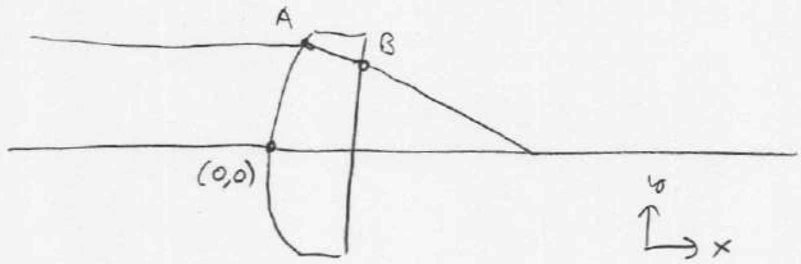
Phys 531 Assignment 6 Solutions

1. "Correct" Orientation



First find point A = intersection with first surface

Equation for surface $(x-R)^2 + y^2 = R^2$

$$R = 50 \text{ mm}$$

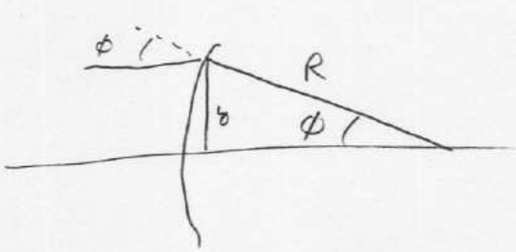
$$x - R = -\sqrt{R^2 - y^2}$$

$$x = R - \sqrt{R^2 - y^2}$$

at $y = 10$ mm, $x = 50 - \sqrt{50^2 - 10^2} = 1.010$ mm

So $A = (1.01, 10)$ mm

At A, normal to surface makes angle ϕ



$$\sin\phi = \frac{y}{R} = 0.2$$

$$\phi = 11.537^\circ$$

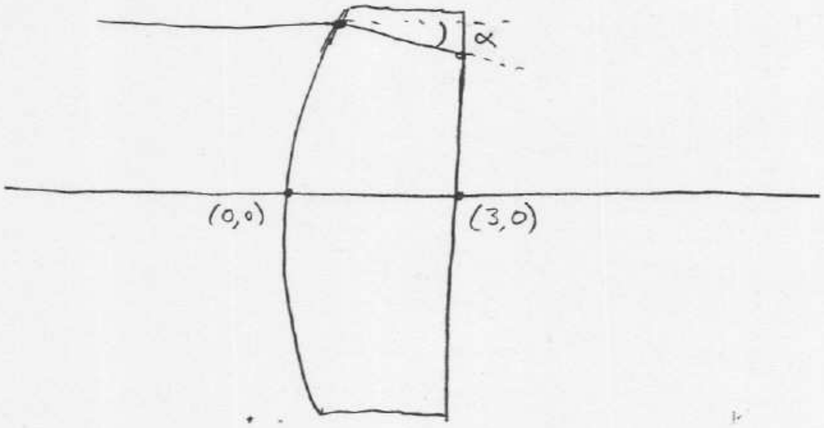
So $\theta_i = 11.537^\circ$

use $\sin\theta_i = n \sin\theta_t$

$$\sin\theta_t = \frac{0.2}{n} = 0.1333$$

$$\theta_t = 7.662^\circ$$

So have



Angle of ray in glass from horizontal is

$$\alpha = \Theta_t - \phi = -3.875^\circ$$

Ray travels a distance $\Delta x = 3 - 1.01 = 1.99\,mm$

Drops a distance $\Delta y = \Delta x \tan\alpha$

$$= -1.99\,mm \cdot \tan(3.875^\circ)$$

$$= -0.1348\,mm$$

So point $B = (3,\ 9.865)\,mm$

Angle of incidence on second surface $= \alpha$

So $\quad n \sin\alpha = \sin\alpha_{out}$

$$\sin\alpha_{out} = 1.5 \sin(-3.875^\circ)$$

$$= -0.1014$$

$$\alpha_{out} = -5.818^\circ$$

Calculate back focal length

$$M_v = \begin{bmatrix} 1 & 0 \\ \frac{3}{1.5} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1-1.5}{50} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.01 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.01 \\ 2 & 0.98 \end{bmatrix}$$

$$bfl = -\frac{D}{B} = 98 \text{ mm}$$

So need height of our ray after travelling 98 mm

Again, $\Delta y = \tan \alpha_{out} \Delta x$

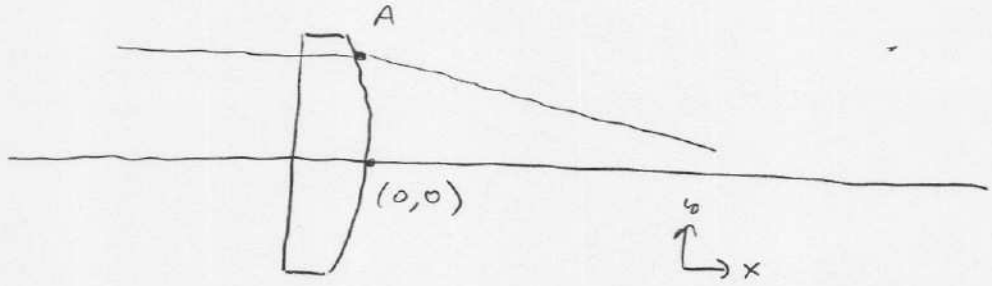
$$= -0.1019 \times 98 = -9.986 \text{ mm}$$

So $h = y_{out} + \Delta y$

$$= 9.865 - 9.986 = \boxed{-0.121 \text{ mm}}$$

Estimate geometrical spot size = 120 μm

Now "incorrect" orientation



First surface has no effect

Find intersection point A

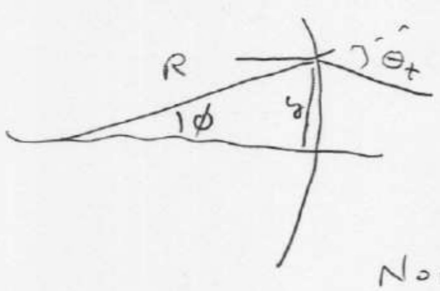
$$(x+R)^2 + y^2 = R^2$$

$$x = \sqrt{R^2 - y^2} - R$$

$$R = 50 \text{ mm} \qquad y = 10 \text{ mm}$$

$$\Rightarrow x = -1.01 \text{ mm} \quad (\text{as before})$$

Surface normal at angle ϕ



$$\sin\phi = \frac{y}{R} = \frac{10}{50}$$

$$\phi = 11.537^\circ \text{ as before}$$

$$= \theta_i$$

Now $n \sin\theta_i = \sin\theta_t$

$$\Rightarrow \theta_t = 17.458^\circ$$

Then $\alpha_{out} = \phi - \theta_t = -5.921^\circ$

Again need back focal length

= front focal length from previous orientation

$$ffl = -\frac{A}{B} = 100\text{ mm}$$

So want h at 100 mm travel

$$\Delta y = 100\text{mm} \tan \alpha_{out}$$

$$= -10.370\text{ mm}$$

$$h = y_{out} - \Delta y = 10\text{mm} - 10.37\text{mm}$$

$$= \boxed{-0.37\text{ mm}}$$

Estimated spot size now $370\mu m$

$\approx 3\times$ bigger!

Again need back focal length

= front focal length from previous orientation

$$ffl = -\frac{A}{B} = 100\ mm$$

Measure bfl from back vertex, so travel distance

$\Delta x = 100mm + 1.01mm = 101mm$

$$\Delta y = 101\ mm \tan \alpha_{out}$$

$$= -10.470\ mm$$

$$h = y_{out} + \Delta y = 10mm - 10.47mm$$

$$= \boxed{-0.47\ mm}$$

Estimated spot size now $470 \mu m$

$\approx 4\times$ bigger!

2. $E_1 = A e^{i(\vec{k}_1 \cdot \vec{r} - \omega t)}$

$E_2 = A e^{i(\vec{k}_2 \cdot \vec{r} - \omega t)}$

$$E_1 + E_2 = A e^{-i\omega t}\left[e^{i\vec{k}_1 \cdot \vec{r}} + e^{i\vec{k}_2 \cdot \vec{r}} \right]$$

$$|E_{TOT}|^2 = |A|^2 \left[1 + 1 + e^{i(\vec{k}_1 - \vec{k}_2)\cdot\vec{r}} + e^{i(\vec{k}_2 - \vec{k}_1)\cdot\vec{r}} \right]$$

$$= |A|^2 \left\{ 2 + 2\cos\left[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} \right] \right\}$$

$$\vec{k}_1 - \vec{k}_2 = k_0(\cos\theta\,\hat{z} + \sin\theta\,\hat{x}) - k_0(\cos\theta\,\hat{z} - \sin\theta\,\hat{x})$$

$$= 2k_0 \sin\theta\,\hat{x}$$

So

$$|E_{TOT}|^2 = 2|A|^2 \left[1 + \cos(2k_0 x \sin\theta) \right]$$

use $1 + \cos 2\beta = 2\cos^2\beta$

$$\boxed{|E_{TOT}|^2 = 4|A|^2 \cos^2(k_0 x \sin\theta)}$$

Nodes when $k_0 x \sin\theta = (n + \frac{1}{2})\pi$

$$\boxed{\Delta x = \frac{\pi}{k_0 \sin\theta} = \frac{\lambda}{2\sin\theta}}$$

3. Just calculate the interference pattern

$$|E_{TOT}|^2 = E_{TOT}\, E_{TOT}^*$$

$$= A\left(e^{i(k_1 z - \omega_1 t)} + e^{i(k_2 z - \omega_2 t)}\right) A^*\left(e^{-i(k_1^* z - \omega_1 t)} + e^{-i(k_2^* z - \omega_2 t)}\right)$$

$$= |A|^2 \Big\{ e^{i(k_1 - k_1^*)z} + e^{i(k_2 - k_2^*)z}$$

$$+ e^{i[(k_1 - k_2^*)z - (\omega_1 - \omega_2)t]} + e^{i[(k_2 - k_1^*)z - (\omega_2 - \omega_1)t]} \Big\}$$

where $k = \frac{\tilde{n}\omega}{c} = \frac{n\omega}{c} + i\alpha$

So
$$k_1 - k_1^* = 2i\alpha$$
$$k_2 - k_2^* = 2i\alpha$$

$$k_1 - k_2^* = \frac{n_1\omega_1}{c} + i\alpha - \frac{n_2\omega_2}{c} + i\alpha$$

$$\equiv k_m + 2i\alpha \qquad k_m = \frac{n_1\omega_1 - n_2\omega_2}{c}$$

$$k_2 - k_1^* = \frac{n_2\omega_2}{c} + i\alpha - \frac{n_1\omega_1}{c} + i\alpha$$

$$= -k_m + 2i\alpha$$

$$|E_{TOT}|^2 = |A|^2 \Big[e^{-2\alpha z} + e^{-2\alpha z} + e^{-2\alpha z} e^{i(k_m z - \omega_m t)}$$
$$+ e^{-2\alpha z} e^{-i(k_m z - \omega_m t)} \Big]$$

$$\omega_m = \omega_1 - \omega_2$$

So $$\boxed{|E_{TOT}|^2 = 2|A|^2 e^{-2\alpha z}\left[1 + \cos(k_m z - \omega_m t)\right]}$$

Propagates at speed $v_g = \frac{\omega_m}{k_m}$

$$= \frac{\omega_1 - \omega_2}{n_1\omega_1 - n_2\omega_2} c$$

$\therefore$ Depends only on real part of n's.

4. a) $f(t) = e^{-\gamma|t|}$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$= \int_{-\infty}^{0} e^{\gamma t} e^{i\omega t} dt + \int_{0}^{\infty} e^{-\gamma t} e^{i\omega t} dt$$

$$= \left.\frac{e^{(\gamma + i\omega)t}}{\gamma + i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{(-\gamma + i\omega)t}}{-\gamma + i\omega}\right|_{0}^{\infty}$$

$$= \frac{1}{\gamma + i\omega} - \frac{1}{-\gamma + i\omega}$$

$$= \frac{(-\gamma + i\omega) - (\gamma + i\omega)}{(\gamma + i\omega)(-\gamma + i\omega)} = \frac{-2\gamma}{-\gamma^2 - \omega^2}$$

$$\boxed{F(\omega) = \frac{2\gamma}{\gamma^2 + \omega^2}}$$

b) $\Delta t =$ FWHM $e^{-\gamma|t|}$

$f(t) = \frac{1}{2} f(0)$ when $\gamma |t| = \ln 2$

$t = \pm \frac{\ln 2}{\gamma}$

FWHM $= \frac{2 \ln 2}{\gamma} \approx \frac{1.4}{\gamma}$

$\Delta\omega =$ FWHM $\frac{2\gamma}{\omega^2 + \gamma^2}$

$F(\omega) = \frac{1}{2} F(0)$ when $\frac{2\gamma}{\omega^2+\gamma^2} = \frac{1}{\gamma}$

$\omega^2 + \gamma^2 = 2\gamma^2$

$\omega^2 = \gamma^2$

$\omega = \pm\gamma$

FWHM $= 2\gamma$

Then $\Delta t \, \Delta\omega = \frac{1.4}{\gamma} \cdot 2\gamma = \boxed{2.8}$

c) $\int_{-\infty}^{\infty} |f|^2 dt = \int_{-\infty}^{\infty} e^{-2\gamma|t|} dt = 2\int_0^{\infty} e^{-2\gamma t} dt$

$= \frac{2}{2\gamma} \int_0^{\infty} e^{-u} du$

$$\boxed{\int_{-\infty}^{\infty} |f|^2 dt = \frac{1}{\gamma}}$$

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} \frac{4\gamma^2}{(\omega^2+\gamma^2)^2} \, d\omega$$

$$= \frac{4}{\gamma} \int_{-\infty}^{\infty} \frac{1}{(1+u^2)^2} \, du$$

$$= \frac{8}{\gamma} \int_{0}^{\infty} \frac{1}{(1+u^2)^2} \, du$$

$$u = \tan\theta$$

$$du = \sec^2\theta \, d\theta$$

$$1 + u^2 = 1 + \tan^2\theta = \sec^2\theta$$

$$= \frac{8}{\gamma} \int_{0}^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^4\theta}$$

$$= \frac{8}{\gamma} \int_{0}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{8}{\gamma} \int_{0}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= \frac{8}{\gamma} \cdot \frac{\pi}{4} = \frac{2\pi}{\gamma}$$

$$\boxed{\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{2\pi}{\gamma} = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt}$$

d) $g(t) = e^{-i\omega_0 t} f(t)$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{i\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega_0 t} e^{-\gamma|t|} e^{i\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(\gamma + i\omega - i\omega_0)t} dt + \int_{0}^{\infty} e^{(-\gamma + i\omega - i\omega_0)t} dt$$

$$= \frac{1}{\gamma + i\omega - i\omega_0} - \frac{1}{-\gamma + i\omega - i\omega_0}$$

$$= \frac{[-\gamma + i(\omega - \omega_0)] - [\gamma + i(\omega - \omega_0)]}{[\gamma + i(\omega - \omega_0)][-\gamma + i(\omega - \omega_0)]}$$

$$= \frac{-2\gamma}{-\gamma^2 - (\omega - \omega_0)^2}$$

$$\boxed{G(\omega) = \frac{2\gamma}{\gamma^2 + (\omega - \omega_0)^2} = F(\omega - \omega_0)}$$