Midterm solutions

Short questions

1. Beam is too big: $W_0 \approx \frac{\lambda}{\theta}$ for focussing angle θ

   So increase $\theta \approx \frac{W_{in}}{f}$ for incident waist W_{in}

   so should decrease f: $\boxed{\text{shorter focal length}}$

2. $\Delta \nu_L = \frac{c}{nL}$ L: round-trip length $= 10\,cm$, $n = 1.5$

   $$= \frac{3\times 10^{10}\ cm/s}{1.5\times 10\ cm} = \boxed{2\ GHz}$$

3. a) Since collisions are slow, $\Delta \nu_{homo} \sim \frac{1}{2\pi}\frac{1}{t_{decay}}$

   lower state decays fastest, so

   $$\boxed{\Delta \nu_{homo} \sim \frac{1}{2\pi}\frac{1}{10ns} \sim 10^7 Hz}$$

   b) Doppler width

   $$\boxed{\Delta \nu_{inhomo} \sim \frac{\Delta V}{\lambda} \sim \frac{1000 m/s}{1 \mu m} \sim 10^9 Hz}$$

4. In steady-state operation, gain = loss

   So, gain per pass = loss per pass = $\boxed{2\%}$

Extended problems

1. First, solve for mode:

$R_1 = \infty$ $\qquad\qquad$ $R_2 = -100\,\text{cm}$

Round trip matrix, starting at flat mirror is

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ \frac{2}{R} & 1 + \frac{2d}{R} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{2d}{R} & 2d + \frac{2d^2}{R} \\ \frac{2}{R} & 1 + \frac{2d}{R} \end{bmatrix}$$

So $$q = \frac{1}{C}\left[\frac{A-D}{2} \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1} \right]$$

$$= \pm \frac{R}{2} \sqrt{\left(1 + \frac{2d}{R}\right)^2 - 1}$$

$$= \pm \frac{R}{2} \sqrt{\frac{4d}{R} + \frac{4d^2}{R^2}}$$

$$= \pm 50\,\text{cm} \sqrt{-2 + 1}$$

$$= \pm i\, 50\,\text{cm}$$

So focus at flat mirror, $z_0 = 50\,\text{cm}$

$$W_0 = \sqrt{\frac{\lambda z_0}{\pi}} = 0.89\,\text{mm}$$

Width at front mirror:

$$W(z) = W_0 \sqrt{1 + \frac{z^2}{z_0^2}}$$

$$= W_0 \sqrt{1+1}$$

$$= \sqrt{2}\, W_0 = 1.26\,mm$$

So, estimate $\boxed{W = 1mm}$ in cavity.

<u>Second</u>, get medium characteristics.

density: $P = N k_B T$

$P = 1\ atm = 101325\ N/m^2$
$T = 300\,k$

so $N_a = \dfrac{101325\ N/m^2}{1.38 \times 10^{-23} J/k \cdot 300k}$

$$\boxed{= 2.45 \times 10^{25}\ m^{-3}}$$

linewidth:

Radiation: $\Delta\nu = \frac{1}{2\pi \tau_1} = \frac{1}{2\pi \times 1ms} = 160\,kHz$

collisions: $\Delta\nu = \frac{1}{\pi} f_{col} = 320\,MHz$

Doppler: $\Delta\nu = 2.35 \sqrt{\frac{kT}{M\lambda^2}} = 52\,MHz$

$(M = 200 \times 1.66 \times 10^{-27} kg = 3.32 \times 10^{-25}\,kg)$

So, $\boxed{\text{collisional broadening dominates.}}$

<u>Third</u>, find inversion

For ideal four-level system with no degeneracies,

$$\Delta N = R(\tau_2 - \tau_1) = R\tau_2 \quad \text{and} \quad R = W_p N_0$$

Since W_p is very weak, expect that we can ignore depletion of ground state.

So $$\Delta N = W_p \tau_2 N_a$$
$$= (10^{-6} s^{-1})(10^{-3} s)(2.45 \times 10^{25} \ m^{-3})$$
$$= \boxed{2.45 \times 10^{16} \ m^{-3}}$$

So small signal gain is

$$\gamma_0 = \frac{\lambda^2}{8\pi t_{sp}} g(\nu) \Delta N$$

Assume $\nu = \nu_0$, so
$$g(\nu) \rightarrow \frac{2}{\pi \Delta \nu} = \frac{2}{f_{col}}$$

$$\gamma_0 = \frac{\lambda^2 \cdot \Delta N}{4\pi^2 t_{sp} \Delta \nu}$$

$$= \frac{(5 \times 10^{-6} m)^2 \cdot 2.45 \times 10^{16} \ m^{-3}}{4\pi^2 (10^{-3} s)(320 \ MHz)}$$

$$\boxed{\gamma_0 = 0.048 \ m^{-1}}$$

and $g_0 = 2\gamma_0 d = 0.048 > T$, so above threshold.

Forth output power (5)

Also need $I_{sat} = \dfrac{4\pi^2 hc\, t_s\, \Delta V}{\lambda^3\, \tau_s}$

For ideal four-level system, $\tau_s = \tau_2 = t_s$

$$I_s = \frac{4\pi^2 (6.63\times10^{-34}\,J\cdot s)(3\times10^8\,m/s)(320\,MHz)}{(5\,\mu m)^3}$$

$$\boxed{= 20\ W/m^2}$$

So $P_{out} \approx \pi w^2 I_s T \left(\dfrac{g_0}{T} - 1\right)$

$$= \pi (1mm)^2 \left(20\frac{W}{m^2}\right)(0.02)\left(\frac{0.048}{0.02} - 1\right)$$

$$\boxed{P_{out} = 1.7\ \mu W}$$

2. Rate equations

$$\frac{dN_2}{dt} = +W_p N_0 - W_p N_2 - \frac{1}{\tau_2} N_2 = 0$$

$$\frac{dN_1}{dt} = +\frac{1}{\tau_{21}} N_2 - \frac{1}{\tau_1} N_1 = 0 \implies N_1 = \frac{\tau_1}{\tau_{21}} N_2$$

$$\frac{dN_4}{dt} = +\frac{1}{\tau_{24}} N_2 - \frac{1}{\tau_4} N_4 = 0 \implies N_4 = \frac{\tau_4}{\tau_{24}} N_2$$

$$N_2 = N_0 \frac{W_p}{W_p + \frac{1}{\tau_2}} = N_0 \frac{\tau_2 W_p}{1 + \tau_2 W_p}$$

In terms of N_a:

$$N_a = N_0 + N_1 + N_2 + N_4$$

$$= N_0 \left(1 + \frac{\tau_2 W_p}{1+\tau_2 W_p} \left[1 + \frac{\tau_4}{\tau_{24}} + \frac{\tau_1}{\tau_{21}} \right] \right)$$

$$= \frac{N_0}{1+\tau_2 W_p} \left[1 + \tau_2 W_p \left(2 + \frac{\tau_4}{\tau_{24}} + \frac{\tau_1}{\tau_{21}} \right) \right]$$

$$\Delta N = N_2 - N_1 = N_2 \left(1 - \frac{\tau_1}{\tau_{21}} \right)$$

$$= N_0 \frac{\tau_2 W_p}{1 + \tau_2 W_p} \left(1 - \frac{\tau_1}{\tau_{21}} \right)$$

$$\Delta N = \frac{\tau_2 W_p (1 - \tau_1/\tau_{21})}{1 + \tau_2 W_p \left(2 + \frac{\tau_4}{\tau_{24}} + \frac{\tau_1}{\tau_{21}} \right)} N_a.$$

3. $P_o = \pi W^2 I_s T \left(\frac{g_o}{T+L} - 1 \right)$

If $g_o \gg T + L$, then

$$P_{out} = \pi W^2 I_s g_o \frac{T}{T+L}$$

But $g_o = \frac{\lambda^2}{8\pi t_s \Delta\nu} \Delta N$. using $g(\nu) \approx \frac{1}{\Delta\nu}$

and $I_s = \frac{8\pi t_s h\nu}{\lambda^2 \tau_s} \Delta\nu_H$ homogeneous linewidth $\Delta\nu_H$

So $P_{out} = \pi W^2 \frac{h\nu}{\tau_s} \frac{\Delta\nu_H}{\Delta\nu} \Delta N \frac{T}{T+L}$

Also, $\Delta N = \frac{R(\tau_2 - \tau_1)}{1 + W_p \tau_2} \approx \frac{R\tau_2}{1 + W_p \tau_2}$.

and $\tau_s = \frac{\tau_2}{1 + W_p \tau_2}$

So $P_{out} = \pi W^2 h\nu \frac{\Delta\nu_H}{\Delta\nu} R \frac{T}{T+L}$

If $\Delta\nu + \Delta\nu_H$ don't depend on t_s, then

P_{out} is $\boxed{\text{independent of } t_s}$

Also note $\nu = \frac{c}{\lambda}$, and $W \propto \sqrt{\lambda}$ in general

So P_{out} is also $\boxed{\text{independent of } \lambda}$