1. For a Kerr medium, $\Delta n = \frac{1}{2} n^3 s E_x^2$

$$= \frac{1}{2} n^3 s \frac{V^2}{d^2}$$

· For light polarized along x, $\Delta n = \frac{1}{2} n^3 s_{xxxx} \frac{V^2}{d^2} = \frac{1}{2} n^3 s_{11} \frac{V^2}{d^2}$

For light polarized along y, $\Delta n = \frac{1}{2} n^3 s_{yyxx} \frac{V^2}{d^2} = \frac{1}{2} n^3 s_{12} \frac{V^2}{d^2}$

So phase shift $\Delta\phi = k \ell \Delta n$

* If you ignored s_{12}, that's fine.

$$= \frac{2\pi}{\lambda} \ell \frac{1}{2} n^3 (s_{11} - s_{12}) \frac{V^2}{d^2}$$

$$\equiv \pi \left(\frac{V}{V_\pi} \right)^2$$

for $\boxed{V_\pi^2 = \frac{\lambda d^2}{n^3 \ell (s_{11} - s_{12})}}$

Just as for Pockels effect, transmission through polarizers is

$$T = \sin^2 \frac{\phi}{2}$$

So $\boxed{I_{out} = I_{in} \sin^2 \frac{\pi}{2} \left(\frac{V}{V_\pi} \right)^2 = I_{in} \sin^2 \left(\frac{\pi}{2} \frac{n^3 \ell (s_{11} - s_{12}) V^2}{\lambda d^2} \right)}$

2. Define $\hat{z}' = \hat{k} = \sin\theta \hat{x} + \cos\theta \hat{z}$

and $\hat{x}' = -\sin\theta \hat{z} + \cos\theta \hat{x}$

$\hat{y}' = \hat{y}$

Then $\hat{y}'$ direction is along a crystal axis,

so $\hat{y}'$ is a principal polarization with index n_y

So other axis will be orthogonal to $y' = x'$

To get index ellipse:

ellipsoid is $\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$

Project to x', y' plane ($z' = 0$ plane)

$$x = x'\cos\theta - z'\sin\theta \quad \rightarrow \quad x'\cos\theta$$
$$y = y'$$
$$z = z'\cos\theta + x'\sin\theta \quad \rightarrow \quad x'\sin\theta$$

So $x'^2\left(\frac{\cos^2\theta}{n_x^2} + \frac{\sin^2\theta}{n_z^2}\right) + \frac{y'^2}{n_y^2} = 1$

Effective index along x' is

$$\boxed{\frac{1}{n_{x'}^2} = \frac{\cos^2\theta}{n_x^2} + \frac{\sin^2\theta}{n_z^2}}$$

$$\boxed{n_{y'} = n_y}$$

3. Have $r_{33} = r_{zzz}$

$r_{13} = r_{xxz} = r_{yyz}$

$r_{22} = r_{yyy} = -r_{xxy} = -r_{xyx}$

$r_{51} = r_{xzy} = r_{yzx}$

Symmetries from Table 18.2-2

So we have many choices.

For integrated optical component, need to apply $\vec{E} \perp \vec{k}$, since no access to faces

A. Apply E_z, light polarized along z

Use r_{33}, $V_\pi = \frac{d}{\ell} \frac{\lambda_0}{r_{33} n_e^3} = \frac{\lambda_0 d}{\ell} \times 3.17 \times 10^9 \frac{V}{m}$

$= 13.5\text{ V}$

B. Apply E_z, light polarized along y (or x)

Use r_{13}, $V_\pi = \frac{d}{\ell} \frac{\lambda_0}{r_{13} n_0^3} = \frac{\lambda_0 d}{\ell} \times 9.68 \times 10^9 \frac{V}{m}$

$= 41.1\text{V}$

C. Apply E_y

Index ellipsoid →

$$\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} - x^2 r_{22} E_y + y^2 r_{22} E_y - 2xz\, r_{51} E_y = 1$$

xz term has only 2^{nd} order effect since $n_x \neq n_z$

So get $\Delta n_x = +\frac{1}{2} n_0^3 r_{22} E_y$

$\Delta n_y = -\frac{1}{2} n_0^3 r_{22} E_y$

So for light polarized along either x or y, get

$V_\pi = \frac{d}{\ell} \frac{\lambda_0}{r_{22} n_0^3} = \frac{\lambda_0 d}{\ell} \times 34.7 \times 10^9 \frac{V}{m}$

$= 147.5\text{V}$

D. Apply E_x

Index ellipsoid →

$$\frac{x^2}{n_0^2} + \frac{y^2}{n_0^2} + \frac{z^2}{n_e^2} - 2xy\, r_{22} E_x + 2yz\, r_{51} E_x = 1$$

yz term has small effect, but xy term does have linear effect since $n_x = n_y$

New axes $x' = \frac{x+y}{\sqrt{2}}$ $y' = \frac{x-y}{\sqrt{2}}$

$$\left.\begin{array}{l} n_{x'} = n_o + \frac{1}{2} n_o^3 r_{22} E_y \\ n_{y'} = n_o - \frac{1}{2} n_o^3 r_{22} E_x \end{array}\right] \text{(Just like KDP example)}$$

So for light polarized along x' or y', get

$$V_\pi = \frac{d}{l} \frac{\lambda_0}{r_{22} n_o^3} = \frac{\lambda_0 d}{l} \times 34.7 \times 10^9 \frac{V}{m}$$

$$= 147.5 \text{ V}$$

So best is clearly case A

Crystal oriented with z perpendicular to $\vec{E}$, and light polarized along z

Get $V_\pi = \frac{850\text{nm} \times 5\mu\text{m}}{1\text{mm}} \times 3.17 \times 10^9 \frac{V}{m}$

$$= \boxed{13.5 \text{ V}}$$

4. a) After the crystal, have $E_{out} = E_{in} e^{-ikl}$

$$k = nk_o$$

$$n = n_e - \frac{1}{2} n_e^3 r_{33} E_1 \cos\Omega t$$

So $E_{out} = E_0 e^{i(\omega_0 t - n_e k_o l)} e^{+i \frac{k_o l}{2} n_e^3 r_{33} E_1 \cos\Omega t}$

$$= (E_0 e^{-in_e k_o l})\left[e^{i(\omega_0 t + \frac{k_o l}{2} n_e^3 r_{33} E_1 \cos\Omega t)} \right]$$

Has desired form, with $\boxed{\delta = \pi \frac{l}{\lambda} n_e^3 r_{33} E_1}$

b) Then $\dfrac{dE_L}{dt} = i(\omega_0 - \delta\Omega \sin\Omega t)\, e^{i(\omega_0 t + \delta\cos\Omega t)}$

$$\omega(t) = \omega_0 - \delta\Omega \sin\Omega t$$

ranges from $\boxed{\omega_0 - \delta\Omega \text{ to } \omega_0 + \delta\Omega}$

c) For small δ, $e^{i\delta\cos\Omega t} \approx 1 + i\delta\cos\Omega t$

$$= 1 + \frac{i\delta}{2}\left(e^{i\Omega t} + e^{-i\Omega t}\right)$$

So $$\boxed{E_L = E_0\left[e^{i\omega_0 t} + \frac{i\delta}{2} e^{i(\omega_0+\Omega)t} + \frac{i\delta}{2} e^{i(\omega_0-\Omega)t}\right]}$$

Three frequencies, as claimed

c. Want
$$e^{i(\omega_0 t + \delta\cos\Omega t)} = \sum_n A_n e^{i(\omega_0 t + n\Omega t)}$$

$$e^{i\delta\cos\Omega t} = \sum_n A_n e^{in\Omega t}$$

Fourier series

$$\int_{-\pi/\Omega}^{\pi/\Omega} e^{-im\Omega t} e^{i\delta\cos\Omega t}\, dt = \sum_n A_n \underbrace{\int_{-\pi/\Omega}^{\pi/\Omega} e^{i(n-m)\Omega t}\, dt}_{= \frac{2\pi}{\Omega}\delta_{nm}}$$

So $$A_m = \frac{\Omega}{2\pi}\int_{-\pi/\Omega}^{\pi/\Omega} e^{i(\delta\cos\Omega t - m\Omega t)}\, dt$$

$$A_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(\delta\cos\phi - m\phi)}\, d\phi$$

Bessel's integral is $J_m(z) = \frac{(i)^{-n}}{\pi}\int_0^{\pi} e^{iz\cos\phi}\cos(m\phi)\, d\phi$

We have $A_m = \frac{1}{2\pi}\left[\int_0^{\pi} e^{i(\delta\cos\phi - m\phi)}\, d\phi + \int_{-\pi}^{0} e^{i(\delta\cos\phi - m\phi)}\, d\phi\right]$

$$\int_{-\pi}^{0} e^{i(\delta\cos\phi - m\phi)}\, d\phi = \int_0^{\pi} e^{i(\delta\cos\phi + m\phi)}\, d\phi$$

So $A_m = \frac{1}{2\pi}\int_0^{\pi} e^{i\delta\cos\phi}\left(e^{-im\phi} + e^{im\phi}\right) d\phi$

$$= \frac{1}{\pi}\int_0^{\pi} e^{i\delta\cos\phi}\cos m\phi\; d\phi$$

$$\boxed{A_m = i^m J_m(\delta)}$$

So to get $A_0 = A_1$, need δ such that $J_0(\delta) = J_1(\delta)$

From tables, occurs at $\boxed{\delta \approx 1.4}$