Phys 532 Assignment 8 Solutions

1. As usual, start with aperture function.

Here incident wave is

$$E_{inc} = E_0 e^{i(k\cdot r - \omega t)}$$
$$= E_0 e^{i(kz\cos\theta + kx\sin\theta - \omega t)}$$
$$\simeq E_0 e^{i(kz + k\theta x - \omega t)}$$

At $z=0$ (plane of aperture), and dropping $e^{-i\omega t}$

get $E(x,y,0) = E_0 e^{ik\theta x}$

So aperture function is

$$A(x,y) = \begin{cases} E_0 e^{ik\theta x} & (|x| < \frac{b}{2},\ |y| < \frac{L}{2}) \\ 0 & (\text{else}) \end{cases}$$

Need transform $\mathcal{A}(k_x, k_y)$

Note $A(x,y) = E_0 e^{ik\theta x} \times$ pulse function

So $\mathcal{A}(k_x,k_y) = E_0 bL \operatorname{sinc}\left[\frac{(k_x - k\theta)b}{2}\right] \operatorname{sinc}\left[\frac{k_y L}{2}\right]$

using translation property

Fraunhofer formula gives

$$A_d(x,y) = \frac{C}{\lambda d}\,\mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right)$$

for $C = -i e^{ikd} e^{ik(x^2+y^2)/2d}$

Here

$$\mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right) = E_0 bL \operatorname{sinc}\left[\frac{\left(\frac{kx}{d} - k\Theta\right)b}{2}\right]\operatorname{sinc}\left[\frac{kyL}{2d}\right]$$

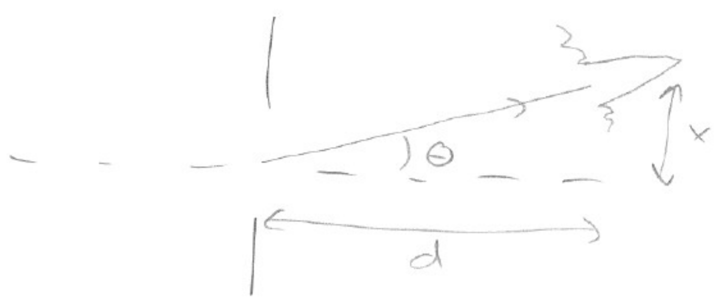
$$= E_0 bL \operatorname{sinc}\left[\frac{k(x-\Theta d)b}{2d}\right]\operatorname{sinc}\left[\frac{kyL}{2d}\right]$$

and

$$\boxed{|A_d(x,y)|^2 = \frac{b^2L^2}{\lambda^2 d^2}|E_0|^2 \operatorname{sinc}^2\left[\frac{k(x-\Theta d)b}{2d}\right]\operatorname{sinc}^2\left[\frac{kyL}{2d}\right]}$$

Centered at $x = \Theta d$, $y = 0$

So center travels at angle Θ, as claimed



2. a) Fresnel, use

$$E(x,y,z) = A_z(x,y) \iint A(X,Y)\, h(x-X, y-Y)\, dX\, dY$$

for $h(x,y) = -\frac{i}{\lambda d} e^{ikd} e^{ik(x^2+y^2)/2d}$

$$A_d = -\frac{i}{\lambda d} e^{ikd} \left\{ \iint \delta(X-a)\,\delta(Y)\, e^{ik[(x-X)^2+(y-Y)^2]/2d}\, dX\, dY + \iint \delta(X+a)\,\delta(Y)\, e^{ik[(x-X)^2+(y-Y)^2]/2d}\, dX\, dY \right\}$$

$$= -\frac{i}{\lambda d} e^{ikd} \left\{ e^{ik[(x-a)^2+y^2]/2d} + e^{ik[(x+a)^2+y^2]/2d} \right\}$$

$$= -\frac{i}{\lambda d} e^{ikd}\, e^{ik(x^2+y^2)/2d} \left\{ e^{-ikax/d} + e^{ikax/d} \right\}$$

$$\boxed{A_d = -\frac{2i}{\lambda d} e^{ikd}\, e^{ik(x^2+y^2)/2d} \cos\left(\frac{kax}{d}\right)}$$

b) Fraunhofer, need $\mathcal{A}(k_x, k_y)$

From table, $\delta(x-a) \to e^{-ik_x a}$ $\qquad \delta(y) \to 1$

$\delta(x+a) \to e^{+ik_x a}$

So $\mathcal{A}(k_x, k_y) = e^{ik_x a} + e^{-ik_x a} = 2\cos(k_x a)$

Then $A_d(x,y) = -\frac{i}{\lambda d} e^{ikd} e^{ik\frac{(x^2+y^2)}{2d}} \mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right)$

$$\boxed{A_d = -\frac{2i}{\lambda d} e^{ikd} e^{ik\frac{(x^2+y^2)}{2d}} \cos\left(\frac{kxa}{d}\right)}$$

Same as Fresnel!

Makes sense, if hole size $\to 0$, Fraunhofer valid for any d

3. Image of distant point source is Airy pattern

$$A_d(\rho) = C\left[\frac{2}{k_\rho a} J_1(k_\rho a)\right]$$

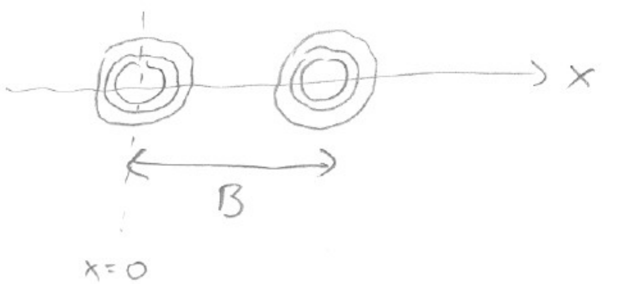
for $k_\rho \to \frac{k\rho}{f}$

$a \to \frac{D}{2}$

$C =$ constant, doesn't matter

$$A_d(\rho) = \frac{4f}{k\rho D} J_1\left(\frac{k\rho D}{2f}\right)$$

Consider field along line $y = 0$, with points displaced along x



Then $A_1(x) = \frac{4f}{kxD} J_1\left(\frac{kxD}{2f}\right)$ = first spot

$A_2(x) = \frac{4f}{k(x-B)D} J_1\left[\frac{k(x-B)D}{2f}\right]$ = second spot

For a) total $I \propto |A_1 + A_2|^2$

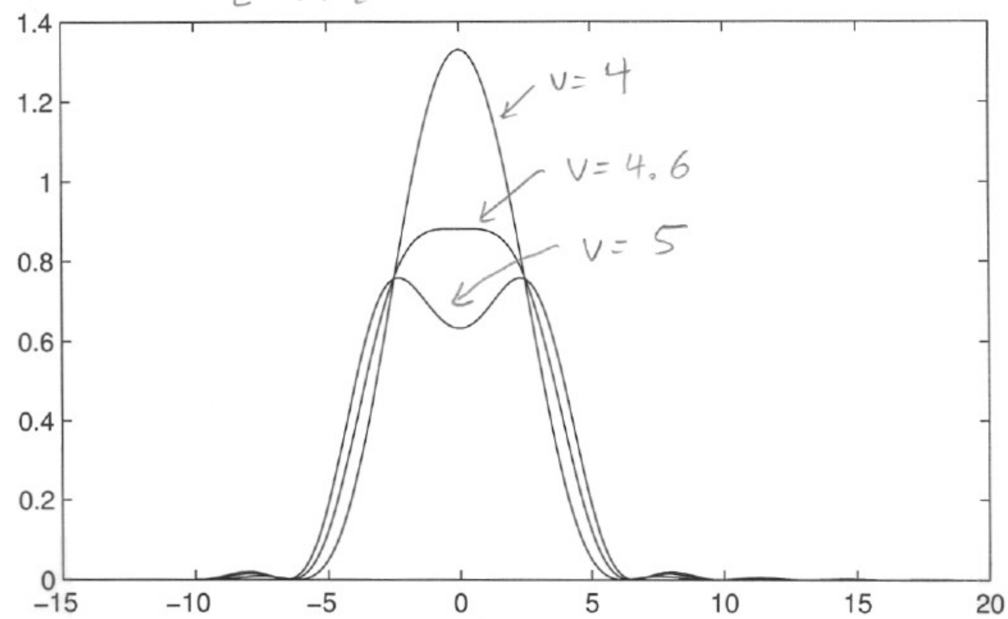
b) $I \propto |A_1|^2 + |A_2|^2$

So plot $\left| \frac{2J_1(u+\frac{v}{2})}{(u+\frac{v}{2})} + \frac{2J_1(u-\frac{v}{2})}{(u-\frac{v}{2})} \right|^2$ for various v:

(Make symmetric just so graph is pretty)

$$\left[\frac{2J_1(u+\frac{v}{2})}{u+\frac{v}{2}} + \frac{2J_1(u-\frac{v}{2})}{u-\frac{v}{2}} \right]^2$$

a)



See that peaks separate at $v = 4.6$

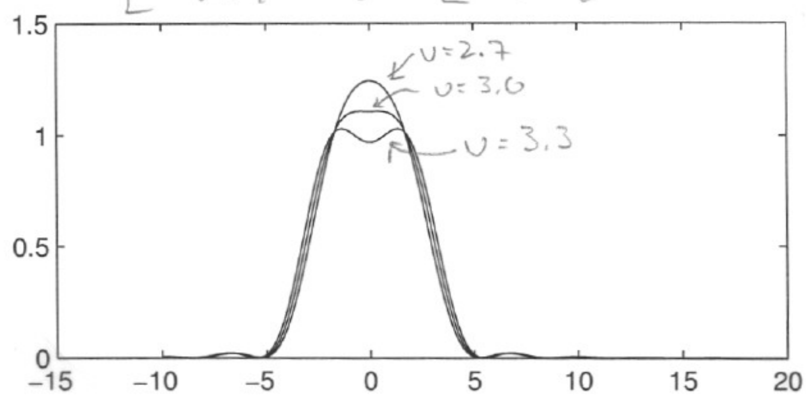
Or $$\frac{kBD}{2f} = 4.6$$

$$B = \frac{9.2f}{kD} = \frac{9.2}{2\pi}\frac{\lambda f}{D} = \boxed{1.47\,\frac{\lambda f}{D}}$$

Then plot $$\left[\frac{2J_1(u+\frac{v}{2})}{u+\frac{v}{2}}\right]^2 + \left[\frac{2J_1(u-\frac{v}{2})}{u-\frac{v}{2}}\right]^2$$

b)

$$\left[\frac{2J_1(u+\frac{v}{2})}{u+\frac{v}{2}}\right]^2 + \left[\frac{2J_1(u-\frac{v}{2})}{u-\frac{v}{2}}\right]^2$$



Peaks separate at $v = 3.0$

So $$\frac{kBD}{2f} = 3.0$$

$$B = \frac{6f}{kD} = \frac{6}{2\pi}\frac{\lambda f}{D} = \boxed{0.96\,\frac{\lambda f}{D}}$$

Matlab code:

a)

```
u = [-100.1:200.1]/10;
v = 4.6;
a1 = 2*besselj(1,u+v/2)./(u+v/2);
a2 = 2*besselj(1,(u-v/2))./(u-v/2);
ir = (a1+a2).^2; <--
plot(u,ir);
hold on
v = 4;
a1 = 2*besselj(1,u+v/2)./(u+v/2);
a2 = 2*besselj(1,(u-v/2))./(u-v/2);
ir = (a1+a2).^2;
plot(u,ir);
v = 5;
a1 = 2*besselj(1,u+v/2)./(u+v/2);
a2 = 2*besselj(1,(u-v/2))./(u-v/2);
ir = (a1+a2).^2;
plot(u,ir);
```

b)

```
u = [-100.1:200.1]/10;
v = 3;
a1 = 2*besselj(1,u+v/2)./(u+v/2);
a2 = 2*besselj(1,(u-v/2))./(u-v/2);
ir = (a1.^2+a2.^2); <--
plot(u,ir);
hold on
v = 3.3;
a1 = 2*besselj(1,u+v/2)./(u+v/2);
a2 = 2*besselj(1,(u-v/2))./(u-v/2);
ir = (a1.^2+a2.^2);
plot(u,ir);
v = 2.7;
a1 = 2*besselj(1,u+v/2)./(u+v/2);
a2 = 2*besselj(1,(u-v/2))./(u-v/2);
ir = (a1.^2+a2.^2);
plot(u,ir);
```

4. Fresnel approximation

Use convolution form

$$A_d(x,y) = \iint A(X,Y)\, h(x-X, y-Y)\, dX\, dY$$

$$h = -\frac{i}{\lambda d}\, e^{ikd}\, e^{ik\frac{(x^2+y^2)}{2d}}$$

Here $A(X,Y) = E_0 \dfrac{e^{-ik\sqrt{X^2+Y^2+d^2}}}{k\sqrt{X^2+Y^2+d^2}}$

but in Fresnel approximation, should expand square root

$\sqrt{d^2+X^2+Y^2} \to d + \dfrac{X^2+Y^2}{2d}$ in phase

$\to d$ in denominator

So $$A(X,Y) \approx E_0 \frac{e^{-ikd}\, e^{-ik(X^2+Y^2)/2d}}{kd}$$

$$A_d(x,y) = -\frac{i}{\lambda d} e^{ikd} E_0 \frac{e^{-ikd}}{kd}$$

$$\times \iint e^{-i\frac{k}{2d}(X^2+Y^2)}\, e^{+i\frac{k}{2d}\left[(x-X)^2+(y-Y)^2\right]} dX\,dY$$

Expand squared terms:

$$= -\frac{i}{2\pi d^2} E_0\, e^{i\frac{k}{2d}(x^2+y^2)} \iint e^{-i\left(\frac{kx}{d}X + \frac{ky}{d}Y\right)} dX\,dY$$

Recognize integral as Fourier transform

of plain circular aperture

Just like what we did in class, could cite that result

Go through it again here:

Write $k_x = \frac{kx}{d}$ $\quad k_y = \frac{ky}{d}$

$X = \rho\cos\phi \quad Y = \rho\sin\phi$

Do calculation for $k_y = 0$, since

$A(k_x, k_y)$ is cylindrically symmetric

Need

$$\int_0^{2\pi}\int_0^{D/2} e^{-ik_x\rho\cos\phi}\,\rho\,d\rho\,d\phi$$

ϕ integral gives Bessel function

$$\int_0^{2\pi} e^{-ik_x\rho\cos\phi}\,d\phi = 2\pi J_0(k_x\rho)$$

Then ρ integral gives J_1:

$$2\pi\int_0^{D/2} J_0(k_x\rho)\,\rho\,d\rho = \frac{2\pi}{k_x^2}\int_0^{\frac{k_xD}{2}} J_0(u)\,u\,du$$

$$= \frac{2\pi}{k_x^2} J_1\left(\frac{k_xD}{2}\right)\frac{k_xD}{2}$$

$$= \frac{\pi D}{k_x} J_1\left(\frac{k_xD}{2}\right)$$

So, $$A_d(x,y) = -\frac{i}{2\pi d^2} E_0\, e^{i\frac{k}{2d}(x^2+y^2)}\,\frac{\pi D d}{k\rho} J_1\left(\frac{k\rho D}{2d}\right)$$

$$\boxed{A_d(x,y) = -i\,\frac{D^2}{4d^2} E_0\, e^{i\frac{k}{2d}(x^2+y^2)}\left[\frac{2d}{k\rho D} J_1\left(\frac{k\rho D}{2d}\right)\right]}$$

Just like Fraunhofer result for plane wave incident on lens with $f=d$.

5 a) Need transform of 2D array

$$A(x,y) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} A_1(x-na,\, y-ma)$$

where
$$A_1(x,y) = E_0 \qquad \left(|x| < \tfrac{b}{2},\ |y| < \tfrac{b}{2}\right)$$
$$= 0 \qquad (\text{else})$$

Then
$$\mathcal{A}(k_x,k_y) = \iint A(x,y)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

$$= \sum_{n,m} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1(x-na,\, y-ma)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

$$x' = x - na \Rightarrow x = x' + na$$
$$y' = y - ma \Rightarrow y = y' + ma$$

$$\mathcal{A}(k_x,k_y) = \sum_{n,m} e^{-i(k_x na + k_y ma)} \int_{-b/2}^{b/2}\int_{-b/2}^{b/2} E_0\, e^{-i(k_x x' + k_y y')}\, dx'\, dy'$$

$$= \sum_{n,m} e^{-i(k_x na + k_y ma)}\ b^2 \operatorname{sinc}\frac{k_x b}{2} \operatorname{sinc}\frac{k_y b}{2}$$

$$= b^2 \operatorname{sinc}\frac{k_x b}{2} \operatorname{sinc}\frac{k_y b}{2} \sum_{n=0}^{N-1} e^{-ik_x na} \sum_{m=0}^{N-1} e^{-ik_y ma}$$

$$= b^2 \operatorname{sinc}\frac{k_x b}{2} \operatorname{sinc}\frac{k_y b}{2} \left(\frac{1 - e^{-ik_x aN}}{1 - e^{-ik_x a}}\right)\left(\frac{1 - e^{-ik_y aN}}{1 - e^{-ik_y a}}\right)$$

Just like in class. So

$$A(k_x,k_y) = b^2 e^{-ik_xa\left(\frac{N-1}{2}\right)} e^{-ik_ya\left(\frac{N-1}{2}\right)} \frac{\sin\frac{Nk_xa}{2}}{\sin\frac{k_xa}{2}} \cdot \frac{\sin\frac{Nk_ya}{2}}{\sin\frac{k_ya}{2}}$$
$$\times \operatorname{sinc}\left(\frac{k_xb}{2}\right)\operatorname{sinc}\left(\frac{k_yb}{2}\right)$$

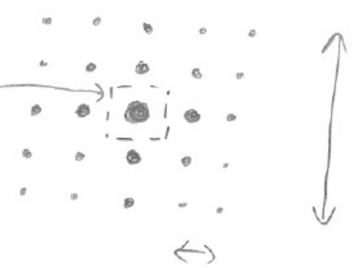
Diffraction pattern

$$\boxed{|A_d|^2 = \frac{b^4}{\lambda^2 d^2}\operatorname{sinc}^2\left(\frac{kxb}{2d}\right)\operatorname{sinc}^2\left(\frac{kyb}{2d}\right) \times \frac{\sin^2\left(\frac{Nkxa}{2d}\right)\sin^2\left(\frac{Nkya}{2d}\right)}{\sin^2\left(\frac{kxa}{2d}\right)\sin^2\left(\frac{kya}{2d}\right)}}$$

b) In focal plane, get pattern $|A_d|^2$ with $d \to f$

Looks like:

filter square



Total width $\approx \frac{4\pi f}{kb} = \frac{2\lambda f}{b}$

from sinc functions

Spacing

$$= \frac{2\pi f}{ka} = \frac{\lambda f}{a}$$

To get smooth image, want to block all but zero order spot

So want $\boxed{\rho \approx \frac{\lambda f}{a}}$

Put in numbers: $\rho = \frac{500\,\text{nm} \times 100\,\text{mm}}{0.5\,\text{mm}}$

$$\boxed{\rho = 100\,\mu\text{m} = 0.1\,\text{mm}}$$