Write $f(x) = e^{i\beta x}\left[f_1(x) + f_2(x)\right]$

$f_1(x) =$ pulse width a centered at zero

So transform $F_1(k) = a \operatorname{sinc} \frac{ka}{2}$

$f_2(x) =$ pulse with a centered at $x = b$

So $F_2(k) = e^{-ikb} \operatorname{sinc} \frac{ka}{2}$

using translation property of Fourier transform

[From notes: $f(t+\tau) \to e^{-i\omega\tau} F(\omega)$
So $f(x+b) \to e^{+ikb} F(k)$ since signs opposite
Then $f(x-b) \to e^{-ikb} F(k)$]

Finally, multiplying by $e^{i\beta x}$ shifts $k \to k-\beta$

So
$$F(k) = a \operatorname{sinc}\left[\frac{(k-\beta)a}{2}\right] + a e^{-i(k-\beta)b} \operatorname{sinc}\left[\frac{(k-\beta)a}{2}\right]$$

$$\boxed{F(k) = a \operatorname{sinc}\left[\frac{(k-\beta)a}{2}\right]\left[1 + e^{-i(k-\beta)b}\right]}$$

2. Have $g(t) = \int_{-\tau_1}^{\tau_1} f_2(t-T)\,dT$

$$u = t - T$$

$$g(t) = \int_{t-\tau_1}^{t+\tau_1} f_2(u)\,du$$

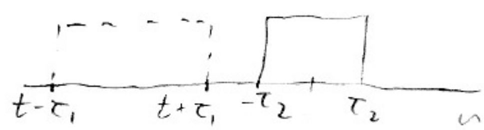
Value depends on how $t+\tau_1$ and $t-\tau_1$ compare to $\pm\tau_2$

Break into parts as suggested.

a) If $t < -\tau_1 - \tau_2$

then $t + \tau_1 < -\tau_2$, and $f(u) = 0$ over entire range

So $\boxed{g(t) = 0}$



b) If $-\tau_1 - \tau_2 < t < \tau_2 - \tau_1$

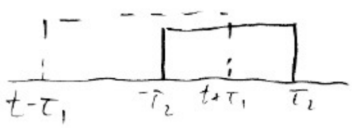
Then $t + \tau_1 > -\tau_2$

but $t - \tau_1 < \tau_2 - 2\tau_1 < -\tau_2$

(since $\tau_1 > \tau_2$)

So $f(u) = 1$ at top of range, but $f(u) = 0$ at bottom

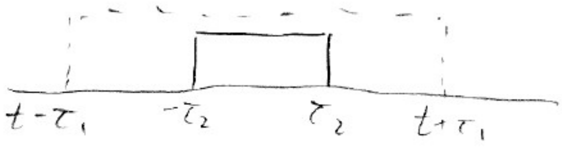
So have $g(t) = \int_{-\tau_2}^{t+\tau_1} du = \boxed{t + \tau_1 + \tau_2}$



c) If $\tau_2 - \tau_1 < t < \tau_1 - \tau_2$

Then $t + \tau_1 > \tau_2$: $f(u) = 0$ at top of range

and $t - \tau_1 < -\tau_2$: $f(u) = 0$ at bottom of range



$$g(t) = \int_{-\tau_2}^{\tau_2} dt = \boxed{2\tau_2}$$

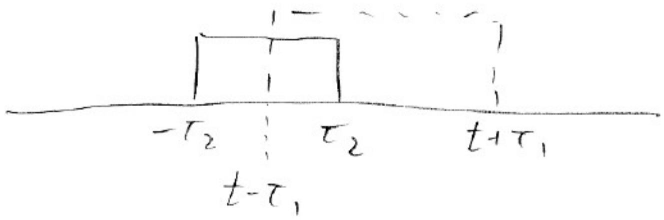
d) If $\tau_1 - \tau_2 < t < \tau_1 + \tau_2$

Then $t + \tau_1 > 2\tau_1 - \tau_2 > \tau_2$

so $f(u) = 0$ at top of range

$t - \tau_1 < \tau_2$

so $f(u) = 1$ at bottom of range



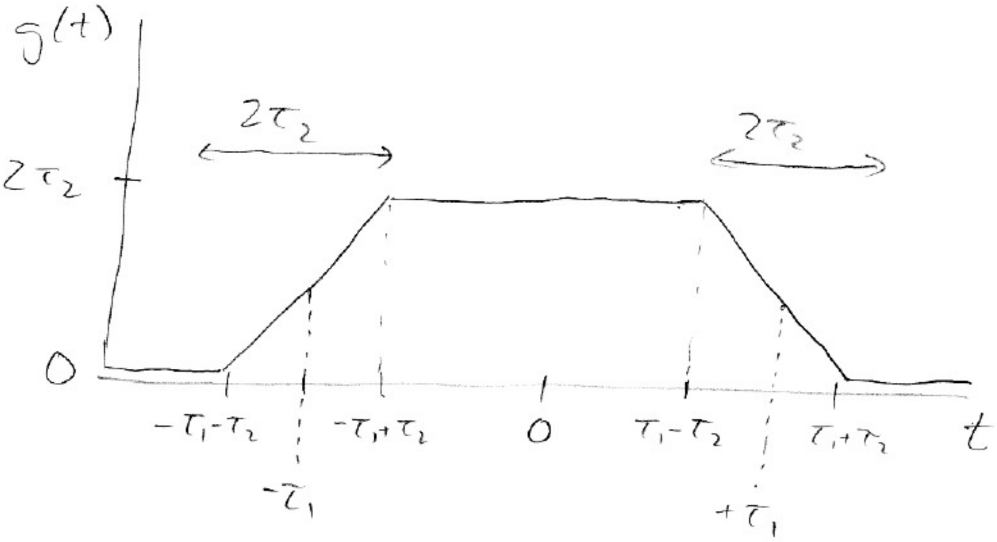
$$g(t) = \int_{t-\tau_1}^{\tau_2} dt = \boxed{\tau_2 + \tau_1 - t}$$

e) If $t > \tau_1 + \tau_2$

Then $t - \tau_1 > \tau_2$: $f(u) = 0$ accross whole range

$$g = 0$$

Thus have:



Full width at half max is $2\tau_1$

Edges "blurred" over time $2\tau_2$

3. For all, use $e^{i(k_x x + k_y y)} \to e^{i(k_x x + k_y y + \varkappa z)}$

$\varkappa = \sqrt{k^2 - k_x^2 - k_y^2}$

a) $k_x = 0 \quad k_y = \frac{k}{4} \Rightarrow \varkappa = \sqrt{k^2 - \frac{k^2}{16}} = k\,\frac{\sqrt{15}}{4}$

$$\boxed{E(\vec{r}) = e^{i\frac{k}{4}[y + z\sqrt{15}]}}$$

b) $k_x = -\frac{1}{2} \quad k_y = -\frac{1}{2} \Rightarrow \varkappa = \sqrt{k^2 - \frac{k^2}{4} - \frac{k^2}{4}} = k\sqrt{\frac{1}{2}}$

$$\boxed{E(\vec{r}) = e^{ik\left[-\frac{x}{2} - \frac{y}{2} + \frac{z}{\sqrt{2}}\right]}}$$

c) $k_x = k \quad k_y = k \Rightarrow \varkappa\ \sqrt{k^2 - k^2 - k^2} = ik$

$$\boxed{E(\vec{r}) = e^{ik(x+y)}\,e^{-kz}}$$

d) $\sin^2 \frac{ky}{4} = \frac{1}{2}\left(1 - \cos\frac{ky}{2}\right)$

$= \frac{1}{4}\left[2 - e^{i\frac{ky}{2}} - e^{-i\frac{ky}{2}}\right]$

First term: $k_x = k_y = 0 \Rightarrow \varkappa = k$

Second, third terms: $k_x = 0,\ k_y = \pm\frac{k}{2} \Rightarrow \varkappa = \sqrt{k^2 - \frac{k^2}{4}} = k\,\frac{\sqrt{3}}{2}$

$$E(\vec{r}) = \frac{1}{4}\left[2e^{ikz} - e^{i\frac{k}{2}(y + z\sqrt{3})} - e^{i\frac{k}{2}(-y + z\sqrt{3})}\right]$$

$$= \boxed{\frac{1}{2}\left[e^{ikz} - e^{ikz\frac{\sqrt{3}}{2}}\cos\frac{ky}{2}\right]}$$

4. In Fraunhoffer pattern,

$$I(x,y) \propto \left|\mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right)\right|^2$$

If $\mathcal{A}(k_x, k_y) = 0$ for $\sqrt{k_x^2 + k_y^2} > k_{max}$

Then $I(x,y) = 0$ for $\frac{k^2x^2}{d^2} + \frac{k^2y^2}{d^2} > k_{max}^2$

$$\frac{x^2+y^2}{d^2} > \frac{k_{max}^2}{k^2}$$

But $\tan\theta = \frac{\sqrt{x^2+y^2}}{d}$

So need $\tan\theta < \frac{k_{max}}{k} = \frac{10^6\,m^{-1}\cdot\lambda}{2\pi}$

$$= 0.1 \text{ rad}$$

$$\boxed{\theta < 0.1 \text{ rad}}$$

So diffraction pattern fits inside cone with this angle

5. Need transform $\mathcal{A}(\omega)$

From tables:

$$\mathcal{A}(\omega) = E_0 \tau \sqrt{\pi}\, e^{-\tau^2(\omega-\omega_0)^2/4}$$

So $$A(t) = E_0 \tau \sqrt{\pi} \int_{-\infty}^{\infty} e^{-\tau^2(\omega-\omega_0)^2/4} e^{-i\omega t} \frac{d\omega}{2\pi}$$

Each $e^{-i\omega t} \to e^{-i\omega t} \times e^{ik(\omega)d}$

So

$$A_{out}(t) = E_0 \tau \sqrt{\pi} \int_{-\infty}^{\infty} e^{-\frac{\tau^2(\omega-\omega_0)^2}{4}} e^{-i\omega t} e^{ik(\omega)d} \frac{d\omega}{2\pi}$$

Using form $k = k_0 + k_1 \Delta + \frac{1}{2} k_2 \Delta^2$

$$A_{out} = E_0 \tau \sqrt{\pi} \int_{-\infty}^{\infty} e^{-\frac{\tau^2 \Delta^2}{4}} e^{-i\omega t} e^{ik_0 d} e^{idk_1\Delta} e^{i\frac{1}{2}dk_2\Delta^2} \frac{d\omega}{2\pi}$$

Change variables to Δ:

$$e^{-i\omega t} \to e^{-i\omega_0 t} e^{-i\Delta t}$$

$$d\omega \to d\Delta$$

$$= E_0 \tau \sqrt{\pi}\, e^{i(k_0 d - \omega_0 t)} \int_{-\infty}^{\infty} e^{-\frac{\Delta^2}{4}(\tau^2 - 2idk_2)} e^{-i\Delta(t-dk_1)} \frac{d\Delta}{2\pi}$$

Define $q^2 = \tau^2 - 2idk_2$

$t' = t - dk_1$

Then

$$A_{out} = E_0 \tau \sqrt{\pi}\, e^{i(k_0 d - \omega_0 t)} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{q^2\Delta^2}{4}} e^{-i\Delta t'} \frac{d\Delta}{2\pi}}$$

Inverse transform of Gaussian

$$\rightarrow \frac{1}{q\sqrt{\pi}} e^{-t'^2/q^2}$$

$$A_{out}(t) = E_0 \frac{\tau}{q} e^{i(k_0 d - \omega_0 t)} e^{-t'^2/q^2}$$

Use $\frac{1}{q^2} = \frac{1}{\tau^2 - 2idk_2} = \frac{\tau^2 + 2idk_2}{\tau^4 + 4d^2k_2^2}$

$$= \frac{1 + 2idk}{\tau^2 + \frac{4d^2k_2^2}{\tau^2}}$$

$$= \frac{1 + 2idk_2}{T^2}$$

$$T^2 = \tau^2 + \frac{4d^2k_2^2}{\tau^2}$$

Then

$$\boxed{A_{out}(t) = E_0 \frac{\tau}{q} e^{i(k_0 d - \omega_0 t)} e^{-\frac{(t-dk_1)^2}{T^2}} e^{-i\frac{(t-dk_1)^2 2dk_2}{T^2}}}$$

a) Peak exits medium at $\boxed{t = dk_1}$

Group velocity $v_g = \frac{d}{t} = \frac{1}{k_1} = \frac{d\omega}{dk}$

b) Duration after medium is

$$\boxed{T = \sqrt{\tau^2 + \frac{4d^2k_2^2}{\tau^2}} > \tau}$$

So pulse lengthens in medium