Phys 532 Assignment 12 Solutions

1. In class, derived threshold intensity

$$I_3 = \frac{n_1 n_2 n_3}{2\omega_1\omega_2}\ \frac{1}{\zeta_0^3 d'^2}\ \frac{\Gamma^2}{\ell^2} \qquad \Gamma = \text{cavity loss} = L + T$$

here
$$= \frac{1.67\cdot 1.66\cdot 1.67}{2\,(2\pi c_0)^2}\ \frac{(550\,\text{nm})(890\,\text{nm})}{(377\Omega)^3 (1.6\times10^{-23}\,\text{C}/\text{V}^2)^2}\ \frac{0.1}{(0.01\,\text{m})^2}$$

$$= 2.4\times10^{10}\ \frac{\text{W}}{\text{m}^2}$$

Focus spot area $\pi w_0^2 = \dfrac{\lambda \ell}{2}$ use $\lambda_{max} = \dfrac{890\,\text{nm}}{1.66}$

$$= 2.7\times10^{-9}\ \text{m}^2$$

$$\boxed{P_t = 66.7\,\text{W}} = \frac{n_1 n_3}{4\omega_1\omega_2}\ \frac{\lambda_2}{\zeta_0^3 d'^2}\ \frac{\Gamma}{\ell}$$

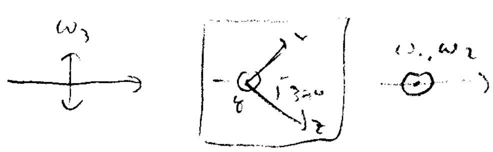
Also have

$$P_1(\text{out}) = \frac{T}{\Gamma}\ \frac{\omega_1}{\omega_3}\ (P_3 - P_t)$$

$$= \frac{1}{2}\ \frac{355\,\text{nm}}{550\,\text{nm}}\ (1000 - 66.7)$$

$$= \boxed{280\,\text{W}}$$

Oops, need d':



$d_{21} = d_{eff}$

$d' = d_{21}\times\cos 37^\circ = 1.6\times10^{-23}\ \text{C}/\text{V}^2$

$$\frac{1}{n_3^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2} \Rightarrow n_3 = 1.67$$

2. a) Have $n_3(\theta)\,\omega_3 = n_1(\omega_1)\,\omega_1(\theta) + n_2(\omega_2)\,\omega_2(\theta)$

Take derivative $\frac{d}{d\theta}$:

$$\omega_3 \frac{dn_3}{d\theta} = n_1 \frac{d\omega_1}{d\theta} + \omega_1 \frac{dn_1}{d\omega_1}\frac{d\omega_1}{d\theta} + n_2 \frac{d\omega_2}{d\theta} + \omega_2 \frac{dn_2}{d\omega_2}\frac{d\omega_2}{d\theta}$$

Since $\omega_2 = \omega_3 - \omega_1$ and ω_3 is fixed,

$$\frac{d\omega_2}{d\theta} = -\frac{d\omega_1}{d\theta}$$

$$\boxed{\omega_3 \frac{dn_3}{d\theta} = \left[n_1 - n_2 + \omega_1 \frac{dn_1}{d\omega} - \omega_2 \frac{dn_2}{d\omega}\right]\frac{d\omega_1}{d\theta}}$$

b) $$\frac{1}{n_3^2} = \frac{\cos^2\theta}{n_{3o}^2} + \frac{\sin^2\theta}{n_{3e}^2}$$

Take $\frac{d}{d\theta}$:

$$-2\frac{1}{n_3^3}\frac{dn_3}{d\theta} = \frac{-2\sin\theta\cos\theta}{n_{3o}^2} + \frac{2\sin\theta\cos\theta}{n_{3e}^2}$$

$$\boxed{\frac{dn_3}{d\theta} = \frac{n_3^3}{2}\sin 2\theta\left(\frac{1}{n_{3o}^2} - \frac{1}{n_{3e}^2}\right)}$$

So $$\boxed{\frac{d\omega_1}{d\theta} = \frac{\omega_3}{n_1 - n_2 + \omega_1\frac{dn_1}{d\omega_1} - \omega_2\frac{dn_2}{d\omega_2}} \cdot \frac{n_3^3}{2}\sin 2\theta\left(\frac{1}{n_{3o}^2} - \frac{1}{n_{3e}^2}\right)}$$

c)

Use $$\frac{dn}{d\omega} = \frac{dn}{d\lambda}\frac{d\lambda}{d\omega}$$

$$\lambda = \frac{2\pi c}{\omega}$$

$$\frac{d\lambda}{d\omega} = -\frac{2\pi c}{\omega^2} = -\frac{\lambda}{\omega}$$

$$= -\frac{dn}{d\lambda}\frac{\lambda}{\omega}$$

So $$\omega\frac{dn}{d\omega} = -\lambda\frac{dn}{d\lambda}$$

Also, $$\frac{d\lambda}{d\theta} = \frac{d\omega}{d\theta}\frac{d\lambda}{d\omega} = -\frac{d\omega}{d\theta}\cdot\frac{\lambda}{\omega}$$

So $$\frac{d\lambda_1}{d\theta} = -\frac{\lambda_1}{\omega_1}\left[\frac{\omega_3}{n_1-n_2-\lambda_1\frac{dn_1}{d\lambda_1}+\lambda_2\frac{dn_2}{d\lambda}}\cdot\frac{n_3^3}{2}\sin 2\theta\left(\frac{1}{n_{3o}^2}-\frac{1}{n_{3e}^2}\right)\right]$$

Need n_3: $$\frac{1}{n_3^2} = \frac{\cos^2\theta}{n_{3o}^2} + \frac{\sin^2\theta}{n_{3e}^2}$$

$$\Rightarrow n_3 = 1.668$$

$$\frac{d\lambda_1}{d\theta} = -\underbrace{\frac{(590\,nm)^2}{355\,nm}}_{980}\left[\frac{1}{\underbrace{(1.671-1.660)+(890\,nm-590\,nm)\times 3.3\times10^{-5}\,nm^{-1}}_{47.8}}\times\underbrace{\frac{(1.668)^3}{2}}_{2.32}\times\underbrace{\sin 74^\circ}_{.961}\times\underbrace{\left(\frac{1}{1.72^2}-\frac{1}{1.586^2}\right)}_{0.0595}\right]$$

$$= +6230\ \frac{nm}{rad}$$

So $$\Delta\theta = \frac{10}{6230\ nm/rad} = 1.6\ mrad = \boxed{0.092^\circ}$$

3. For third order process, write

$$P(\omega) = 3\chi^{(3)} E(\omega)\, |E(\omega')|^2$$

for effective $\chi^{(3)}$

Total polarization is

$$P(\omega) = \varepsilon_0 \chi_0 E(\omega) + 3\chi^{(3)} |E(\omega')|^2 E(\omega)$$

χ_0 = unperturbed susceptibility

So, define $\chi = \chi_0 + 3\frac{1}{\varepsilon_0}\chi^{(3)}|E(\omega')|^2$

But,

$$\chi = \tilde{n}^2 - 1$$

$$= n^2 - \frac{\gamma^2}{4k_0^2} + i\frac{\gamma n}{k_0} - 1$$

$$= n^2 - 1 + i\frac{\gamma n}{k_0}$$

So, identify $i\frac{\gamma n}{k_0} = \frac{1}{\varepsilon_0}\chi^{(3)}|E(\omega')|^2$

$$\chi^{(3)} = i\frac{\varepsilon_0}{3k_0} n\gamma \frac{1}{|E(\omega')|^2}$$

$$= i\frac{\varepsilon_0}{3k_0} n \frac{\lambda^2}{8\pi t_s \Delta\nu} \cdot t_s N_a \frac{\lambda'^2}{8\pi t_s' \Delta\nu'} \frac{I'}{\hbar\omega'} \frac{1}{|E(\omega')|^2}$$

use $I' = \frac{n'|E(\omega')|^2}{2\eta_0}$ $\qquad k_0 = \frac{2\pi}{\lambda}$

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

$$\chi^{(3)} = i\frac{\varepsilon_0}{6\pi}\frac{n\lambda^3\lambda'^2}{(8\pi)^2\Delta\nu\Delta\nu' t_s'}\frac{1}{\hbar\omega'}\frac{n'}{2\eta_0}N_a$$

$$\boxed{\chi^{(3)} = i\frac{\varepsilon_0 n n' \lambda^3 \lambda'^2}{3(8\pi)^3 \eta_0\, \hbar\omega'\, \Delta\nu\Delta\nu' t_s'} N_a}$$

4. a) Equation for mode is

$$X \frac{J_1(X)}{J_0(X)} = Y \frac{K_1(Y)}{K_0(Y)}$$

with $X^2 + Y^2 = V^2 = 4$

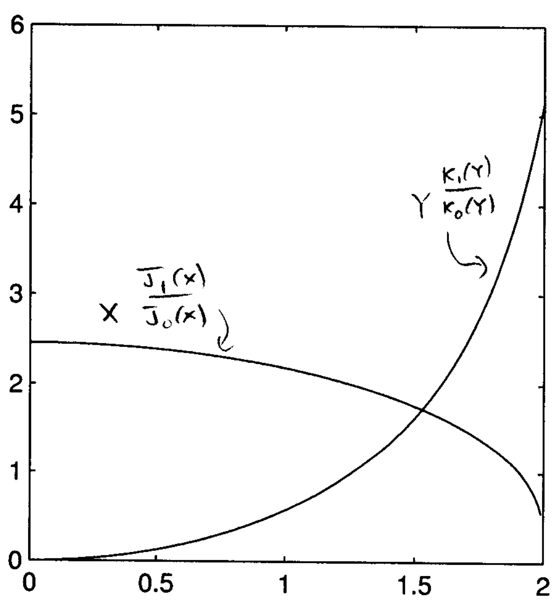
$X = k_T a$

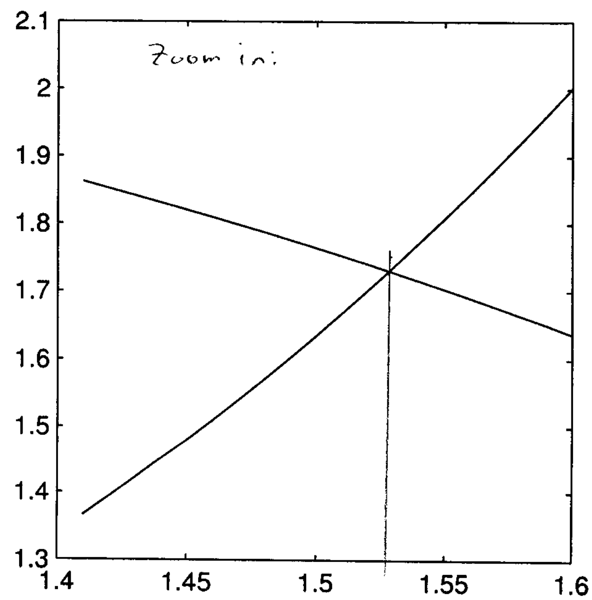
$Y = \gamma a$

So need to solve

$$X \frac{J_1(X)}{J_0(X)} = \sqrt{4-X^2} \; \frac{K_1(\sqrt{4-X^2})}{K_0(\sqrt{4-X^2})}$$

Plot in matlab, get $X = 1.53 \Rightarrow Y = 1.29$





So, $\boxed{\begin{aligned} k_T &= \frac{X}{a} = 1.02\,\mu m^{-1} \\ \gamma &= \frac{Y}{a} = 0.86\,\mu m^{-1} \end{aligned}}$

b) $\beta^2 = n_1^2 k_0^2 - k_T^2 = (1.5)^2(2\pi \times 1\,\mu m^{-1})^2 - (1\,\mu m^{-1})^2 \Rightarrow \beta = 9.4\,\mu m^{-1}$
$\approx n_1 k_0$

$$2\beta \frac{d\beta}{d\omega} = 2 n_1^2 k_0 \frac{dk_0}{d\omega} - 2 k_T \frac{dk_T}{d\omega}$$

$$k_0 = \frac{\omega}{c_0} \qquad k_T = \frac{X}{a}$$

so $$\frac{d\beta}{d\omega} = \frac{1}{\beta}\left[n_1^2 k_0 \frac{1}{c_0} - \frac{k_T}{a}\frac{dX}{d\omega}\right]$$

Then $$\frac{dX}{d\omega} = \frac{dX}{dV}\frac{dV}{d\omega}$$

$$V = k_0 a \cdot NA$$
$$= \omega \frac{a}{c_0} NA$$

So $$\frac{dV}{d\omega} = \frac{a}{c_0} NA$$

$$\frac{d\beta}{d\omega} = \frac{1}{c_0 \beta}\left[n_1^2 k_0 - k_T \cdot NA \cdot \frac{dX}{dV}\right]$$

To get $\frac{dX}{dV}$, find X again for $V = 2.1$

Get $X = 1.56$

So $$\frac{dX}{dV} \approx \frac{\Delta X}{\Delta V} = \frac{0.03}{0.1} = 0.3$$

So $$\frac{d\beta}{d\omega} = \frac{1}{c_0}\left[n_1 - \frac{k_T \cdot NA}{n_1 k_0}\, 0.3\right]$$

$$\text{Get } NA = \frac{V}{k_0 a} = \frac{V}{2\pi} \cdot \frac{\lambda}{a} = 0.212$$

$$\frac{d\beta}{d\omega} = \frac{1}{c_0}\left[n_1 - \frac{1.02\,\mu m \times 0.21}{9.4\,\mu m}\, 0.3\right]$$

$$\left[1.5 - 0.0068\right]$$

$$\approx \frac{1.493}{c_0}$$

So $\boxed{v \approx 0.670\, c_0}$