Phys 531 Lecture 22 8 November 2005

Coherence Theory: Spatial

Last time, developed theory for incoherent sources

Temporal incoherence:

Wave fluctuates randomly in time

Today, generalize to spatial incoherence:

Wave fluctuates randomly in space

Get spatial fluctuations from extended sources

Outline:

- Review temporal coherence
- Interference with extended sources
- van Cittert-Zernike theorem
- Mutual coherence function
- Michelson interferometer

Material from Hecht Ch 12

Next time: Polarization

Review Temporal Coherence

Random waves characterized by coherence time τ_c

Over times $\ll \tau_c$: oscillations are regular

Over times $\gg \tau_c$: wave fluctuates

Characterize with temporal coherence function

$$\Gamma(\tau) = \langle E(t+\tau)E^*(t)\rangle$$

or complex degree of temporal coherence

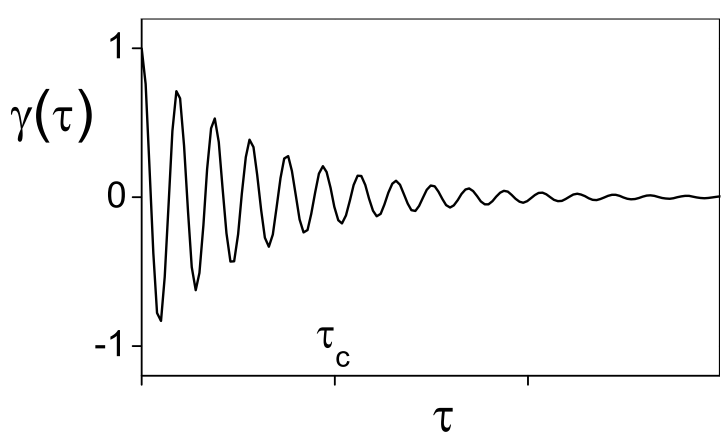
$$\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)}$$

Have $|\gamma(0)| = 1$

Amplitude decreases over time scale τ_c

Phase of γ oscillates at average frequency ω_0

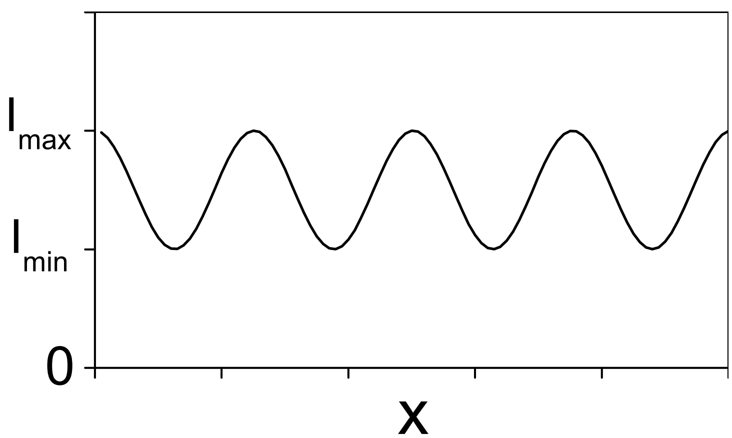
Typical behavior:



For interferometer with path-length difference h:

Get fringe visibility $\mathcal{V} = \left|\gamma\left(\frac{h}{c}\right)\right|$

where $\mathcal{V} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$

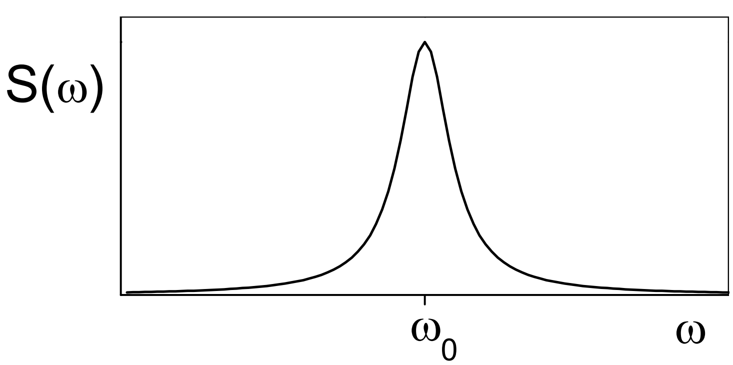


Γ also related to spectrum of light

Power spectral density

$$S(\omega) = \frac{1}{2\eta_0}\int_{-\infty}^{\infty}\Gamma(\tau)e^{-i\omega\tau}d\tau$$

Then $S(\omega)\Delta\omega$ = total irradiance
in frequency range ω to $\omega + \Delta\omega$



Extended Sources (Hecht 12.2)

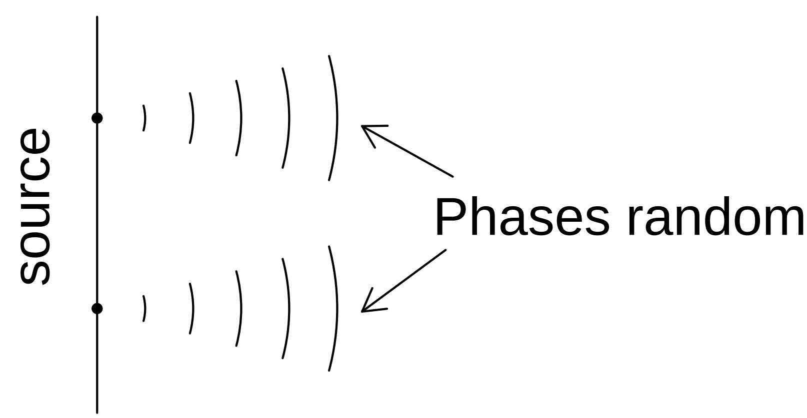
So far, considered waves from point source
(recall plane wave = point source at ∞)

No problem writing down $E(\mathbf{r})$
Have spherical wave, plane wave, or dipole pattern

What if source is extended object?
= collection of many points

Suppose source monochromatic, frequency ω

Still have incoherence:
phase of source varies from point to point



Average over phases:
No interference between fields from different points

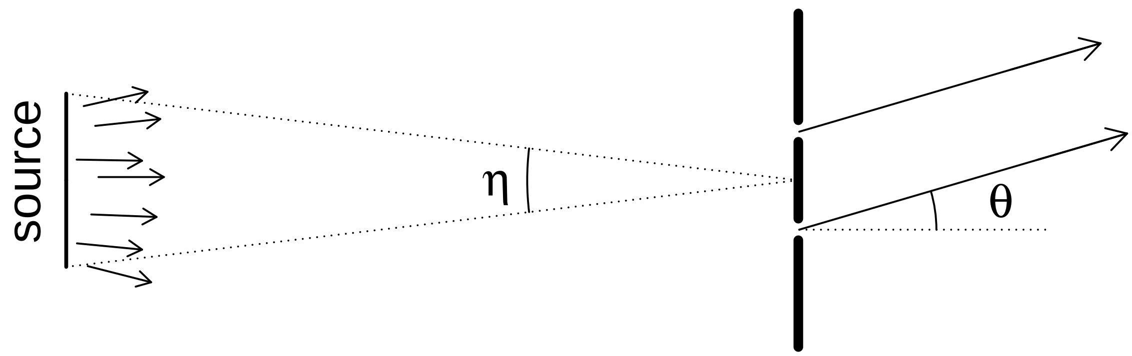
Add irradiances to get total pattern

Question: If the phases are random but constant, don't you get some unpredictable but steady interference pattern?

Reduces visibility of interference

Example: two slit interference

Distant extended source: subtends angle η



Slit spacing a

For point source at normal incidence,
have interference pattern

$$I(\theta) \approx I_0 \left[1 + \cos(ka\theta)\right]$$

For point source at angle η' have

$$I(\theta; \eta') \approx I_0 \left[1 + \cos ka(\theta - \eta')\right]$$

For extended source, average over η':

$$I_{\rm tot}(\theta) = \frac{1}{\eta} \int_{-\eta/2}^{\eta/2} I(\theta; \eta')\, d\eta'$$

We can evaluate this:

$$I_{\text{tot}}(\theta) = I_0 \left[1 + \frac{1}{\eta} \int_{-\eta/2}^{\eta/2} \cos ka(\theta - \eta')\, d\eta'\right]$$

Set $u = ka(\theta - \eta')$

$$I_{\text{tot}}(\theta) = I_0 \left[1 + \frac{1}{ka\eta} \int_{ka(\theta-\eta/2)}^{ka(\theta+\eta/2)} \cos u\, du\right]$$

$$= I_0 \left[1 + \frac{\sin ka(\theta + \eta/2) - \sin ka(\theta - \eta/2)}{ka\eta}\right]$$

Expand sines using

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$

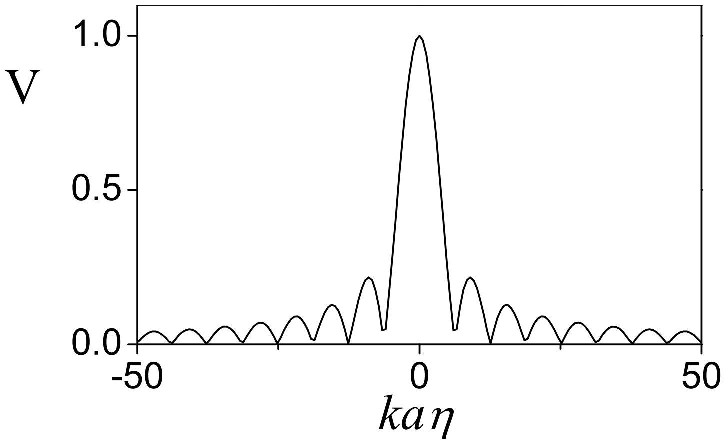
Then $\sin(ka\theta)$ terms cancel, leaves

$$I_{\text{tot}}(\theta) = I_0 \left[1 + \frac{2}{ka\eta} \sin\left(\frac{ka\eta}{2}\right) \cos(ka\theta)\right]$$

$$= I_0 \left[1 + \text{sinc}\left(\frac{ka\eta}{2}\right) \cos(ka\theta)\right]$$

Get visibility

$$\mathcal{V} = \left|\text{sinc}\left(\frac{ka\eta}{2}\right)\right|$$



See that $\mathcal{V} < 1$

Decreases for large η and large a

To have large $\mathcal{V}$, need $a \ll \lambda/\eta$

Leads to idea of spatial coherence:

Field at slit 1 is not completely coherent
with field at slit 2

= no definite phase relationship

- Due to indefinite phase between different points on source

In two-slit example, see that field is coherent
over distance $\rho_c \approx \lambda/\eta$

For point source $\eta \to 0$ so $\rho_c \to \infty$

Call ρ_c = lateral (or transverse) coherence length

$a \ll \rho_c$, can treat as point source

$a \gg \rho_c$, no interference

Generally true that

$$\boxed{\rho_c \approx \frac{\lambda}{\eta}}$$

for source subtending angle η

Question: For a source consisting of two points separated by angle η, the interference pattern is lost when the slit separation $a = \lambda/2\eta$. Is there a simple way to see this?

Example: star α-Centauri

distance $d = 4.1 \times 10^{16}$ m

diameter $D = 1.7 \times 10^{9}$ m

So $\eta = D/d = 4 \times 10^{-8}$ rad

$\rho_c \approx 12$ m for visible light ($\lambda = 500$ nm)

Could observe interference with slits 12 m apart

(hard to achieve in practice)

Also:

Require 12 m diameter telescope to resolve disc

(over distances < 12 m, acts like point source)

Spatial Coherence Function (Hecht 12.3)

General way to calculate visibility:

Suppose interferometer samples source fields
$E_1 = E(\mathbf{r}_1)$
$E_2 = E(\mathbf{r}_2)$

In two slit interferometer, $\mathbf{r}_1$ and $\mathbf{r}_2$
= positions of slits

Let fields propagate, acquire phases ϕ_1 and ϕ_2
before overlapping

Final field

$$E_{\text{tot}} = E_1 e^{i\phi_1} + E_2 e^{i\phi_2}$$

Interference depends on $\epsilon = \phi_1 - \phi_2$:

$$\left\langle |E_{\text{tot}}|^2 \right\rangle = \left\langle |E_1|^2 \right\rangle + \left\langle |E_2|^2 \right\rangle + \langle E_1 E_2^* \rangle e^{i\epsilon} + \langle E_1^* E_2 \rangle e^{-i\epsilon}$$

In two-slit example, $\epsilon = ka\theta$
from extra propagation distance

Define *spatial coherence function*

$$\Gamma_{12} = \langle E_1 E_2^* \rangle = \langle E(\mathbf{r}_1) E^*(\mathbf{r}_2) \rangle$$

Analogous to $\Gamma(\tau)$

Here average =
average over random phases of source

Write $\Gamma_{12} = |\Gamma_{12}| e^{i\phi}$:

Then

$$\left\langle |E_{\text{tot}}|^2 \right\rangle = \Gamma_{11} + \Gamma_{22} + |\Gamma_{12}| e^{i(\phi+\epsilon)} + |\Gamma_{12}| e^{-i(\phi+\epsilon)}$$

$$= \Gamma_{11} + \Gamma_{22} + 2|\Gamma_{12}| \cos(\phi + \epsilon)$$

Here $\Gamma_{11} = \left\langle |E_1|^2 \right\rangle = 2\eta_0 I_1$
similar for Γ_{22}

So Γ_{12} determines interference pattern

For point source, don't need to average:

$$|\Gamma_{12}| = |E_1||E_2| = \sqrt{\Gamma_{11}\Gamma_{22}}$$

So define *complex degree of spatial coherence*

$$\gamma_{12} = \frac{\Gamma_{12}}{\sqrt{\Gamma_{11}\Gamma_{22}}}$$

Have $0 < |\gamma_{12}| < 1$

Visibility of interference pattern is

$$\mathcal{V} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |\gamma_{12}|$$

just as for temporal coherence

van-Cittert–Zernike Theorem (Hecht 12.3.1)

For temporal coherence, can't calculate $\Gamma(\tau)$

But can calculate Γ_{12} from source geometry

Characterize source by brightness $B(X,Y)$

Recall $B = \dfrac{\text{power}}{\text{solid angle} \cdot \text{area}}$ emitted by source

So $B(X,Y)\,dX dY$ = W/srad
emitted by area $dX\,dY$ on source

Suppose $B(X,Y)$ has transform $\mathcal{B}(k_x,k_y)$

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are a distance d from source
with $d \gg |\mathbf{r}_1 - \mathbf{r}_2|$ and $d \gg$ source size

Then find

$$\boxed{\gamma_{12} = \frac{1}{\mathcal{L}}\,\mathcal{B}\left[\frac{k(x_1-x_2)}{d}, \frac{k(y_1-y_2)}{d}\right]}$$

with $\mathcal{L} = \mathcal{B}(0,0) = \iint B(X,Y)\,dX\,dY$

Same as Fraunhofer diffraction pattern produced by
aperture equal to source

Won't prove, but saw example already:

Line source length b subtends angle $\eta = b/d$

From previous calculation

$$\gamma_{12} = \text{sinc}\left(\frac{ka\eta}{2}\right) = \text{sinc}\left[\frac{k(x_1 - x_2)b}{2d}\right]$$

Same as single-slit diffraction pattern

So circular source diameter D gives

$$\gamma_{12} = \frac{4d}{k\rho D} J_1\left(\frac{k\rho D}{2d}\right)$$

for $\rho = |\mathbf{r}_1 - \mathbf{r}_2|$

Mutual Coherence Function (Hecht 12.3)

In general, both spatial and temporal fluctuations

Define *mutual coherence function*

$$\Gamma_{12}(\tau) = \langle E(\mathbf{r}_1, t + \tau) E^*(\mathbf{r}_2, t) \rangle$$

Here average over time and source phases

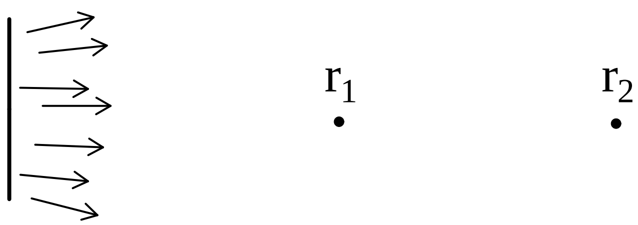
Define *complex degree of coherence*

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}}$$

If interferometer samples source field at $\mathbf{r}_1$, $\mathbf{r}_2$
and interfers fields with time delay τ:

Get visibility $\mathcal{V} = |\gamma_{12}(\tau)|$

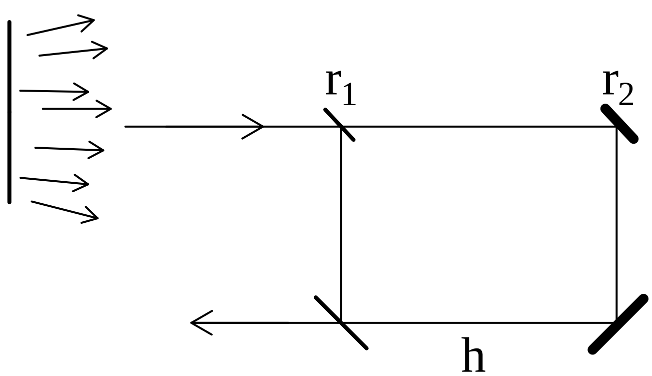
Note that space and time coherence intermixed



Here $\mathbf{r}_1$ sees field at earlier time that $\mathbf{r}_2$
temporal fluctuations contribute to $\gamma_{12}(0)$

Use τ only for time delay after sampling field

Example:



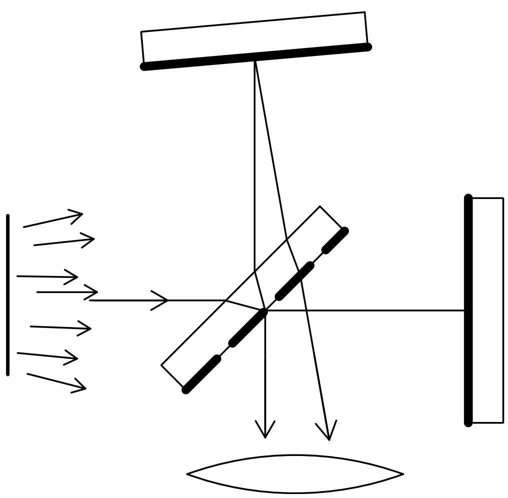
Here want $\gamma_{12}(\tau)$ for $\tau = h/c$

Note van Cittert-Zernike theorem only applies to
nearly monochromatic source

Michelson Interferometer

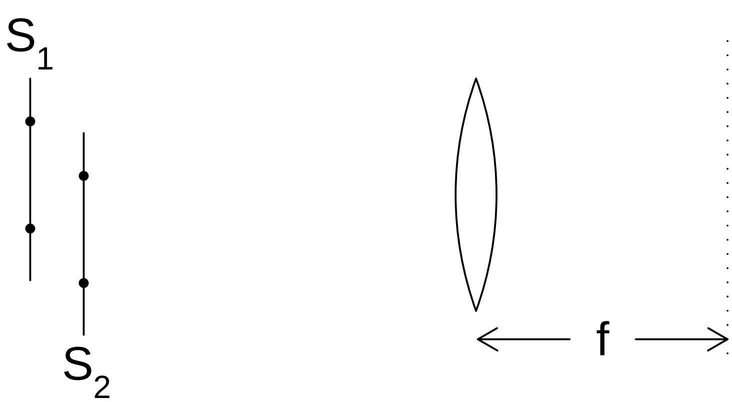
Michelson is special case:

spatial coherence unnecessary



27

To analyze, look at virtual sources produced by mirrors:



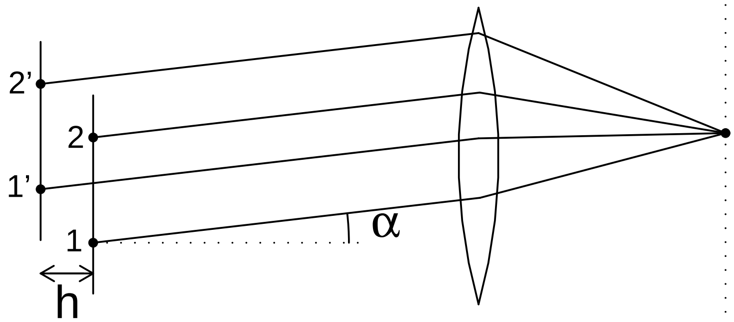
Sources separated by path length difference h

And displaced by θd

d = arm length, θ = mirror tilt

Detect at focal plane of lens

All rays at angle α imaged to point $\mathbf{r}$



Light from $\mathbf{r}_1$ interfers only with light from $\mathbf{r}_1'$

Path difference 11' $= h/\cos\alpha$

Same as path difference 22'

So light from all points reaching $\mathbf{r}$ has same phase

See interference pattern as function of α

In Michelson demo:
source = frosted glass illuminated with laser
lens = camera lens
detector = CCD

Note, no interference observed without lens

Lens is good trick to recover interference

Summary:

- Extended source usually decreases $\mathcal{V}$
- Characterize spatial coherence with Γ_{12}
  $\Gamma_{12} = \langle E(\mathbf{r}_1)E(\mathbf{r}_2)^*\rangle$
- Lateral coherence length $\rho_c \approx \lambda/\eta$
  $\eta =$ angle subtended by source
- Get Γ_{12} from van Cittert-Zernike
- Both time and space coherence: $\Gamma_{12}(\tau)$
- Michelson works with extended source
  when using lens