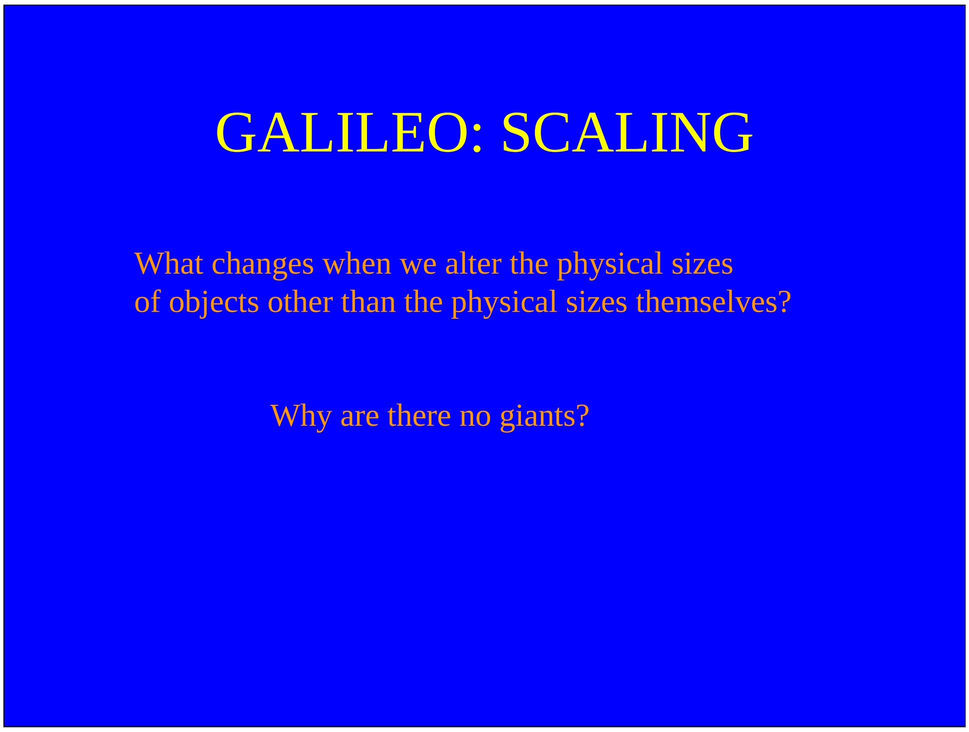


Physical size is important for most of the things we deal with in life. Buildings, animals, cars, trees, … For example the movie King Kong is intriguing first and foremost because a 60 foot tall gorilla does not exist. Why not? A six inch tall gorilla could exist (small monkeys, baby gorillas.) Other animals that size do, so there is nothing impossible about a miniature gorilla. (Probably not as a box office hit, however). Why are there no giants?

Galileo was the first person to think about such questions carefully. He developed a way of thinking about the sizes of objects in a systematic way that is called scaling. Today scaling is used widely in comparative anatomy and engineering. For example if you are building an airplane you might first want to build a scale model and test it in a wind tunnel. You then need to know how to scale up the test results to find the lift and drag forces that will exist for the full size aircraft.

Let's begin by thinking about the physical size of simple objects. As we scale up or down the dimensions of a sphere for example, how do its surface area and volume change?

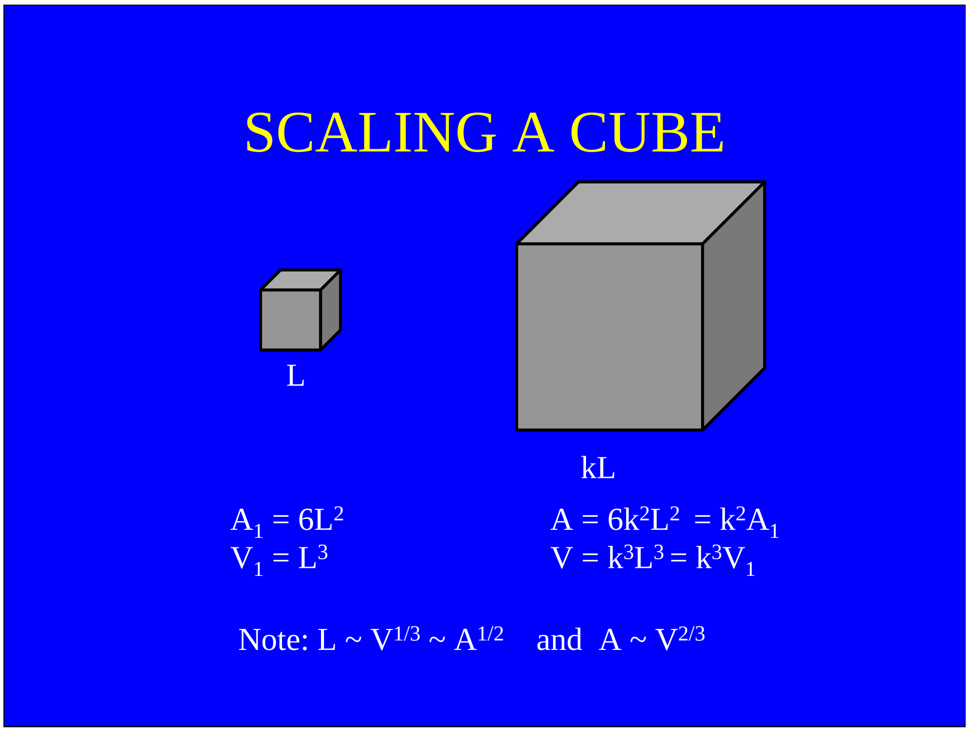
SHOW weight comparison of one inch and two inch diameter steel spheres. For a sphere, volume = $4/3\pi r^3$, so doubling r multiplies the volume and hence weight by 8.

VOLUME OF ONE MILLION GRAINS OF RICE

1 m^3
0.3 m^3
0.1 m^3
0.03 m^3
0.01 m^3

Here we see some containers with rice in them. The largest has one million grains of rice in it. Make a rough estimate of what you think is the volume of one million grains of rice.

Answer: 0.02 m^3. The volume of an object is the product of its length, width, and depth. In this case all are a fraction of a meter, so the volume is a small fraction of a cubic meter.



Here we have a cube with side of length L. If we multiply the length by k we create a cube of larger or smaller size. What happens to its surface area and volume when we do this?

The area is multiplied by k^2 and the volume by k^3. In fact this is true for any three dimensional object. If we multiply all the dimensions of an automobile by k, its surface area or cross sectional area is multiplied by k^2 and its volume by k^3. So far this is just geometry. Galileo took this thinking one step further.

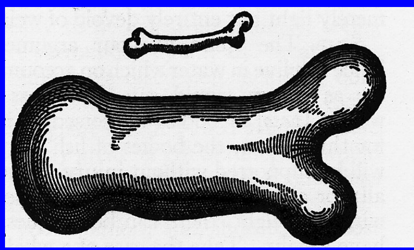
Suppose you see a chandelier hanging from the ceiling of a small room and you like its design. You want to make a larger one to hang in a larger room. Let's say you decide to double each of its dimensions. Then since you multiplied its volume by 8, you also multiplied its weight by 8. Now consider the rope that supports the chandelier. Its strength is proportional to its cross sectional area which is a measure of how many fibers it is made of. If you double each of the dimensions of the rope, you have multiplied its cross sectional area by 4. So it can support four times the weight of the original rope, but it needs to hold 8 times as much. If the original rope was close to its limit, then the scaled rope will not support the scaled chandelier.

Galileo realized that simple geometric scaling does not work when the mechanical strength of structures is taken into account.

Note also that L scales (is proportional to) the cube root of V or the square root of A. And A scales as V raised to the 2/3 power. These results are valid for any three-dimensional object.

KING KONG

Multiply the dimensions of a gorilla by 10. Then its weight is multiplied by 1000, and bone cross sectional area by 100.



The King Kong of the movie is simply a geometrically scaled up gorilla. It has the same proportions as a normal gorilla, but is about 10 times as tall, thick, and wide. This has the effect of multiplying its weight by 1000 and its bone cross sectional areas by 100.

If normal gorillas have bones just strong enough to carry their bodies in normal locomotion, then King Kong is in big trouble. The intrinsic strength of the material bone is made from in animals of all sizes (Calcium apatite embedded in a matrix of collagen) is about the same. The strength of a bone is proportional to its cross sectional area. Compressive failure occurs for a stress of about $2*10^8$ N/m^2 (29,000 lbs/in^2). Most animals produce stresses nearly this large in vigorous activity. This means that a scaled up gorilla King Kong's size would not be able to move around at all. Evolution could not produce such a creature.

Galileo realized that larger animals need thicker bones compared with smaller animals. Above is a sketch taken from *Two New Sciences* illustrating this idea. The larger bone has three times the length of the smaller bone. By what factor should its diameter be scaled to be strong enough to support the animal's weight? Answer: It's cross sectional area must be larger by 27 so its diameter must be scaled by $(27)^{1/2}$ or a little over 5. Three is not enough. In the sketch Galileo exaggerated, making the diameter larger by about 7.

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EXPECTED BONE SHAPE VARIATION WITH ANIMAL MASS

If bone cross sections are determined by animal weight, then we expect cross sectional area A ~ m where m is the total animal mass.

Bone length should be proportional to $m^{1/3}$

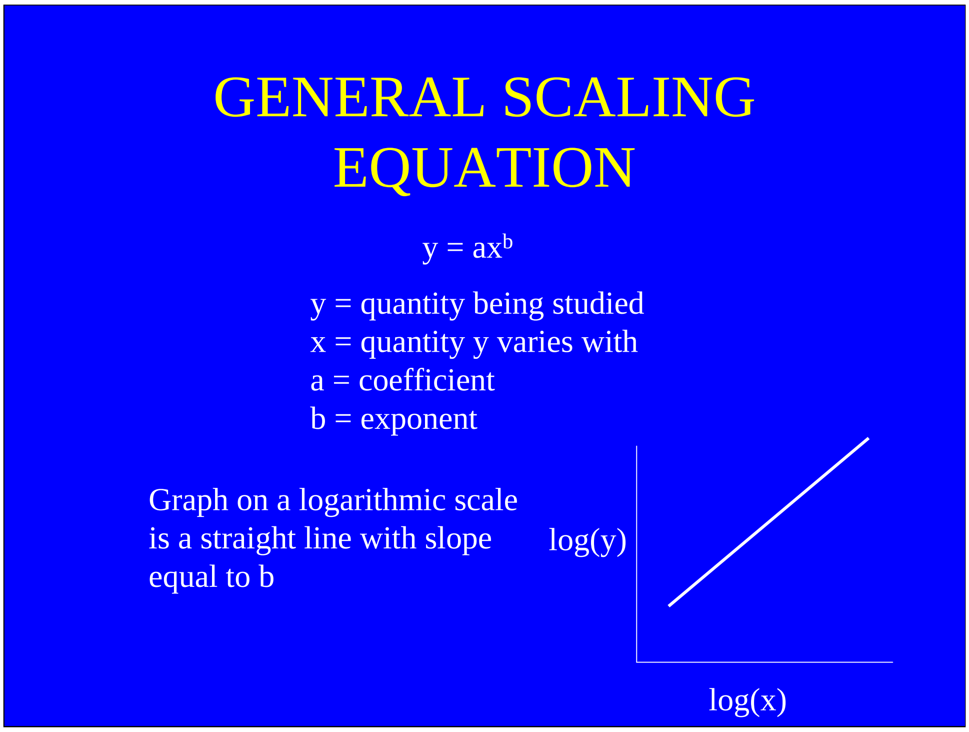
So skeletal mass should be proportional to $(m^{1.0})*(m^{0.33}) = m^{1.33}$

How do we expect bone shape to change with animal size? Since weight can be readily measured, we will use animal mass as a measure of size. Skeletal mass can also be readily measured so we use this as a measure of the size of the bones.

Then if the cross sectional area of bones is determined by the need to support the animal's weight, we expect the area to be proportional to m, the animal's mass.

Animal volume, and hence mass, scale as the cube of the linear dimension of the animal, and hence as the cube of bone length. So bone length should be proportional to $m^{0.33}$. Therefore the skeletal volume, and hence its mass should be proportional to $m^{1.33}$.

This is our expectation based on the assumption that bone cross section is determined primarily by animal weight. How does skeletal mass actually vary with animal mass? How can we go about finding out?

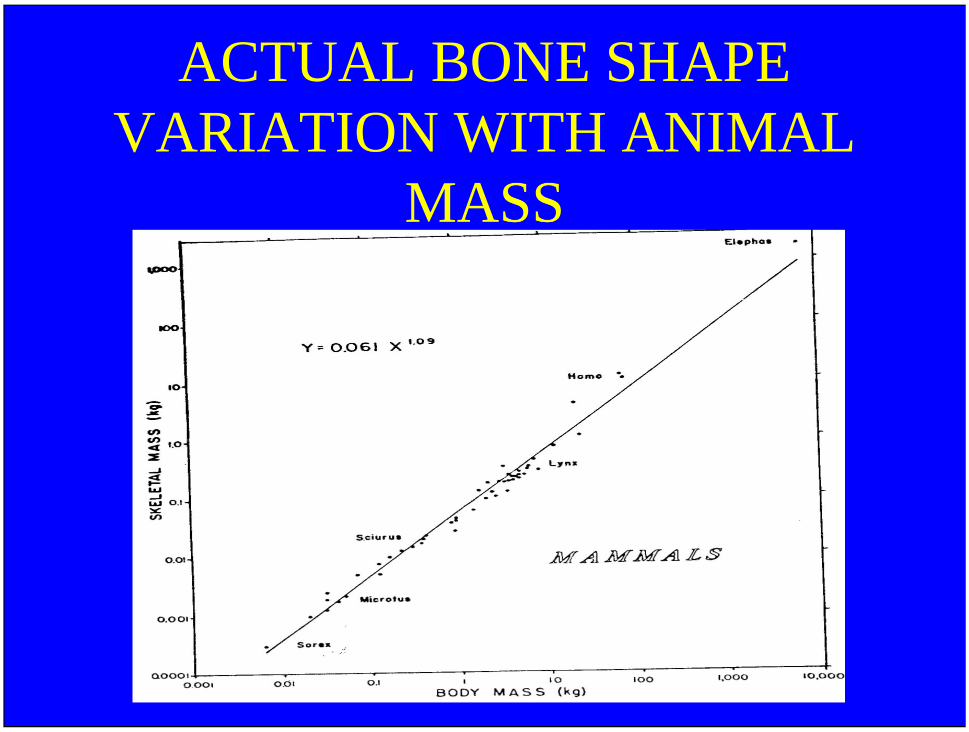


We have seen that when we scale an object geometrically, keeping its shape the same, often one quantity varies as a power of another quantity, e.g. volume varies as the cube of the length scale. A general equation that can include all the examples we have discussed can be written as above. y is the quantity we want to study, such as the skeletal mass of animals. x is the quantity we think y varies with, such as animal size or total mass. a is a coefficient and b an exponent.

When y = skeletal mass, and x = total mass, our model predicts that b = 1.33.

To find out whether animals are actually built like our model says, what we need to do is collect data on many animals with a wide range of sizes. Then when y is plotted against x on a logarithmic scale, the above equation forms a straight line with a slope of b.

The equation above is general in the sense that it can be used to find out whether any quantity y scales with any other quantity x. They do not need to be geometric quantities as our skeletal mass example shows, and the coefficient and exponent can have any values that the data require. They are not limited to geometric values.



Above is shown a graph of measured skeletal masses in kg plotted versus total animal mass in kg. Both scales are logarithmic. The data points do fall on a straight line, the slope of which is 1.09. This is less than our expected slope of 1.33. What does this mean? Simply put, the skeletal mass varies more slowly with animal size than our model predicted.

If the bones of a mouse have the right thickness to support its weight, the bones of an elephant must be too thin. Conversely, if the bones of an elephant have the right thickness to support it, then the bones of a mouse must be too thick.

Our conclusion is that weight is not the only factor affecting animal skeletal shape. Animals do not just stand around. They also walk, run, jump, etc. Studies show that animals of all sizes stress their bones to near the breaking point in normal, vigorous activity. We know this is true for humans as well: Athletes are frequently at risk for breaking bones.

Small and large animals therefore behave differently. Mice and squirrels jump and run; Elephants can gallop, but usually walk, and do not jump.

The coefficient in the above equation tells us that a 1 kg animal has a skeletal mass of 0.061 kg (6.2 kg for a 70 kg human).

RESTING METABOLIC RATES OF MAMMALS

Maintains body temperature

Heat loss proportional to surface area ~ $M^{2/3}$

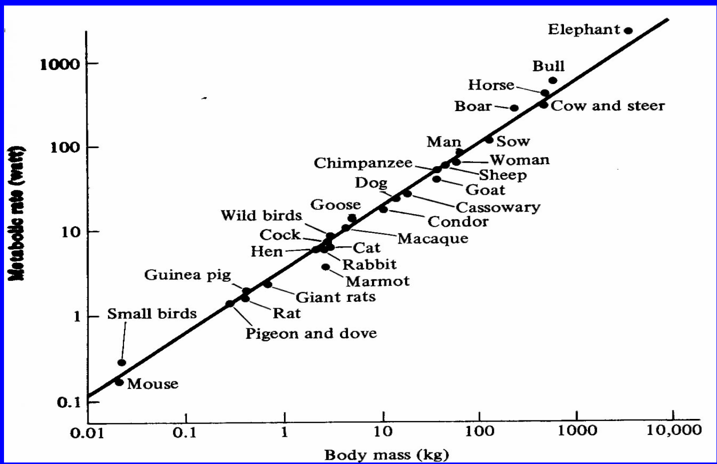
So we expect Metabolic Rate ~ $M^{2/3}$

An intriguing problem in comparative anatomy that has remained controversial for more than a century has to do with the resting metabolic rate of mammals. When an animal is not exercising, its metabolism is needed mainly to maintain body temperature. Assuming the outside temperature is below body temperature, which is true over most of the earth, then metabolic energy is being generated to replace heat lost to the surrounding air.

When study of this question began during the early 1800s, it was assumed that the body surface scaled as body size to the 2/3 power. That is, it was assumed that animals have the same shape, and are scaled versions of one another. We know that this is not so now, but at the time it was a reasonable place to start.

This leads directly to the expectation that the resting metabolism of mammals should scale as $M^{2/3}$.

RESTING METABOLIC RATES VS MAMMAL SIZE



Once again we have developed a simple model for how a measurable quantity should scale with animal size, and we can test the model as we have done before. Data has been collected on the metabolic rates of mammals from mice to elephants and is shown on the graph above using logarithmic scales. Indeed the points do fall close to a straight line so the scaling idea itself is supported.

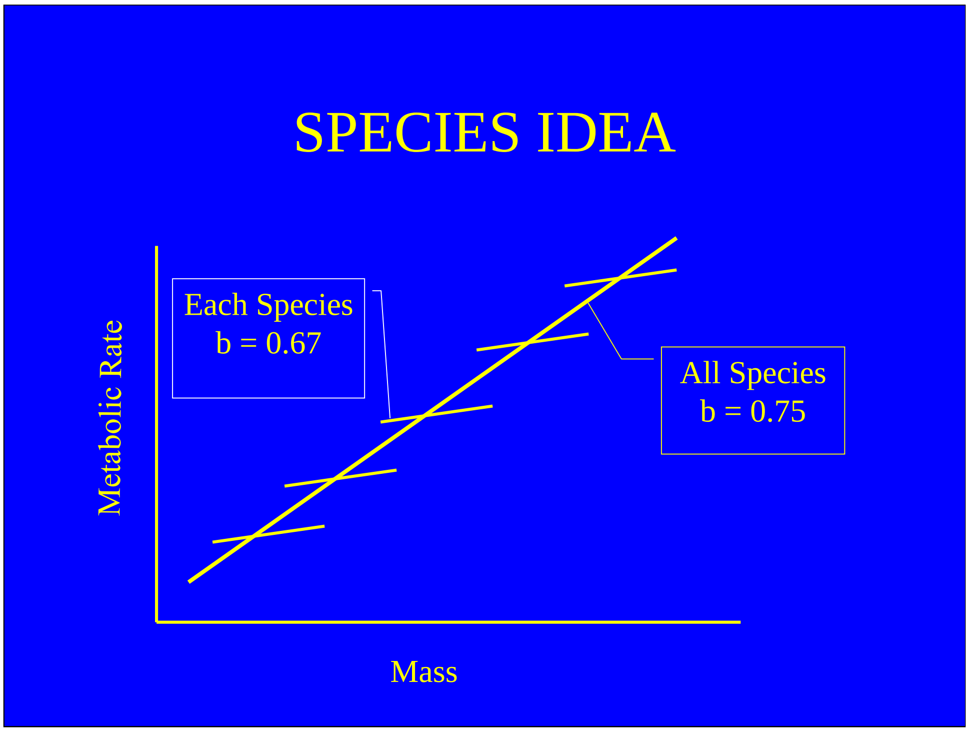
The straight line that fits the data, however has a slope of 0.75 instead of the expected 0.67. This discrepancy is well outside the uncertainties evident by the scatter of the measured points. Many proposals have been made during the past several decades to explain the slope of this graph. Evidently 0.67 is not the slope of the graph, but what is the argument for 0.75?

RESTING METABOLIC RATE FOR HUMANS

Metabolic Rate (Watts) = $4M^{0.75}$

For a 70 kg person, Rate = 70 Watts

The regression line from the previous slide for mammals can be described by Rate (Watts) = $4M^{0.75}$ with mass in kg. For a 70 kg person, this corresponds to a resting metabolic rate of 70 Watts. So when you are sitting in your room reading at a 100 W light, you are heating the room less than is your light bulb.



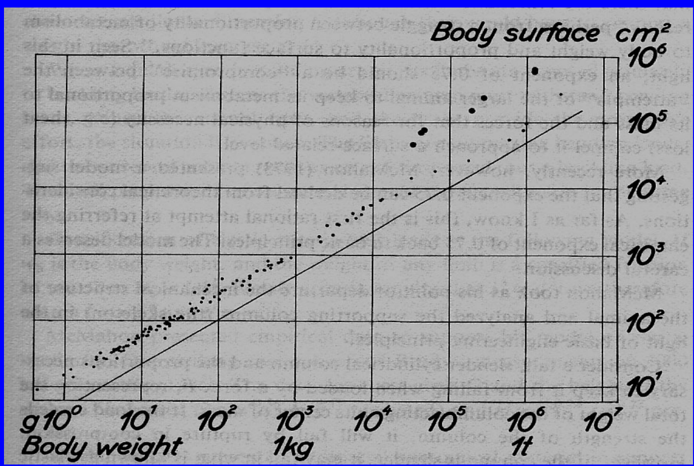
One proposal that somewhat clarifies the situation is this. We know that all animals are not geometrically scaled versions of one another. Within a given species however, this is much closer to being the case. So if we separate the metabolic data among different species, what do we find?

The result is that each species can be represented by a line with a slope of 0.67, which is what we expect for animals that are simply scaled. Upon going from one species to another, the various species lines group along the line with a slope of 0.75. What changes between species is the coefficient a in the general scaling equation. Large animals have larger values of a but within each species, the scaling value of b.

This is interesting, but it still does not explain the value of 0.75 for all mammals.

Question: We started by assuming that animal surface area scales as $M^{0.67}$. Is this accurate?

BODY SURFACE AREA VS MASS



Above is a test of the surface area scaling of mammals. The points fall close to a straight line with a slope of 0.67. So in spite of differences in appearance of different species of animals, our body areas scale in the same uniform way with body weight.

So in spite of the naiveté of the early assumption that all animals are geometric scale models of one another, we find that surface areas do scale that way even though different animals look different. This result by itself is a curiosity at least, and perhaps an indication that we are missing something important. At this point we do not understand why surface areas scale so simply.

In addition of course this leaves us without an explanation for the metabolic exponent of 0.75. This a puzzle yet to be explained.

An equation describing the above straight line is $A = aM^{0.67}$ with $a = 10$ when A is in cm^2 and M in grams.

The heavy dots in the upper right part of the graph above are for beech trees(!).

SKIN AREA FOR HUMANS

Area = $0.1M^{0.67}$
with area in m^2,
M in kg.

For M = 70 kg, Area = 1.76 m^2

From the regression line on the previous slide, the surface area of animals can be described by the equation given above. This means (in case you were curious) that the skin area of a 70 kg person is about 1.76 square meters.

Someone weighing more or less can evaluate their skin areas from the above equation.

ENERGY COST OF RUNNING

Work done by leg muscles per step =
(Force)*(Contraction distance) ~ $(M^{2/3})*(M^{1/3}) \sim M$

Step length ~ $M^{1/3}$ so steps per km ~ $M^{-1/3}$

Work per km ~ $(M)*(M^{-1/3}) \sim M^{0.67}$

So work per kg per km ~ $(M^{0.67})*(M^{-1}) \sim M^{-.33}$

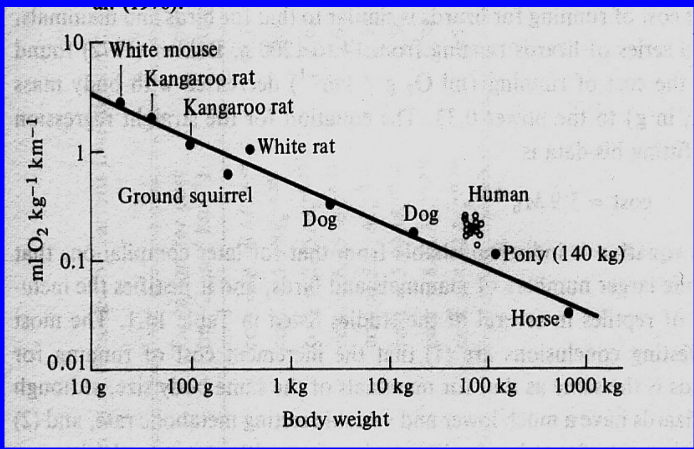
The energy cost of locomotion of animals is a fascinating one with some surprising results. What can we say about it in a general way?

The work done by any force equals the magnitude of the force times the distance through which it acts. In the case of leg muscles this is the muscle force times the distance of contraction. This is the work done per step by leg muscles. Since muscle force is proportional to its cross section, and contraction distance proportional to its length, the work done per step scales as muscle volume or mass. Muscle mass is about 40% of body mass for nearly all animals.

Distance covered per step is proportional to leg length which scales as body mass to the one-third power. So the number of steps per km scales as $M^{-1/3}$.

So based on these general arguments, we expect that the energy cost per kg per km will scale as $M^{-1/3}$.

COST OF RUNNING VS BODY WEIGHT



Here we see the energy cost of running expressed as ml of Oxygen metabolized per kg of animal mass per km plotted vs body weight in kg. The data fall close to a straight line with a slope of – 0.33 as expected.

The negative slope shows that a larger animal can more efficiently move a kg of its mass over a distance of a kilometer than can a smaller animal. This difference is due entirely to the fact that a smaller animal must take more steps to go a km.

The cluster of points for humans lies somewhat above the line, probably because we are bipedal runners and all the other animals shown have four legs.

The above line is described by the equation Cost (liters of O_2per kg per km) = $0.8M^{-0.33}$

The cost for birds to fly can be evaluated similarly with the result Cost (liters of O_2per kg per km) = $0.26M^{-.23}$.

Surprisingly, birds transport themselves more efficiently than running animals (using the same units of cost). This is mainly due to the fact that birds fly so much faster than animals of a similar size run.

COST OF RUNNING FOR HUMANS

Energy cost = 1 Cal/kgkm
= 70 Cal/km for 70 kg

On the previous slide the points for humans was above the line, with the cost of running being about twice that for quadrupeds(!). Changing the energy units to Calories (ie nutritional calories) we find the energy cost as shown above.

This is remarkably efficient transportation. This result is independent of speed, provided, of course that the person is running.

HOW HIGH CAN YOU JUMP?

Energy needed to jump height h = mgh ~ M

Energy available from muscles also ~ M

So height h is independent of body size!

Jumper	mass	height
Flea	0.5 mg	20 cm
Click Beetle	40 mg	30 cm
Locust	3 g	59 cm
Man	70 kg	60 cm

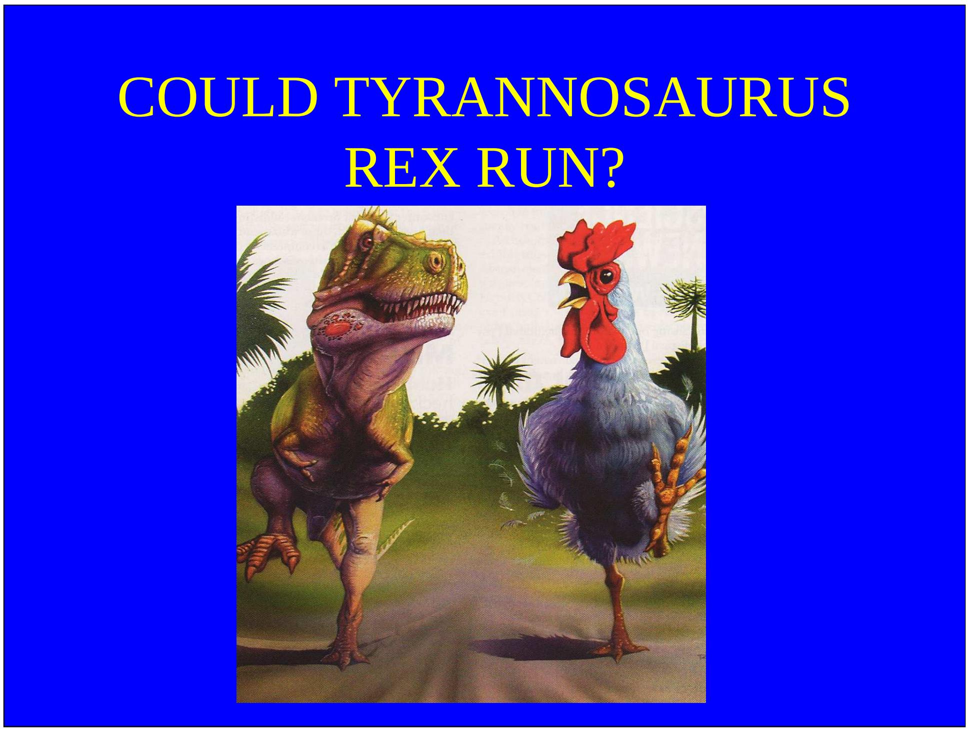
How high can different animals jump? Think of examples you are familiar with: dogs, cats, a horse, yourself. Here I am referring to a standing high jump. In a running jump, part of the kinetic energy of running is converted into height.

In fact there is not much difference in the heights these animals can jump. Why is that? It is simply a matter of scaling. We have not talked about energy yet, but we will see that to raise your center of gravity a height h requires an energy of mgh. This clearly scales as body weight M.

Where does this energy come from? From leg muscles, and we have already seen that the work they can do also scales as M. So when we set these two equal to each other, body mass M cancels out and h is independent of animal size.

Well, not quite. Very small creatures are limited by air friction. As body mass increases, air friction becomes less important and we see that locusts and people both jump to about the same height.

At the large end of the scale, elephants don't jump well because they are pushing the strength limits of their bones and muscles just to walk around. Mid-range size animals do, however, all jump nearly to the same height.



We started with King Kong, so let’s finish with another famous monster. Tyrannosaurus rex was a bipedal carnivore, and we have seen in the movie Jurassic Park, that it could chase down jeeps at 40 mph. But could this 6000 kg monster really run? This is meant in the track and field sense of having both feet off the ground at times.

The chicken is a 3 kg bipedal carnivore that can run, and very well. So let’s put them side by side to see what we think.

From the picture above we can’t immediately tell whether we are seeing a chicken scaled geometrically up until its mass is 6000 kg, or a Tyrannosaur scaled down until its mass is 3 kg. I think we can see from the discussions today that a 6000 kg chicken could not run.

Estimates have been made that for a Tyrannosaur to run it would need to devote about 85% of its body weight to leg muscle. The chicken devotes about 15% of its body weight to leg muscle, and the average for all animals is 40% for all muscles.

The Tyrannosaur could walk, at perhaps a brisk 9 mph with its long legs, and a 3 kg Tyrannosaur could definitely run. But a full size version could not.