PHYS 725 HW #5. Due 6 December 2001

1. Riley 15.8

   Solution:

   Problem was to find eigenvalues and eigenfunctions of

   $$Ly \stackrel{df}{=} x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \tfrac{1}{4}y = \lambda y\,.$$

   Substitute $y = e^t$ and get

   $$\ddot{y} + \dot{y} + \tfrac{1}{4}y = \lambda y\,.$$

   Now eliminate the first derivative term with the substitution $y = e^{-t/2}v$ to get

   $$\ddot{v} = \lambda v$$

   subject to $v(0) = v(1) = 0$. Hence

   $$v(t) = \sin(n\pi t)$$

   and

   $$\lambda = -n^2\pi^2\,.$$

   Therefore

   $$y_n(x) = x^{-1/2}\sin(n\pi \ln x)\ .$$

   To express the solution of

   $$Ly = x^{-1/2}$$

   we expand $y(x)$ in the (complete set of) eigenfunctions of L:

   $$y(x) = \sum_{n=1}^{\infty} c_n y_n(x)\,;$$

then

$$-\sum_{n=1}^{\infty} c_n n^2 \pi^2 y_n(x) = x^{-1/2} .$$

Multiplying both sides by $y_n(x)$ and integrating from $x = 1$ to $x = e$ we get (after changing variables to $t = \ln x$)

$$-c_n n^2 \pi^2 \int_0^1 dt \sin^2(n\pi t) = \int_0^1 dt \sin(n\pi t)$$

or

$$c_n = \frac{-2}{n^3\pi^3}\left[1 - (-1)^n\right] .$$

2. Riley 15.9

   Solution:

   In physicists' units we have

   $$\nabla^2 \varphi = -4\pi\rho(\vec{x}) .$$

   This is most easily done by Fourier transform, although we could also solve the equation

   $$\nabla^2 G = \delta(\vec{x}) .$$

   To find the Green's function directly, note that

   $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) = \frac{A}{r^2}\delta(r)$$

   (the form of the above follows from the fact that the volume element in spherical polar coordinates is $d^3r = r^2 \sin\theta dr d\theta d\phi$); then

   $$\frac{\partial G}{\partial r} = \frac{A}{r^2}$$

and

$$G = -\frac{A}{r} = -\frac{1}{4\pi r}\,.$$

To do it by Fourier transform, we multiply the equation

$$\nabla^2 \varphi = -4\pi \rho\left(\vec{x}\right)$$

by $\exp\left(i\vec{k}\cdot\vec{r}\right)$ and integrate over all space:

$$\int d^3r\, e^{i\vec{k}\cdot\vec{r}} \nabla^2 \varphi\left(\vec{r}\right) = -4\pi \int d^3r\, e^{i\vec{k}\cdot\vec{r}} \rho\left(\vec{r}\right)$$

Then, manifestly,

$$\varphi\left(\vec{r}\right) = 4\pi \int d^3r' \rho\left(\vec{r}'\right) \int \frac{d^3k}{\left(2\pi\right)^3} \frac{1}{k^2}\, e^{i\vec{k}\cdot\left(\vec{r}' - \vec{r}\right)}\,,$$

where we have interchanged the orders of integration. Performing the integral over $\vec{k}$ we have

$$\varphi\left(\vec{r}\right) = \int d^3r' \frac{\rho\left(\vec{r}'\right)}{\left|\vec{r}' - \vec{r}\right|} \frac{2}{\pi} \int_0^\infty \frac{dk}{k} \sin\left(k\left|\vec{r}' - \vec{r}\right|\right) \equiv \int d^3r' \frac{\rho\left(\vec{r}'\right)}{\left|\vec{r}' - \vec{r}\right|}\,.$$

3. Riley 15.10

   Solution:

   This one was essentially done in class: we want the outgoing-wave solution of

$$\left(-\nabla^2 - K^2\right) \Psi\left(\vec{r}\right) = F\left(\vec{r}\right)\,.$$

   We Fourier transform and eventually must perform the integral

$$\int \frac{d^3k}{\left(2\pi\right)^3} \frac{e^{i\vec{k}\cdot\vec{s}}}{k^2 - K^2 - i\varepsilon} = \frac{1}{4\pi^2 is} \int_{-\infty}^{\infty} dk\, k \frac{e^{iks}}{k^2 - K^2 - i\varepsilon}\,.$$

In the limit as $\varepsilon \to 0$ we have

$$\frac{e^{iks}}{4\pi s},$$

or in other words,

$$\Psi(\vec{r}) = \int d^3r' \, \frac{F(\vec{r}')\, e^{ik|\vec{r}-\vec{r}'|}}{4\pi\,|\vec{r}-\vec{r}'|} \, .$$

4. Riley 21.1

Solution:

We were to solve the equation

$$\int_0^\infty dv\, \cos(uv)\, y(v) = \exp\left(-u^2/2\right) \, .$$

We notice that

$$\int_0^\infty dv\, \cos(uv)\, y(v) = \frac{1}{2}\int_{-\infty}^\infty dv\, e^{iuv}\left[y(v)\,\theta(v) + y(-v)\,\theta(-v)\right]$$

so we can solve for the unknown function

$$h(v) \stackrel{df}{=} y(v)\,\theta(v) + y(-v)\,\theta(-v)$$

by Fourier-transforming both sides. Thus

$$h(v) = \sqrt{\frac{4}{\pi}}\, \exp\left(-v^2/2\right) ,$$

which is the solution for $v > 0$.

5. Riley 21.2

   Solution:

   We recognize

$$\int_0^\infty f(x)\; e^{-sx} dx = \frac{a}{a^2+s^2}$$

   as a Laplace transform—therefore we inverse transform (or use a table of inverse Laplace transforms) to find

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{sx}\frac{a}{a^2+s^2} = \sin ax\,.$$

6. Riley 21.3

   Solution:

   We want to solve the Volterra equation

$$f(x) = e^x + \int_0^x (x-y)\, f(y)\; dx\,.$$

   Differentiate twice with respect to x to obtain

$$\frac{d^2 f}{dx^2} - f = e^x$$

   subject to initial conditions $f(0) = f'(0) = 1$. Laplace transform to solve:

$$\left(s^2-1\right)\tilde{f}(s) = \frac{1}{s-1} + sf(0) + f'(0) = \frac{s^2}{s-1}\,,$$

   or

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\frac{s^2 e^{sx}}{(s+1)(s-1)^2} \\ &\equiv e^x\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\frac{(s+1)^2 e^{sx}}{(s+2)s^2} \\ &= \frac{e^{-x}}{4} + e^x\left(\frac{3}{4}+\frac{x}{2}\right)\,. \end{aligned}$$

Note we could also have Laplace transformed the integral equation and used the convolution theorem, since it has a difference kernel.

7. Riley 21.5

Solution:

Expand $\psi = \sum_n a_n h_n$ and substitute:

$$\sum_n a_n h_n = \lambda \sum_n h_n (g_n, \psi) \equiv \lambda \sum_n \sum_m h_n (g_n, h_m) a_m$$

or

$$a_n = \lambda \sum_m M_{nm} a_m \, .$$

Therefore the eigenvalues are given by $\det [I - \lambda M] = 0$ which is the same thing as $\det [\lambda^{-1} I - M] = 0$. The coefficients a_n are the components of the corresponding eigenvector in discrete representation.

Applying this to

$$\psi (x) = \lambda \int_0^{2\pi} dy \, K (x, y) \, \psi (y)$$

where

$$K (x, y) = \sum_{n=1}^{\infty} \tfrac{1}{n} \cos (nx) \cos (ny) \, ,$$

we have

$$a_n \stackrel{df}{=} \int_0^{2\pi} dx \, \psi (x) \cos (nx) = \frac{\pi \lambda}{n} a_n \, .$$

Therefore the eigenvalues are $\lambda = n/\pi$ and the eigenfunctions are $\cos(nx)$.

8. Riley 21.8

Solution:

We are to solve

$$f(x) = x^2 + 2\int_0^1 (x+y)\, e^{x-y} f(y)\, dy$$

by the Fredholm method. The kernel $(x+y)\ \exp(x-y)$ is separable so we do not really need Fredholm theory. However, let us compute the Fredholm resolvent. From Riley, *et al.* we use the recursion relations

$$\begin{aligned} d_n &= \mathrm{Tr}\, D_{n-1} \\ D_n(x,y) &= K(x,y)\, d_n - \int dz\, K(x,z)\, D_{n-1}(z,y) \end{aligned}$$

to compute the coefficients

$$\begin{array}{ll} d_0 = 1 & D_0(x,y) = (x+y)\, e^{x-y} \\ d_1 = \mathrm{Tr}\, D_0 = \int_0^1 2x dx = 1 & D_1 = e^{x-y}\left(\frac{x}{2} + \frac{y}{2} - xy - \frac{1}{3}\right) \\ d_2 = \mathrm{Tr}\, D_1 = \frac{-1}{6} & D_2 = 0 \end{array}$$

The resolvent is therefore

$$R(x,y) = e^{x-y}\left[\frac{x+y-\lambda\left(\frac{x}{2}+\frac{y}{2}-xy-\frac{1}{3}\right)}{1-\lambda-\lambda^2/12}\right].$$

To solve the equation we set $\lambda = 2$ and thus get

$$f(x) = x^2 - e^x\,(3xI_3 + I_2)\ ,$$

where

$$I_n = \int_0^1 x^n e^{-x} dx\,.$$

9. What is the differential equation satisfied by

$$y(x) = x^\alpha J_{\pm m}\left(\beta x^\gamma\right)?$$

Use this result to find the eigenvalues of the swinging chain. (This is a uniform suspended chain whose lower end is free—thus the restoring force is gravitational. The horizontal displacement of a point along the chain obeys a linear, second-order pde that we discussed in class:

$$\frac{\partial^2 \Psi}{\partial t^2} = g \frac{\partial \Psi}{\partial x} + g x \frac{\partial^2 \Psi}{\partial x^2},$$

where $x = 0$ is the lower—free—end of the chain and $x = L$ is the upper end.)

Solution: To find the differential equation satisfied by

$$y(x) = x^\alpha J_{\pm m}\left(\beta x^\gamma\right)$$

we let $u = x^\gamma$ so that

$$J_{\pm m}(\beta u) = u^{-\alpha/\gamma} y\left(u^{1/\gamma}\right).$$

Since

$$u^2 \frac{d^2 J}{du^2} + u \frac{dJ}{du} + \left(\beta^2 u^2 - m^2\right) J = 0,$$

we see that

$$\frac{u^{2/\gamma}}{\gamma^2} \frac{d^2 y}{dx^2} + \frac{u^{1/\gamma}}{\gamma}\left(\frac{1}{\gamma} - \frac{2\alpha}{\gamma}\right) \frac{dy}{dx} + \left(\beta^2 u^2 + \frac{\alpha^2}{\gamma^2} - m^2\right) y = 0,$$

which may be rewritten

$$x^2 \frac{d^2 y}{dx^2} + x(1 - 2\alpha) \frac{dy}{dx} + \left(\beta^2 \gamma^2 x^{2\gamma} + \alpha^2 - m^2 \gamma^2\right) y = 0.$$

To solve the pde we let $\Psi(x,t) = e^{i\omega t}\psi(x)$ giving

$$x \frac{d^2 \psi}{dx^2} + \frac{d\psi}{dx} + \frac{\omega^2}{g}\psi = 0$$

which has the solution

$$\psi\left(x\right) = AJ_0\left(2\omega\sqrt{x/g}\right)\,.$$

Since $\psi(x = L) = 0$ the eigenvalues ω must be given by the roots of

$$J_0\left(2\omega\sqrt{L/g}\right) = 0\,;$$

the smallest of these is $2.4048\ldots$ so the smallest value of ω is

$$1.20241\ldots \times \sqrt{g/L}\,.$$

10. Energy in a star is produced in nuclear reactions initiated by collisions. If the number of collisions per unit time, of particles with CM kinetic energy between E and $E + dE$ is $NEe^{-E/\Theta}$ where $\Theta = k_BT$ is the absolute temperature in energy units and N is a constant; and if the probability that a collision will result in a reaction is $Me^{-\alpha\big/\sqrt{E}}$ (again M and α are constants), use the method of steepest descents to estimate the rate of nuclear reactions, assuming

$$\left(\Theta\big/\alpha^2\right)^{1/6} \ll 1\,.$$

Solution:
The reaction rate is

$$\frac{dn}{dt} = MN\int_0^\infty dE\, Ee^{-E/\Theta}e^{-\alpha\big/\sqrt{E}}$$

so we identify the function $f(E)$ whose saddle point we must seek as

$$f(E) = \ln E - \frac{E}{\Theta} - \frac{\alpha}{\sqrt{E}}\,.$$

Setting the derivative to 0 we find

$$\frac{df}{dE} = \frac{1}{E} - \frac{1}{\Theta} + \frac{\alpha}{2E^{3/2}} = 0$$

or with

$$u = \frac{\alpha}{\sqrt{E}}$$

we have

$$2u^2 + u^3 = \frac{2\alpha^2}{\Theta} \gg 1\,,$$

whose approximate solution is

$$u \approx \left(\frac{2\alpha^2}{\Theta}\right)^{1/3} - \frac{2}{3} \approx \left(\frac{2\alpha^2}{\Theta}\right)^{1/3}\,.$$

The second derivative is

$$\left.\frac{d^2 f}{dE^2}\right|_{E=E_0} \approx -\frac{3\alpha}{4E_0^{5/2}} < 0\,,$$

hence the saddle point is a maximum and lies on the real positive $E-$axis. We may therefore replace the integral with

$$\int_0^\infty dE\, e^{f(E)} \approx e^{f(E_0)} \int_{-\infty}^{\infty} dE e^{(E-E_0)^2 f_0''/2} = \sqrt{\frac{2\pi}{|f_0''|}}\, e^{f(E_0)}$$

giving

$$\frac{dn}{dt} \approx MN\alpha\sqrt{\frac{\pi\Theta^3}{3}}\, \exp\left[-\frac{3}{2}\left(\frac{2\alpha^2}{\Theta}\right)^{1/3}\right]\,.$$

11. Apply the Gram-Schmidt orthogonalization method to the monomials x^0, x^1, and x^2 to derive the first 3 Hermite polynomials H_0, H_1, and H_2, where these polynomials are orthogonal on the interval $(-\infty, +\infty)$ with respect to the weight function e^{-x^2}. Do not bother to normalize the polynomials.

Solution:

$$\int_{-\infty}^{\infty} dx e^{-x^2} H_m(x) H_n(x) = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} .$$

To evaluate integrals like

$$I_k(\lambda) = \int_{-\infty}^{\infty} dx\, x^{2k} e^{-\lambda x^2}$$

we note that

$$\begin{aligned} I_0(\lambda) &= \sqrt{\frac{\pi}{\lambda}} , \\ I_1(\lambda) &= -\frac{dI_0(\lambda)}{d\lambda} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} , \\ &\cdots \end{aligned}$$

Manifestly,

$$\begin{aligned} H_0(x) &= (\pi)^{-1/4} , \\ H_1(x) &= x\left(\frac{\pi}{4}\right)^{-1/4} , \end{aligned}$$

and we can then find H_2 by insisting it be orthogonal to H_0 and H_1:

$$\begin{aligned} & H_2(x) = x^2 - H_0(x)\left(H_0, x^2\right) - H_1(x)\left(H_1, x^2\right) = \\ & \quad x^2 - (\pi)^{-1/4} \int_{-\infty}^{\infty} ds e^{-s^2} (\pi)^{-1/4} s^2 \\ & \quad -x\left(\frac{\pi}{4}\right)^{-1/4} \int_{-\infty}^{\infty} ds e^{-s^2} s \left(\frac{\pi}{4}\right)^{-1/4} s^2 \\ & \quad N\left(x^2 - \frac{1}{2}\right) . \end{aligned}$$

12. The Schrdinger equation of the hydrogenic atom is

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{Ze^2}{r}\psi = E\psi\,.$$

Estimate the ground state energy using the trial function $e^{-\lambda r^2}$.

Solution:
We want to minimize the functional

$$E\left\{\psi\right\} = \frac{\iiint d^3r\left(\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - \frac{Ze^2}{r}\left|\psi\right|^2\right)}{\iiint d^3r\left|\psi\right|^2}\,;$$

with our trial function we have

$$\nabla e^{-\lambda r^2} = -2\lambda\vec{r}\,e^{-\lambda r^2}$$

hence we must minimize (we let $2\lambda = \alpha$)

$$\begin{aligned} E\left(\alpha\right) &= \frac{\int_0^\infty dr\,r^2\left(\frac{\hbar^2}{2m}\alpha^2 r^2 - \frac{Ze^2}{r}\right)e^{-\alpha r^2}}{\int_0^\infty dr\,r^2 e^{-\alpha r^2}} \\ &= \frac{\frac{3\hbar^2}{16m}\alpha^{-1/2}\sqrt{\pi} - \frac{Ze^2}{2\alpha}}{\frac{1}{4}\sqrt{\pi}\,\alpha^{-3/2}} = \frac{3\hbar^2}{4m}\alpha - \frac{2Ze^2}{\sqrt{\pi}}\alpha^{1/2}\,, \end{aligned}$$

whose minimum is

$$E_{\text{min}} = -\frac{4Z^2e^4m}{3\pi\hbar^2} = -\left(\frac{8}{3\pi}\right)Z^2\,\text{Ry} \approx -0.85Z^2\,\text{Ry}\,.$$

13. Solve the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-y|}f(y)\,dy = \frac{1}{x^2+1}$$

Solution:
This is a difference kernel on an infinite interval. Hence we can apply

the convolution theorem for Fourier transforms. Defining the Fourier transform as

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx$$

we have

$$\mathcal{F}\left(\int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy\right) \equiv \mathcal{F}(K \odot f) = \mathcal{F}(K)\,\mathcal{F}(f)\ .$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+1}\, dx &= \pi e^{-|k|} \\
\int_{-\infty}^{\infty} e^{ikx} e^{-|x|}\, dx &= \frac{2}{k^2+1} \\
\mathcal{F}(f) &= \frac{\pi}{2} e^{-|k|} \left(k^2+1\right)
\end{aligned}$$

and we can see easily that the inverse transform gives

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-ikx} \frac{\pi}{2} e^{-|k|} \left(k^2+1\right) \\
&= \frac{1}{2} \int_{0}^{\infty} dk\, \cos(kx)\, e^{-k} \left(k^2+1\right) \\
&= \frac{1}{2} \left(1 - \frac{d^2}{dx^2}\right) \frac{1}{x^2+1} = \frac{1}{2(x^2+1)} \left(1 + \frac{2-6x^2}{(x^2+1)^2}\right)
\end{aligned}$$

14. Consider the operator defined by the kernel

$$K(x,y) = \frac{1}{x^2+y^2+1}$$

(a) Show that it is Hermitian.

$$\begin{aligned}\left(\int dx\,dy\,\varphi^*(x)\,K(x,y)\,\varphi(y)\right)^\dagger &= \int dx\,dy\,\varphi(x)\,K^*(x,y)\,\varphi^*(y)\\ &\equiv \int dx\,dy\,\varphi(y)\,K(y,x)\,\varphi^*(x)\\ &= \int dx\,dy\,\varphi^*(x)\,K(y,x)\,\varphi(y)\end{aligned}$$

That is, its diagonal matrix elements in any basis are real.

(b) Show that it represents a positive-definite operator.

$$\int dx\,dy\,\varphi^*(x)\,K(x,y)\,\varphi(y) = \int_0^\infty ds\,e^{-s}\left|\int dx\,e^{-sx^2}\varphi(x)\right|^2 > 0$$

(c) Show it is bounded.
By the secret theorem (p. 284ff)

$$\|K\| \leq \sup_x \frac{1}{\sigma(x)}\int dy\,|K(x,y)|\,\sigma(y)\,.$$

Let us take the interval to be $(-\infty,+\infty)$ and let $\sigma(x)=1$. Then

$$\|K\| \leq \sup_x \int_{-\infty}^{\infty} dy \frac{1}{x^2+y^2+1} = \sup_x \frac{\pi}{\sqrt{x^2+1}} = \pi$$

(d) Show that it is compact.
The Schmidt norm is defined by

$$\|K\|_S^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(\frac{1}{x^2+y^2+1}\right)^2 = \pi\int_0^\infty du\left(\frac{1}{u+1}\right)^2 = \pi\,.$$

Since this is finite, the kernel is compact.

(e) Show that its eigenvalue spectrum is countably infinite.
Compact kernels have a countable spectrum that accumulates only at 0.

15. Evaluate the integral

$$\Lambda^{2-d}\int d^d p\left(\Delta^2+p^2\right)^{-1}=\Lambda^{2-d}\Omega\left(d\right)\int_0^\infty dp\,p^{d-1}\left(\Delta^2+p^2\right)^{-1}.$$

Solution:
Following the hint we evaluate the angular factor $\Omega\left(d\right)$:

$$\Omega\left(d\right)=\frac{\int_0^\infty dp\,p^{d-1}e^{-p^2}}{\left(\int_{-\infty}^\infty dp\,e^{-p^2}\right)^d}=\frac{\frac{1}{2}\Gamma\left(d/2\right)}{\left(\pi\right)^{d/2}}$$

so

$$\Lambda^{2-d}\int d^d p\left(\Delta^2+p^2\right)^{-1}=\frac{\Gamma\left(d/2\right)}{4\left(\pi\right)^{d/2-1}\sin\left(\pi d/2\right)}\Lambda^{2-d}\Delta^{d-2}.$$

The singularity at $d=4$ is a simple pole.