Phys 532 / 822 HW # 2 solutions

1. a) Peak intensity $I_0 = \frac{2P}{\pi W(z)^2}$

$$W(z)^2 = W_0^2 \left(1 + \frac{z^2}{z_0^2}\right)$$

$$z_0 = \frac{\pi W_0^2}{\lambda}$$

So $I_0(d) = \frac{2P}{\pi} \frac{1}{W_0^2 + \frac{\lambda^2 d^2}{\pi^2 W_0^2}}$

To have maximized peak intensity $I_0(d)$,

$W(z)^2 = W_0^2 + \frac{\lambda^2 d^2}{\pi^2 W_0^2}$ must be minimized

so $\frac{\partial W^2(z)}{\partial W_0} = 2 W_0 - 2\frac{\lambda^2 d^2}{\pi^2 W_0^3} = 0$

so $W_0^4 = \frac{\lambda^2 d^2}{\pi^2}$

$$\boxed{W_0 = \sqrt{\frac{\lambda d}{\pi}}}$$

In other word, $d = \frac{\pi W_0^2}{\lambda} = z_0$

b) $\lambda = 532$ nm

$d = 1$ cm $\Rightarrow$ $W_0 = 41\ \mu m$

$= 1$ m $\Rightarrow$ $W_0 = 0.41$ mm

$= 100$ m $\Rightarrow$ $W_0 = 4.1$ mm.

2. Have

$$W_1^2 = W_0^2 \left(1 + \frac{z_1^2}{z_0^2}\right) = W_0^2 + \frac{\lambda^2 z_1^2}{\pi^2 W_0^2}$$

where W_0 = waist at focus

z_1 = distance from W_1 to focus

are to be determined.

Also $$W_2^2 = W_0^2 + \frac{\lambda^2}{\pi^2 W_0^2}(z_1 + d)^2$$

$d = 0.1\,m$

(take $d > 0$)

Rewrite to simplify:

define $u_0 = \frac{\pi}{\lambda} W_0^2 \quad (= z_0)$

$u_1 = \frac{\pi}{\lambda} W_1^2 = 0.8551\ m$

$u_2 = \frac{\pi}{\lambda} W_2^2 = 3.3842\ m$

So $$u_1 = u_0 + \frac{z_1^2}{u_0}$$

$$u_2 = u_0 + \frac{1}{u_0}(z_1 + d)^2$$

$$= u_0 + \frac{1}{u_0}(z_1^2 + 2z_1 d + d^2)$$

$$= u_1 + \frac{1}{u_0}(2z_1 d + d^2)$$

so $$z_1 = \frac{u_0(u_2 - u_1) - d^2}{2d}$$

Substitute:

$$u_1 = u_0 + \frac{1}{u_0}\left[\frac{u_0(u_2-u_1)-d^2}{2d}\right]^2$$

$$= u_0 + \frac{1}{4u_0 d^2}\left[u_0^2(u_2-u_1)^2 - 2d^2 u_0(u_2-u_1) + d^4\right]$$

$$\left[4 + \frac{(u_2-u_1)^2}{d^2}\right]u_0^2 - 2(u_1+u_2)u_0 + d^2 = 0$$

Solve the quadratic equation:

$$\boxed{u_0 = \frac{d^2}{(u_2-u_1)^2+4d^2}\left[u_1+u_2 \pm \sqrt{u_1 u_2 - d^2}\right]}$$

Evaluate: $u_0 = z_0 = 11.9\ \text{mm}$ or $1.31\ \text{mm}$.

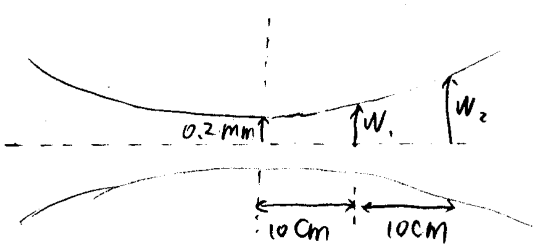
$$W_0 = \sqrt{\frac{\lambda z_0}{\pi}} = \boxed{0.2\ \text{mm} \quad \text{or} \quad 66.5\ \mu\text{m}}$$

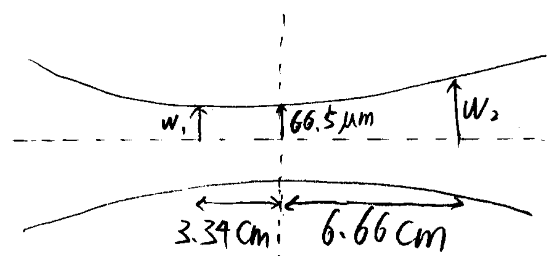
Then

$$z_1 = \frac{u_0(u_2-u_1)-d^2}{2d} = \boxed{10\ \text{cm} \quad \text{or} \quad -3.34\ \text{cm}}$$

$$z_2 = z_1 + d = \boxed{20\ \text{cm} \quad \text{or} \quad 6.66\ \text{cm}}$$

a





3 Input beam has $z = 35\,mm$

$$z_0 = \frac{\pi W_0^2}{\lambda} = 14.76\,mm$$

so $q_1 = 35 + i\,14.76$ mm

Lens matrix is $\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$

so after lens, $$q_2 = \frac{35 + i\,14.76}{-\frac{1}{25}(35 + i\,14.76) + 1}$$

$$= -44.7 + i\,29 \quad mm$$
$$= z' + i z_0'$$

So resulting focus is $\boxed{44.7\,mm}$ from lens

with waist $W_0 = \sqrt{\frac{\lambda z_0'}{\pi}} = \boxed{70\,\mu m}$

Ray optics

i) collimated input gives focus at $z = f = \boxed{25\,mm}$

ii) use $\frac{1}{i} = \frac{1}{f} - \frac{1}{o} = \frac{1}{25\,mm} - \frac{1}{35\,mm} = \frac{1}{87.5\,mm}$

Focus at $\boxed{87.5\,mm}$

Gaussian beam focus is in between.

5. At mirror, input beam has $z = 100\ mm$

$$z_0 = \frac{\pi w_0^2}{\lambda} = 187.5\ mm$$

Mirror has matrix $\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$

so reflected beam has

$$q' = \frac{z + iz_0}{\frac{2}{R}(z + iz_0) + 1}$$

Want R such that $q' = -z + iz_0$

(now $-z$, because converging instead of diverging)

So
$$\frac{2}{R}(z + iz_0) + 1 = \frac{z + iz_0}{-z + iz_0}$$

$$\frac{2}{R} + \frac{1}{z + iz_0} = \frac{1}{-z + iz_0}$$

$$\frac{2}{R} = \frac{1}{-z + iz_0} - \frac{1}{z + iz_0}$$

$$= \frac{(z + iz_0) - (-z + iz_0)}{-(z^2 + z_0^2)}$$

$$= \frac{-2z}{z^2 + z_0^2}$$

$$R = -\frac{z^2 + z_0^2}{z} = -\frac{100mm^2 + 187.5mm^2}{100mm}$$

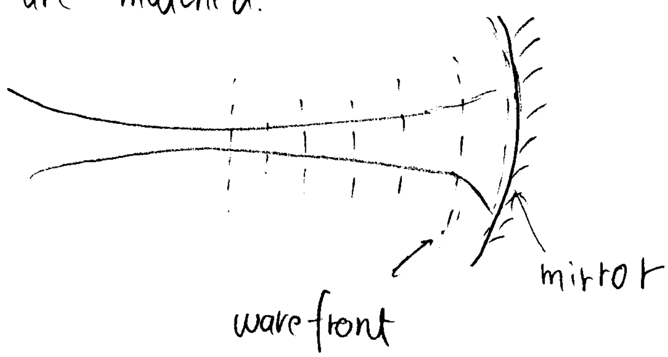
$\boxed{R = -451\ mm}$

At the mirror, beam has curvature.

$$\frac{1}{R_b} = \frac{z}{z^2 + z_0^2}$$

so $\boxed{R_{mirror} = -R_{beam}}$

Follow our sign convention, mirror and wavefront are matched.



5 Let q_0 = input value

q_1 = value after system 1

q_2 = value after system 2

Then $$q_1 = \frac{A_1 q_0 + B_1}{C_1 q_0 + D_1}$$

$$q_2 = \frac{A_2 q_1 + B_2}{C_2 q_1 + D_2}$$

$$= \frac{A_2 \left(\frac{A_1 q_0 + B_1}{C_1 q_0 + D_1}\right) + B_2}{C_2 \left(\frac{A_1 q_0 + B_1}{C_1 q_0 + D_1}\right) + D_2}$$

$$= \frac{(A_1 A_2 + C_1 B_2) q_0 + (B_1 A_2 + D_1 B_2)}{(A_1 C_2 + C_1 D_2) q_0 + (B_1 C_2 + D_1 D_2)}$$

while $\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$

$$= \begin{bmatrix} A_1 A_2 + C_1 B_2 & B_1 A_2 + D_1 B_2 \\ A_1 C_2 + C_1 D_2 & B_1 C_2 + D_1 D_2 \end{bmatrix}$$

so $\boxed{q_2 = \frac{A_3 q_0 + B_3}{C_3 q_0 + D_3}}$