Physics 751 Homework #3

Due Friday September 26, 11:00 am.

1. (a) If U and V are unitary, is UV unitary? Prove your result.

   (b) If A and B are Hermitian, is AB Hermitian? Prove your result.

2. If H is Hermitian, prove

   (a) $U = e^{iH}$ is unitary,

   (b) that log det $U = i\mathrm{Tr}H$.

3. Find the eigenvalues and eigenvectors of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Write down explicitly the unitary matrix that diagonalizes σ_x. Can these two Hermitian matrices σ_x , σ_y be diagonalized simultaneously? Explain.

4. (a) Regarding $U(\theta) = e^{i\frac{1}{2}\theta\sigma_x}$ as a function of σ_x, use your result from problem 3 to write down its eigenvectors and eigenvalues.

   (b) Find another form for U by expanding the exponential and summing the series to give well-known functions. (We shall be using this result later.)

5. Find the eigenvalues and eigenvectors of

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and construct the unitary matrix which diagonalizes L_x.

6. Prove that both the determinant and the trace of a Hermitian matrix are unchanged in a unitary transformation, and hence find simple expressions for them in terms of the eigenvalues. (You may assume detAB = det$A$·det$B$.)