04/18/05 Lecture 32.

Nonlinear optics: the medium responds electric field nonlinearly.

Maxwell's equation: (non magnetic medium)

$$\begin{cases} \vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{D} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \end{cases}$$

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\vec{P}$: induced polarization in medium

$\vec{P}$ = dipole moment / unit volume

So $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E}$

$= -\vec{\nabla}^2 \vec{E}$ (homogeneous & isotropic)

$$= -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \frac{\partial^2}{\partial t^2} \vec{D}$$

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

In linear optics, $\vec{P}$ depends linearly on $\vec{E}$

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

$$\vec{\nabla}^2 - \frac{1}{c^2}(1 + \chi) \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$v = \frac{c}{\sqrt{1+\chi}} = \frac{c}{n} \qquad n = \sqrt{1+\chi}.$$

In a sense, index is a way to handle motion of charge in medium (reflected in the polarization $\vec{P}$)

But, generally a non-linear response as well:

$$\vec{P} = \epsilon_0 \chi \vec{E} + 2d\vec{E}^2 + 4\chi^{(3)} \vec{E}^3 + \cdots \quad \Leftarrow \text{our convention}$$

or $\vec{P} = \epsilon_0 \chi_1 \vec{E} + \epsilon_0 \chi_2 \vec{E}^2 + \epsilon_0 \chi_3 \vec{E}^3 + \cdots$

Typical values: $\chi \sim \chi_1 \sim 1$

$$\chi_{n+1} \sim \left(\frac{1}{E_{atom}}\right)^n \qquad E_{atom} \sim 10^{11} \frac{V}{m}$$

So $$d \sim \frac{\epsilon_0}{E_{atom}} = \frac{10^{-11}\ C/V\cdot m}{10^{11}\ V/m} \sim 10^{-22} \frac{C}{V^2}$$

$$\chi^{(3)} \sim \frac{\epsilon_0}{E_{atom}^2} = \frac{10^{-11}\ C/V\cdot m}{10^{22}\ V^2/m^2} \sim 10^{-33} \frac{C\cdot m}{V^3}$$

other system: $\chi_2 \sim 10^{-11}\ m/V$

$\chi_3 \sim 10^{-22}\ m^2/V^2$

Size of coefficient tells you what convention is being used.

In any case: nonlinear effects are small.

Typically write $\vec{P} = \epsilon_0 \chi \vec{E} + \vec{P}_{NL}$

So wave equation becomes

$$\vec{\nabla}^2 \vec{E}^2 - \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}$$

$$\frac{1}{v^2} = \frac{1+\chi}{c^2} = \frac{n}{c^2}$$

$\vec{P}_{NL}$ acts like source of new waves (generally at new frequency)

Simple approximation (Born approximation):

* Assume new waves are weak.
* use incident field to calculate $\vec{P}_{NL}$
* use $\vec{P}_{NL}$ to get scattered field
  ignore $\vec{P}_{NL}$ produced by scattered field.

Example:

2nd order nonlinear optics

$$P_{NL} = 2dE^2 \qquad E = E_0 \cos(\omega t)$$

$$= 2d E_0^2 \cos^2 \omega t$$

$$= d E_0^2 (1 + \cos 2\omega t)$$

$$= P_{NL}(\omega' = 0) + P_{NL}(\omega' = 2\omega) \cos 2\omega t$$

$\downarrow$ DC polarization $= d E_0^2$

$\downarrow$ Amplitude at 2ω $= d E_0^2$

However, in reality d depends on frequency

$$d(\omega'=0) \neq d(\omega'=2\omega)$$

Generally, d depends on ω as well as ω'.

write $d(\omega,-\omega,0)$, and $d(\omega,\omega,2\omega)$

First term: dc polarization, produces voltage across crystal

2nd term: source at 2ω

$$\mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} = -4\omega^2 \mu_0 d \mathcal{E}_0^2 \cos 2\omega t$$

Treat as given source of new wave.

Exponential notation

Convenient to use $\tilde{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$

rather than $\mathcal{E} = \mathcal{E}_0 \cos(\omega t + \phi)$

have $\mathcal{E}$ = real field $= \mathrm{Re}\{\mathcal{E}_0 e^{i\omega t}\} = \mathrm{Re}\{\tilde{\mathcal{E}}\}$

Exponential trick breaks down with nonlinear terms

$$P_{NL}(t) = P_{NL}(\omega'=0) + \mathrm{Re}\{P_{NL}(2\omega) e^{2i\omega t}\}$$

$$\neq \mathrm{Re}\{2d\tilde{\mathcal{E}}^2\} \quad \text{for } \tilde{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$$

$$= 2d\mathcal{E}_0^2 \cos 2\omega t$$

wrong amp missing dc term

Need to write out

if $E(t) = \mathrm{Re}\, \tilde{E}(t)$ $\qquad\qquad \tilde{E}(t) = E_0 e^{i\omega t}$

$= \frac{1}{2}(\tilde{E} + \tilde{E}^*)$

So $E(t)^2 = \frac{1}{4}(\tilde{E}^2 + \tilde{E}^{*2} + 2\tilde{E}\tilde{E}^*)$

$= \frac{1}{4}(\tilde{E}^2 + \tilde{E}\tilde{E}^*) + \text{c.c.}$

So take $P_{NL}(t) = \frac{1}{2}(\tilde{P}_{NL} + \tilde{P}^*_{NL})$

$\tilde{P}_{NL} = 2d$