2 = |E_1(t) + E_2(t)|^2$$
$$= |E_1(t)|^2 + |E_2(t)|^2$$
$$+ E_1^*(t)E_2(t) + E_1(t)E_2^*(t)$$

But only want to look at averages:

$$\left\langle |E|^2 \right\rangle = \left\langle |E_1|^2 \right\rangle + \left\langle |E_2|^2 \right\rangle + \langle E_1^* E_2 \rangle + \langle E_1 E_2^* \rangle$$

Have $\left\langle |E_1|^2 \right\rangle = \left\langle |E_2|^2 \right\rangle = |\beta|^2 \left\langle |E_0|^2 \right\rangle$

$$= |\beta|^2 \Gamma(0)$$

For interference terms, have

$$\langle E_1(t)E_2^*(t) \rangle = |\beta|^2 \langle E_0(t - d/c)E_0^*(t - d/c - h/c) \rangle$$

With stationary wave can rearrange times:

$$\langle E_0(t - d/c)E_0^*(t - d/c - h/c) \rangle = \langle E_0(t + h/c)E_0^*(t) \rangle$$
$$= \Gamma(\tau)$$

for $\tau = h/c =$ time delay

Also $\langle E_1^*(t)E_2(t) \rangle = |\beta|^2 \Gamma^*(\tau)$

So total output is

$$\left\langle |E|^2 \right\rangle = |\beta|^2 \left[2\Gamma(0) + \Gamma\left(h/c\right) + \Gamma^*\left(h/c\right) \right]$$

Suppose $\Gamma(\tau) = \Gamma(0) e^{-|\tau|/\tau_c} e^{-i\omega_0 t}$

Define $\ell_c = c\tau_c =$ *longitudinal coherence length*

Then

$$\left\langle |E|^2 \right\rangle = |\beta|^2 \Gamma(0) \Big[2 + e^{-|h/\ell_c|} e^{-i\omega_0 h/c} + e^{-|h/\ell_c|} e^{i\omega_0 h/c} \Big]$$

$$= 2|\beta|^2 \Gamma(0) \left[1 + e^{-|h/\ell_c|} \cos\left(\frac{\omega_0 h}{c} \right) \right]$$

See oscillation with h: interference
- but amplitude decays for $|h| \gtrsim \ell_c$

Define *visibility* of interference pattern

$$\mathcal{V} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$$

For perfect fringe, $I_{\text{min}} = 0 \Rightarrow \mathcal{V} = 1$

Generally good measure of fringe contrast

In our example,

$$I_{\text{max}} = \frac{|\beta|^2 \Gamma(0)}{\eta_0} \left(1 + e^{-|ch/\tau_c|} \right)$$

$$I_{\text{min}} = \frac{|\beta|^2 \Gamma(0)}{\eta_0} \left(1 - e^{-|ch/\tau_c|} \right)$$

So $\mathcal{V} = \frac{2e^{-|h/\ell_c|}}{2} = e^{-|h/\ell_c|}$

General result:

$$\boxed{\mathcal{V} = |\gamma(\tau)|}$$

when interfering waves with time delay τ

If amplitudes of E_1 and E_2 are different, get

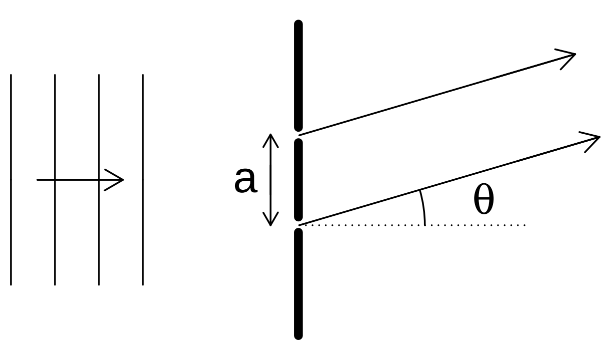
$$\boxed{\mathcal{V} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |\gamma(\tau)|}$$

Question: Is $\mathcal{V}$ higher or lower if $I_1 \neq I_2$?

Michelson interferometer
= good way to measure γ

Demo: Michelson with white light

Another example: Two slit interference



Interference at angle θ:
path length difference = $a\theta$

Time delay = $a\theta/c$

So fringe visibility decays as $|\gamma(a\theta/c)|$

If coherence time τ_c, need $|a\theta/c| < \tau_c$
limits $|\theta| < \ell_c/a$

For light bulb, $\tau_c = 2$ fs
$\Rightarrow \ell_c = 0.6$ μm

If $a = 100$ μm, need $|\theta| < 6$ mrad = 0.3°

But fringe spacing $\Delta\theta = \lambda/a = 5$ mrad
Only observe about one fringe

Generally limited to $N \approx \dfrac{\ell_c}{\lambda}$ fringes

Hard to see interference with natural light

Power Spectral Density (Hecht 11.3.4)

Γ also useful for characterizing spectrum of light

Idea of spectrum:

Polychromatic light has range of frequencies

Want to characterize by $I(\omega)$
- irradiance as function of frequency

For instance: Pass light through filter for freq ω_0, frequency width $\Delta\omega$

Expect to transmit irradiance $I(\omega_0)\Delta\omega$

Obvious approach: Fourier transform

Try to define

$$I(\omega) = \frac{1}{2\eta_0}|\mathcal{E}(\omega)|^2$$

for $\mathcal{E}$ = transform of $E(t)$

Doesn't work – several reasons

- Units wrong:

  $\mathcal{E}(\omega)$ units Vs/m

  So $I(\omega)$ units (W s^2)/m^2

Want units (W s)/m^2 so that $I(\omega)\Delta\omega$ = W/m^2

- Bad for monochromatic light:

  If $E(t) = E_0 e^{-i\omega_0 t}$ then

  $$\mathcal{E}(\omega) = 2\pi E_0 \delta(\omega - \omega_0)$$

  $$I(\omega) = \frac{2\pi^2 |E_0|^2}{\eta_0} \delta(\omega - \omega_0)^2$$

  $\delta()^2$ is nasty

- Bad for random light:

  $$E(\omega) = \int_{-\infty}^{\infty} E(t) e^{i\omega t}\, dt$$

  Need to average $\langle .. \rangle$, don't know how

Attack averaging problem
(solves others as well)

Define $\mathcal{E}_T(\omega) = \int_{-T/2}^{T/2} E(t) e^{i\omega t}\, dt$

Then $|\mathcal{E}_T|^2 \Delta\omega / 2\eta_0$ = energy/m^2 in time T
(Parseval's theorem)

Want power = average energy/time

Define $S(\omega) = \lim_{T\to\infty} \frac{|\mathcal{E}_T(\omega)|^2}{2\eta_0 T}$

Call $S(\omega)$ = *power spectral density*

Write out limit

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E^*(t) e^{-i\omega t}\, dt \int_{-T/2}^{T/2} E(t') e^{i\omega t'}\, dt'$$

Change variables to (t, τ) with $t' = t + \tau$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2-t}^{T/2-t} E(t+\tau) E^*(t) e^{-i\omega t} e^{i\omega(t+\tau)}\, d\tau\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2-t}^{T/2-t} E(t+\tau) E^*(t) e^{i\omega\tau}\, d\tau\, dt$$

Move τ integral outside limit

$$= \int_{-\infty}^{\infty} e^{i\omega\tau} \left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E(t+\tau) E^*(t)\, dt \right] d\tau$$

Recognize expression in brackets:

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E(t+\tau) E^*(t)\, dt = \langle E(t+\tau) E^*(t) \rangle$$

$$= \Gamma(\tau)$$

So $\boxed{S(\omega) = \frac{1}{2\eta_0}\int_{-\infty}^{\infty} \Gamma(\tau) e^{i\omega\tau}\, d\tau}$

Power spectral density =
Fourier transform of temporal coherence function

Called the *Wiener-Khintchine theorem*

Extremely useful, not just in optics

For instance:
    Calculate spectrum of electronic noise
    Or spectrum of stock market fluctuations

Good way to characterize any noisy signal

Examples:

- Monochromatic field $E(t) = E_0 e^{-i\omega_0 t}$

Saw already $\Gamma(\tau) = |E_0|^2 e^{-i\omega_0 \tau}$

So $S(\omega) = \frac{|E_0|^2}{2\eta_0}\int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t} dt$

$$= \frac{\pi |E_0|^2}{\eta_0}\delta(\omega - \omega_0)$$

δ-peak at $\omega = \omega_0$: makes sense, monochromatic

Total irradiance is

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega$$

Here

$$I = \frac{|E_0|^2}{2\eta_0}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)d\omega$$

$$= \frac{|E_0|^2}{2\eta_0}$$

using $\int \delta(\omega)d\omega = 1$

Get expected result

- Atomic radiation

Have $\Gamma(\tau) = \Gamma(0)e^{-|\tau|/\tau_c}\, e^{-i\omega_0\tau}$

$$S(\omega) = \frac{\Gamma(0)}{2\eta_0}\int_{-\infty}^{\infty} e^{-|\tau|/\tau_c}\, e^{i(\omega-\omega_0)\tau}\, d\tau$$

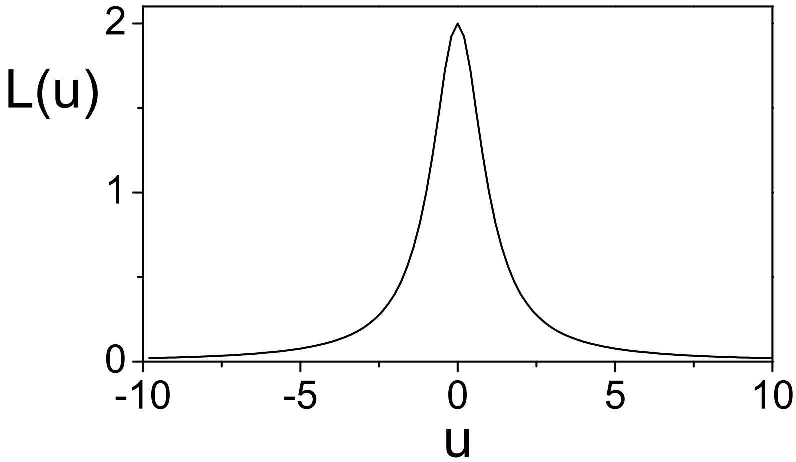
Did this already, HW problem 6.4

Get $S(\omega) = \dfrac{\Gamma(0)}{\eta_0}\dfrac{\tau_c}{1+\tau_c^2(\omega-\omega_0)^2}$

Called *Lorentzian* function

Peak at ω_0
FWHM $\Delta\omega = 2/\tau_c$

Plot $L(u) = \dfrac{2}{1+u^2}$



Normalized: $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} L(u)\, du = 1$

General property of Fourier transforms:

$$\Delta\omega \Delta t \gtrsim \pi$$

Here $\Delta\omega$ is *spectral bandwidth*
$=$ range of frequencies present

So $\Delta\omega$ is related to coherence time τ_c:

$$\tau_c \approx \frac{\pi}{\Delta\omega}$$

Incoherent source has broad bandwidth

Monochromatic source is very coherent

Gives another way to look at loss of visibility

Consider two slit interference

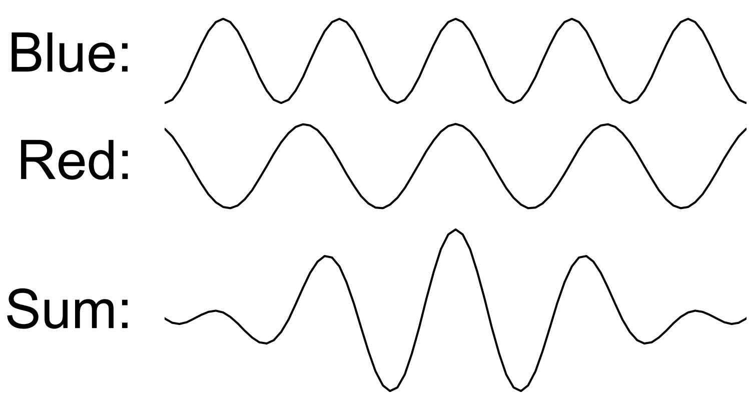
Incoherent source: bandwidth $\Delta\omega$

Imagine we have red and blue light

Red light makes red interference pattern
w/ high visibility

Blue light makes blue pattern
w/ high visibility

Observe sum of red and blue:



Pattern washed out at high angles
peaks of blue cancel troughs of red

Find that interference goes away at $\theta \approx \ell_c/a$
as before

Summary

- Characterize random waves with averages
- Describe coherence with $\Gamma(\tau)$
  $\Gamma = \langle E(t+\tau)E^*(t)\rangle$
- Γ sets visibility of interference
  visibility $\rightarrow 0$ for large path difference
- Describe spectrum with $S(\omega)$
  $=$ Fourier transform of $\Gamma(\tau)$