1. A particle of mass $m$, (nonrelativistic) energy $E$, spin one-half and magnetic moment $\mu\sigma$ is traveling in the positive $x$-direction from large negative $x$. For $x > 0$, there is a constant magnetic field $B$ in the positive $z$-direction. There is no field for $x < 0$.

   (a) Find the transmission coefficient as a function of energy $E$ for spin up in the $B$-direction, and also for spin down in the $B$-direction.

   (b) Suppose now the spin is initially in the $x$-direction. Describe how the spin of the transmitted particle varies, if at all. Include in your discussion the case of very low incoming kinetic energy.

2. Consider two particles of spin-1. Write the spin states as $|m\rangle_{1,2}$ where 1,2 denotes the particle and $m = -1, 0, 1$ denotes the $z$-component of that particle’s spin.

   (a) The total spin of the two particles may be 2, 1, or 0. Give the number of states corresponding to each value. Do you expect them to be symmetric or antisymmetric with respect to exchange of two particles?

   (b) Find explicit expressions for the states of total spin 0 and 1.

   (c) An alternative basis for the spin-1 states is the *Cartesian* basis

\[
|x\rangle = \frac{1}{\sqrt{2}} (|+1\rangle - |-1\rangle) \\
|y\rangle = -\frac{i}{\sqrt{2}} (|+1\rangle + |-1\rangle) \\
|z\rangle = |0\rangle.
\]

Write the total spin 0 and spin 1 states that you found in part 2 in this basis (use the basis for both the constituent spin 1’s and the total spin 1 states). Comment on the rotational properties of your answer.

   (d) By acting with $S^+ = S_1^+ + S_2^+$ show that for two spin-$s$ particles the state

\[
\sum_{m=-s}^{s} (-1)^m |m\rangle_1 |-m\rangle_2
\]

is always a spin singlet (total spin 0).

   (You may find it useful to recall $S^+ |s, m\rangle = \sqrt{s(s+1)-m(m+1)} |s, m\pm 1\rangle$.)
3. The electron states in a certain two-dimensional material (graphene) lying in the plane $z = 0$ can be described by the wavefunction $\Psi(r) = \psi_A(r)f_A(r) + \psi_B(r)f_B(r)$ where $r = (x, y)$. $f_{A/B}(r)$ are two periodic functions and the wavefunctions $\psi_{A/B}(r)$ satisfy the two-component wave equation

$$v_F \left[ p_x \sigma_x + p_y \sigma_y \right] \begin{pmatrix} \psi_A(r) \\ \psi_B(r) \end{pmatrix} = E \begin{pmatrix} \psi_A(r) \\ \psi_B(r) \end{pmatrix}$$

where $p = -i\hbar(\partial_x, \partial_y)$ is the usual momentum operator in two dimensions, $\sigma_{x,y}$ are the usual Pauli matrices, and $v_F$ is a constant having units of velocity. In the presence of a magnetic field we make the usual substitution $p \rightarrow p - eA/c$ (all vectors in this problem are two-dimensional)

(a) By considering the effect of a parity transformation $x \rightarrow -x$, $y \rightarrow -y$ (or otherwise) show that solutions of Eq. (1) come in $\pm E$ pairs (or have $E = 0$).

(b) Choose a convenient vector potential appropriate to describe a uniform magnetic field in the $z$-direction. For the Hamiltonian $H = v_F (p - eA/c) \cdot \sigma$ find the form of the operator $H^2$.

(c) By identifying $H^2$ with a familiar Hamiltonian, deduce the eigenvalues of $H^2$ and hence of $H$.

4. Consider the one-dimensional quantum mechanical system given by the Hamiltonian

$$H = \frac{1}{2} p^2 + \frac{\lambda^2}{2x^2} + \frac{1}{2} x^2$$

where $[x, p] = i$ (take $\hbar = 1$) and $\lambda > 0$.

In analogy with the construction of the eigenvalues of the angular momentum operator $L_3$, we find the eigenvalues of $H$, with the help of raising and lowering operators.

(a) Show that the operators $H, K$ and $D$ all have simple commutation relations with each other, where

$$K = \frac{1}{2} p^2 + \frac{\lambda^2}{2x^2} - \frac{1}{2} x^2$$

$$D = \frac{1}{2} (xp + px) = xp - i/2$$
(b) Show that $C_0 = H^2 - D^2 - K^2$ commutes with $H$, $D$ and $K$ using the commutation relations derived in (a). ($C_0$ is analogous to $L^2$, $H$, $D$, $K$ are analogous to $L_1$, $L_2$ and $L_3$.)

(c) Calculate $C_0$ explicitly in terms of $x$ and $p$. Cancel out as many $x$’s and $p$’s as possible: $C_0$ should look extremely simple!

(d) Show that $K + iD$ applied to an eigenstate of $H$ reproduces an eigenstate of $H$ with higher eigenvalue. This is the raising operator. Write down the lowering operator.

(e) Find all the eigenvalues of $H$.

5. The spin states of the Hydrogen atom in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$ can be described by a Hamiltonian

$$H = A\hat{\mathbf{\sigma}} \cdot \hat{\mathbf{r}} - \mu_e \sigma^z B$$

where $A > 0$ is the hyperfine coupling between the electron and the proton, and $\hat{\sigma}^{el} = \sigma_x^{el}, \sigma_y^{el}, \sigma_z^{el}$ are the spin $\frac{1}{2}$ operators for the electron and proton spins.

(a) Without detailed calculation, sketch the dependence of the eigenenergies on $B$, including both $B < < A$ and $B > > A$ regions.

(b) Find the eigenstates and eigenenergies.

6. As a simple model for the ionization of an atom by a charged particle, consider a one dimensional harmonic oscillator of frequency $\omega$ subject to the perturbation

$$H_1 = \lambda \delta \ x - vt .$$

Find to lowest order the probability that the system, initially in its ground state, is left in the first excited state.

You may find it helpful to know the form of the first two eigenstates:

$$\psi_0 x = \left( \frac{m\omega_0}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega_0 x^2}{2\hbar} \right)$$

$$\psi_1 x = \sqrt{2} \left( \frac{m\omega_0}{\pi \hbar} \right)^{1/4} x \exp \left( -\frac{m\omega_0 x^2}{2\hbar} \right).$$
7. A particle of mass $m$ in three dimensions interacts with a potential

$$V(r) = -\lambda \delta (r-a)$$

that is, a delta function spread over the surface of a sphere of radius $a$, like a spherical shell.

(a) Find the minimum value of $\lambda$ for which this potential has a bound state.

(b) As the strength of the potential is increased, would you expect to see more bound states? Give a qualitative argument only.

(c) Find the phase shift for scattering from this potential in the low energy limit.

(Warning: as $k \to 0$, the phase shift is linear in $k$.)

8. Two electrons (electron mass = $m$, spin = $\frac{1}{2}$) are confined to move on a circle of radius $R$ (as for example in a model for the electronic states of [ring-shaped] benzene).

The Hamiltonian is

$$H_0 = -\frac{\hbar^2}{2mR^2} \left( \frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right)$$

where $\theta_i$ is the angular position of particle $i$ on the circle.

(a) Find the lowest energy, as well as the corresponding eigenfunction(s). Give the degeneracy, if any, explicitly. Do the same for the first excited state.

(b) Include in the energy of part (a) an electron interaction term of the form

$$V = V_0 \cos \theta_1 - \theta_2, \quad V_0 \ll \frac{\hbar^2}{2mR^2}.$$  

Find the corrected energy and degeneracy (if any) of the ground state(s) to lowest order of perturbations in $V_0$. Discuss the first order correction to the wave function(s) of the ground state(s)—in particular, how does/do the zeroth order wave function(s) of the first excited state(s) contribute?

9. (a) Write down the Schrödinger equation for a particle in one dimension with the potential $V(x) = \lambda_1 \delta(x) + \lambda_2 \delta(x-a)$.

(b) Find the boundary conditions satisfied by the wave function at $x = 0$ and $x = a$ for states of negative or zero energy.
(c) For a solution to the Schrödinger equation with negative energy, sketch the possible forms of the wave function of a bound state in the three regions, taking both delta functions to be attractive. (This is not necessarily the lowest lying state.)

(d) If $\lambda_1$ is a negative (attractive potential), what is the positive (repulsive) value of $\lambda_2$ for which there is a zero-energy bound state in this system? Show that your answer makes sense in the limit of $a$ going to zero.

10. A particle of mass $m$ starts ($t = -\infty$) in the ground state of a one-dimensional harmonic oscillator $V(x) = \frac{1}{2}m\omega^2x^2$. At time $t = 0$, the harmonic oscillator potential suddenly disappears, and then it reappears later at a time $t = \tau$. That is, the potential is

$$V(x,t) = \begin{cases} \frac{1}{2}m\omega^2x^2, & t < 0; \\ 0, & 0 < t < \tau; \\ \frac{1}{2}m\omega^2x^2, & \tau < t. \end{cases}$$

A long time later ($t \to \infty$), the energy of the particle is measured. Solve exactly for the probability that this measurement yields $\frac{1}{2}h\omega$.

The ground state wave function of a harmonic oscillator is $\psi_0(x) = (m\omega/\pi\hbar)^{1/4}\exp\left(-m\omega x^2/2\hbar\right)$. An integral you may or may not find useful is

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2/2}e^{-ikx} = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}.$$

11. The electron states in a certain two-dimensional material [Graphene] lying in the plane $z = 0$ can be described by the wavefunction $\Psi(r) = \psi_A(r)f_A(r) + \psi_B(r)f_B(r)$ where $r = (x, y)$, $f_{A/B}(r)$ are two periodic functions and the wavefunctions $\psi_{A/B}(r)$ satisfy the two-component wave equation

$$v_F\left[p_x\sigma_x + p_y\sigma_y\right]\begin{pmatrix} \psi_A(r) \\ \psi_B(r) \end{pmatrix} = E\begin{pmatrix} \psi_A(r) \\ \psi_B(r) \end{pmatrix}$$  \hspace{1cm} (1)

where $p = -i\hbar(\partial_x, \partial_y)$ is the usual momentum operator in two dimensions.
(a) Find the plane-wave solutions of Eq. (1) and the corresponding eigenenergies.

Suppose that every negative energy state is occupied by one electron (ignoring spin) and consider the effect of the plane wave propagating in the $z$-direction

$$A(r,t) = A_0 \hat{e}_x \cos(kz - \omega t).$$

By replacing $p$ with $p - eA(c)$ in the wave equation in the usual way, use the Golden Rule to calculate the fraction of energy absorbed by the beam as follows:

(b) Treat the vector potential term in the wave equation as a perturbation and find the matrix element between a state in the lower band and one in the upper band with the same momentum, assuming that $v_F \ll c$.

(c) Find the density of states with one electron in the upper band, using the eigenenergies calculated in part (a).

(d) Find the energy flux of the beam, and combine with the rate found from the Golden Rule to show that the fraction of incident energy absorbed from the beam is $\pi \alpha / 2$, where $\alpha = e^2 / hc$ is the fine structure constant.

12. A particle of mass $M$ is restricted to move only along the circumference of a fixed circle of radius $R$ lying in the $xy$ plane. The particle moves freely along the circumference.

(a) What are the eigenenergies of this system? What are the corresponding wave functions as functions of the position variable $s = R\theta$?

(b) Write the Hamiltonian in terms of the momentum $p_s$, conjugate to $s$.

(c) Now suppose that the particle is an electron and that there is an external uniform magnetic field $\vec{B}$ in the $+z$ direction. Write down the Hamiltonian. Include spin, but ignore spin-orbit coupling. [Note: $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$.]

(d) What are the eigenenergies of this system? What is the ground state energy $E_0$ as a function of $B$? Sketch $E_0$ vs. $B$. 

13. A simple harmonic oscillator in two dimension with characteristic frequency $\omega$ is perturbed by the potential $V = Axy$. Consider all the states corresponding to the lowest two energy levels of the unperturbed system. Determine their energies to the second order in perturbation theory.

\[
\text{Information you may find useful: } a = \sqrt{\frac{m\omega_0}{2\hbar}} \left(x + i \frac{p}{m\omega_0}\right) .
\]

14. Two identical particles of mass $m$ and spin $\frac{1}{2}$ move in one dimension, in a state of zero total momentum. They interact via the spin-independent potential $V(x_1 - x_2) = g\delta(x_1 - x_2)$, and they may be treated as non-relativistic particles obeying the Schrödinger equation. Solve for their wave functions with energy $E > 0$ for

(a) the singlet state;

(b) the triplet states.

Do not calculate the overall normalization coefficient.

In a collider scattering experiment, the two-particle system is in an energy eigenstate. The particles are spin polarized, that coming in from the left has spin up, that from the right has spin down.

(c) After the collision, what is the probability that the particle going out to the right has spin up? Give your answer in terms of $m$, $g$, $E$ and $\hbar$.

15. This is a one-dimensional nonrelativistic problem. A particle of charge $q$ and mass $m$ in one dimension is bound at $t = 0$ to a delta function potential $V(x) = -\lambda\delta(x)$. A constant electric field $E_0$ in the +x direction is switched on at $t = 0$, off at the later time $t = \tau$. The field is assumed small enough for perturbation theory to be valid.

Compute the probability of ionization as follows:

(a) Find and normalize the initial bound state wave function.

(b) Take the possible final states to be free particle states $\psi_k \propto e^{ikx}$, defined by periodic boundary conditions on a line of length $L$. Write down the normalized wave function $\psi_k$ and find the density of states in energy.
(c) We will be using time-dependent perturbation theory: write down the perturbation Hamiltonian (the electric field term) and find its matrix element between the initial state and a final state $\psi_k$. 

(d) Prove that the transition probability to a final state $f$ at time $t$ is given by

$$|c_f(t)|^2 = \frac{1}{\hbar^2} \left| \int_0^t V_{fi}(t') e^{i\omega t'} dt' \right|^2,$$

where $V_{fi}$ is the matrix element from (c).

(e) Derive an expression for the transition probability to a plane wave state in the energy range $E, E + dE$.

16. A spin one-half particle is in the magnetic field

$$\vec{B}(t) = B_0 \hat{z} + B_1 \left( \hat{x} \cos \omega t + \hat{y} \sin \omega t \right)$$

and has Hamiltonian $H = -(\gamma/2) \hbar \hat{\sigma} \cdot \vec{B}$.

(a) Show that $H$ can be written

$$H = -(\gamma \hbar/2) \left[ B_0 \sigma_z + B_1 e^{-i\omega \sigma_z/2} \sigma_x e^{i\omega \sigma_z/2} \right].$$

(b) Transforming the wave function to a rotating frame $|\psi'\rangle = e^{i\omega \sigma_z/2} |\psi\rangle$ prove that $|\psi'\rangle$ satisfies

$$i\hbar \frac{\partial}{\partial t} |\psi'\rangle = -\left[ \frac{\hbar}{2} (\omega + \gamma B_0) \sigma_z + (\gamma \hbar B_1/2) \sigma_x \right] |\psi'\rangle = H' |\psi'\rangle,$$

say.

(c) Suppose the particle has spin up along the $z$ axis at $t = 0$, and the rotating field has frequency equal to that of spin precession about the field $B_0$ alone, at what later times will measurement of the spin component along the $z$ axis certainly give spin down?

17. (Here you may find it useful to know that $J_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)} |j, m\pm 1\rangle$.)

A Stern-Gerlach apparatus has its magnetic field and the gradient of that field lying in the $(x, z)$ plane, at 45 degrees to both the $x$- and $z$-axes, as shown in the sketch. Call this the $z'$-direction.
A beam of atoms, each with total spin zero and angular momentum \( l = 1 \), passes through the apparatus to be sorted according to the angular momentum component \( m_z \) in the \( z' \) direction.

(a) From the information above, construct the matrix operators \( \hat{L}_x, \hat{L}_y, \hat{L}_z \), in the \( z' \)-basis, and use them to construct \( \hat{L}_z \), also in the \( z' \)-basis.

(b) Show that \( \hat{L}_z \) is Hermitian.

(c) Write down (or calculate) the eigenvalues of \( \hat{L}_z \).

(d) Consider the beam emerging from this Stern-Gerlach apparatus with \( m_z = 0 \).
Suppose this beam is passed through a second Stern-Gerlach apparatus, this time with the magnetic field gradient in the usual \( z \)-direction, so this beam is separated according to the value of \( m_z \). Calculate the probabilities of finding the various possible values of \( m_z \).

18. (Here you may find it useful to know that
\[
x = \sqrt{\hbar/2m\omega}(a^+ + a), \quad p = i\sqrt{m\hbar/2}(a^+ - a), \text{ and } a^+ |n\rangle = \sqrt{n+1} |n+1\rangle.
\]
Consider the following one-dimensional simplified model for Van der Waals forces: Two atoms are treated as one-dimensional harmonic oscillators of angular frequency \( \omega \), the distance between the atoms \( R \gg \sqrt{\hbar/m\omega} \). Each atom has an immovable nucleus of charge \( +e \). The electrons, charge \( -e \) and mass \( m \), are assumed to be bound as in a harmonic oscillator of frequency \( \omega \).

(a) Ignoring the interaction between the atoms, write down the Hamiltonian \( H_0 \).
Label the electron positions \( x_1, x_2 \) relative to their respective nuclei. (This is all in one dimension only.)
(b) For \( R \gg \sqrt{\hbar/\mu \omega} \), and the atoms initially in their ground state, write down the leading nontrivial term in the interaction Hamiltonian \( H_{\text{int}} \) between the atoms (expressed as a power series in the small variables \( x_1/R, x_2/R \)) assuming the interactions between distant charges are Coulomb interactions.

(c) Determine the correction to the ground state energy of the system to lowest non-vanishing order in perturbation theory. Is the force between the two atoms attractive or repulsive? What is its \( R \)-dependence?

19. A particle of mass \( m \) is in a one-dimensional infinitely-deep square well that extends from \( x = a/2 \) to \( x = +a/2 \) (i.e. the particle cannot go outside these limits). At time \( t = 0 \), the particle is equally likely to be anywhere in the right half of the well and has zero probability to be anywhere in the left half of the well. Give an expression for the probability that a measurement at a later time \( t \) will find the particle somewhere—anywhere—in the left half.

20. With the notation from the diagram below, the electrostatic potential energy between two hydrogen atoms far apart compared with their size is

\[
V = -e^2 \left( \vec{r}_A \cdot \vec{\nabla} \right) \left( \vec{r}_B \cdot \vec{\nabla} \right) \frac{1}{R} = e^2 \left[ \frac{\vec{r}_A \cdot \vec{r}_B}{R^2} - \frac{3(\vec{r}_A \cdot \vec{R})(\vec{r}_B \cdot \vec{R})}{R^5} \right].
\]

![Diagram of two hydrogen atoms](image)

**Problem #20**

(The protons are at \( A, B \), the other small circles represent the electrons.)

(a) Taking the \( z \)-direction to be along the line of centers \( \vec{R} \), rewrite \( V \) in terms of the \( x, y, z \) components of \( \vec{r}_A, \vec{r}_B \).

(b) Assume the hydrogen atoms are both in the ground state. Find the first-order correction to the total ground state energy.

(c) For both in the ground state, find an expression for the second-order correction to the total ground state energy, and make a rough, ballpark estimate of its value (in terms of \( e, a_0 \), etc).
(d) If now one of the atoms is in the ground state $|1,0,0\rangle$, but the other is in the state $|2,1,0\rangle$, will the results of a perturbation analysis be similar to those above? That is, will the leading energy correction have the same distance dependence? Don’t do any calculation for this part – just give a reason for any significant difference from the earlier case you expect to see.

21. Suppose two noncommuting Hermitian operators $\hat{A}, \hat{B}$ have expectation values $A, B$ in a particular state $|\psi\rangle$. Write $(\Delta A)^2 = \langle \psi | (\hat{A} - A)^2 | \psi \rangle$.

(a) Prove that $(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \langle \psi | i [\hat{A}, \hat{B}] | \psi \rangle \right|^2$.

*Hint:* Write $\hat{A} = A + \hat{a}$, apply Schwartz’ inequality to $\hat{a} |\psi\rangle, \hat{b} |\psi\rangle$. The inequality is $\left| \langle \lambda | \mu \rangle \right|^2 \leq \langle \lambda | \lambda \rangle \langle \mu | \mu \rangle$ for Hilbert space vectors. Then, write $\langle \psi | \hat{a}\hat{b} | \psi \rangle$ as a real term plus a pure imaginary term.

(b) For a free particle in one dimension, take $\hat{A} = \hat{x}, \hat{B} = \hat{p}$. What is the condition on Hilbert space vectors for the Schwartz inequality to become an equality? Use this condition to deduce the form of a minimum uncertainty wave function.

(c) Now take $\hat{A} = \hat{J}_x, \hat{B} = \hat{J}_y$ for a spin one particle, and $|\psi\rangle$ is the $m = 0$ state, so the right-hand side of the equation in (a) is zero. Does this mean $\hat{A}, \hat{B}$ can be precisely measured at the same time? Explain.

22. (a) Write down the Born approximation to the amplitude $f(\theta, \phi)$ for scattering of a particle of mass $m$ by a potential $V(\vec{r})$, the potential being zero beyond some radius.

(b) Find $f(\theta, \phi)$ in the Born approximation for a particle of mass $m$ scattering off the potential

$$V(r) = \begin{cases} -V_0 & \text{for } r < r_0 \\ 0 & \text{for } r \geq r_0 \end{cases}$$

(c) Prove that in the low energy limit, the scattering from this potential becomes isotropic.

(d) Find a condition on the potential of part (b) for the Born approximation to be valid in the low energy limit. Does this relate to the possibility of a bound state in this potential? Discuss.
23. A hydrogen atom in its ground state $|1,0,0\rangle$ is subject to the time-dependent potential

$$V(\vec{r},t) = V_0 \cos(kz - \omega t).$$

Using time-dependent perturbation theory, find the transition rate for the electron to be emitted into a state with momentum $\vec{p}$, including the directional dependence.

Indicate any difference between this (perhaps unrealistic) problem and the photoelectric effect for the hydrogen atom (ejection of the atomic electron by exposure to monochromatic light).

*Information you might find useful:*

Hydrogen atom ground state:

$$\psi(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

Any plane wave state:

$$\psi(\vec{r}) = (1/L^{3/2}) e^{i\vec{k}\cdot\vec{r}}$$

24. An ionic impurity of charge $e$ in a solid acts as a mass $M$ in a three-dimensional harmonic oscillator potential $\frac{1}{2} kr^2$. Suppose that such an impurity is in the oscillator ground state at large negative time and is subject near $t=0$ to a pulse of electric field in the $z$-direction, specifically

$$\vec{E}(t) = E_0 e^{-(t/\tau)^2} \hat{z}.$$ 

Find, in lowest-order perturbation theory, the probability that the oscillator is in an excited state as $t \to \infty$. Derive any formula you use from first principles; that is, start with

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle.$$

*Information you may find useful:*

$$a = \sqrt{\frac{m\omega_0}{2\hbar}} \left( x + i \frac{p}{m\omega_0} \right)$$

25. (a) Write down the rotation operator which rotates a spin one-half state through an angle $\theta$ about an arbitrary axis.

(b) A spin one-half is initially pointing in the $z$-direction, that is, its spinor representation is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in standard notation. It is then subjected to the following sequence of operations:

(i) it is rotated by $\pi/2$ about the $x$-axis.
(ii) it is rotated by $\phi$ about the $z$-axis.

(iii) it is rotated by $\pi/2$ about an axis in the $xy$-plane chosen so that it ends up again pointing in the $z$-direction.

Find the phase difference between the final ket and the initial one. What is your answer for $\phi = 2\pi$?

Useful info: $\vec{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

26. The projection theorem states that for a vector operator $V$

$$\langle \alpha', j, m' | V^i | \alpha, j, m \rangle = \frac{\langle \alpha', j, m | \vec{J} \cdot \vec{V} | \alpha, j, m \rangle}{\hbar^2 j (j+1)} \cdot \langle j, m' | J^i | j, m \rangle.$$ 

where the $\alpha$'s denote nonangular quantum numbers, $\vec{J}$ is the angular momentum operator of the isolated system, and the superscript $i$ indicates the $i^{th}$ component of the vector.

(a) State briefly (a sentence or so) what is the physical significance of this theorem.

(b) Suppose a system has total angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$, and total magnetic moment $\vec{\mu} = \gamma_1 \vec{J}_1 + \gamma_2 \vec{J}_2$. Find the expectation values of the components of $\vec{\mu}$ in the eigenstate $|l, m\rangle$ of $\vec{J}$.

27. In the Stark effect for a hydrogen atom in its ground state, the energy change $\Delta E$ associated with a small applied electric field is proportional to the square of the field strength $\epsilon$, $\Delta E = -\frac{1}{2} \alpha \epsilon^2$, where $\alpha$ is the polarizability.

(a) Find an expression for $\alpha$ using perturbation theory.

(b) Neglecting continuum states, use bounds on energy differences to establish that

$$4a_0^3 < \alpha < (16/3)a_0^3,$$

where $a_0$ is the Bohr radius. (The ground state wavefunction for hydrogen is $\psi = C e^{-r/a_0}$, with $C$ a normalization constant, while the ground state energy is $-e^2/2a_0$.)
28. We know that for the one-dimensional oscillator,
\[ a = \sqrt{\frac{m\omega}{2\hbar}} (x + i \frac{p}{m\omega}), \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad [a, a^\dagger] = 1. \]

(a) Derive an expression for the commutator \([a, (a^\dagger)^n]\).

(b) Prove that \(|\lambda\rangle = \exp\left(-\frac{1}{2} \lambda^2 \right) \exp(\lambda a^\dagger) |0\rangle\) is an eigenstate of the (non-Hermitian) annihilation operator \(a\), and give its eigenvalue.

(c) Use the identity \(e^A e^B = e^{B e^{[A,B]}}\) to establish that \(|\lambda - \mu\rangle \langle \lambda| = \exp(-|\lambda - \mu|^2)\).

(d) Prove that these states are complete, that is, for \(\lambda = x + iy\),
\[ \int \frac{dxdy}{\pi} |\lambda\rangle \langle \lambda| = I. \]

(Hint: Use polar coordinates.)

29. An electron of charge \(e\) and mass \(m\) is constrained to move on a ring of radius \(R\).

(a) What are the energies and eigenfunctions for the system?

(b) Consider a perturbing potential \(V = a z^2\), where \(z\) is along a diameter of the ring, and find the energies of the two lowest levels to second order in perturbation theory.

30. A light wave
\[ \vec{A}(\vec{r}, t) = \vec{A}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) \]
is incident on a hydrogen atom in its ground state at the origin. The frequency \(\omega\) is sufficient to liberate the electron into a plane wave state, the problem here is to calculate the rate at which this occurs. Treat the incoming light wave as a classical field.

(a) State Fermi’s Golden Rule for calculating transition rates.

(b) Find an expression for the matrix element for the electron to be ejected into a plane wave state with momentum \(\vec{p}_f\).

(c) Give an order of magnitude argument for ignoring the \(k\)-dependence of the incident light wave in the matrix element, assuming the light is in the visible range.
(d) Find explicitly the angular dependence of this matrix element, and therefore of
the relative probability of emission into different solid angles $d\Omega$.

(e) Do the necessary integral to evaluate the matrix element. You can take the
normalization constants of the hydrogen ground state and the plane wave state to
be $N_1, N_2$ if you don’t know them.

31. A particle with electric charge $q$ is confined to move in the $xy$-plane, with a uniform
perpendicular magnetic field, magnitude $B$, and a corresponding vector potential
$\vec{A} = (0, Bx, 0)$.

(a) Write down the time-independent Schrödinger equation for the particle.

(b) Find the commutation relations of the Hamiltonian with the momenta $p_x$ and $p_y$.
Use this information to write the eigenfunction in the form $\psi(x, y) = \psi(x)\varphi(y)$
where one of the functions $\psi, \varphi$ is a plane wave. Which one? Find the
differential equation the other function satisfies, and write down its lowest
energy solution.

(c) Suppose now that the system has finite size $L$ in the $y$-direction, and assume
periodic boundary conditions in that direction. How does that affect possible
solutions of the $x$-direction equation?

(d) If the system can be taken to have infinite extent in the $x$-direction, what is the
density of lowest energy states per unit length in that direction?

(e) Choose one particular ground state wave function and find the probability
current distribution.

32. A spin one (not spin one-half!) particle has its component of spin parallel to the positive
$z$-axis equal to one (that is, it is in the $m = 1$ state).

(a) Suppose the component of spin is measured in the direction defined by the
vector $(1, 1, 1)$. What is the probability of finding $m = 1$ in that direction?

(b) In the same $(1,1,1)$ direction, what is the probability of finding $m = -1$?

Possibly useful info (possibly not!):

$$Y_{1}^{\pm 1} = \mp (3/8\pi)^{1/2} \sin \theta e^{\pm \varphi}$$
$$Y_{1}^{0} = (3/4\pi)^{1/2} \cos \theta$$
\[ [J_i, J_j] = i\hbar \epsilon_{ijk} J_k \]
\[ J^2 \mid jm \rangle = \hbar^2 j(j+1) \mid jm \rangle \]
\[ J \pm = J_x \pm iJ_y \]
\[ J \pm \mid jm \rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} \mid j, m \pm 1 \rangle \]

33. Consider a two-level system with Hamiltonian \( H = H_0 + V(t) \), where \( H_0 \) has energy levels \( E_1, E_2, E_1 < E_2 \). The matrix elements of the perturbation \( V(t) \) in the eigenbasis of \( H_0 \) are:
\[ V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real}). \]

(a) Write down Schrödinger’s equation for the system, and show that it can be expressed in terms of the amplitudes \( c_1(t), c_2(t) \) for the two states as the coupled differential equations:
\[ i\hbar \dot{c}_k = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_0 t} c_n, \quad (k = 1, 2). \]

Given that at \( t = 0 \) only the lower level is populated so \( c_1(0) = 1, \quad c_2(0) = 0 \), find \( |c_1(t)|^2 \) and \( |c_2(t)|^2 \) exactly by solving the coupled differential equations

(b) Do the same problem using time-dependent perturbation theory to lowest non-vanishing order. Compare the two approaches for small values of \( \gamma \).

34. You have a 1-dimensional harmonic oscillator, \( V(x) = \frac{1}{2} m \omega^2 x^2 \), with an angular frequency composed of a constant term plus a small “wobbling” term:
\[ \omega = \begin{cases} 
\omega_0, & t < 0 \\
\omega_0 + \delta \omega \cos \alpha t, & 0 < t < T \\
\omega_0, & t > T
\end{cases} \]
where \( \delta \omega \ll \omega_0 \). The system is in its ground state just before \( t = 0 \).

(a) Use perturbation theory in the small parameter \( \delta \omega \) to find the (time-dependent) amplitude for the probability that the system is in a particular excited state for times larger than \( T \). Leave your answer in terms of unevaluated matrix elements and/or unevaluated time integrals.

(b) Evaluate any matrix elements that you may have found in part (a). To which excited states can transitions occur to this order of perturbation theory?
35. Two electrons scatter from one another. The forces between them are such that \( f(\theta) \) would be the scattering amplitude if the electrons were distinguishable. (\( \theta \) is the center-of-mass scattering angle.) But of course real electrons are not distinguishable, and the following questions apply to real electrons. [Express your answers, when appropriate, in terms of \( f(\theta) \).]

(a) What is the differential cross section at \( \theta = 90^\circ \) if the electrons form a spin singlet?

(b) What is the differential cross section at \( \theta = 90^\circ \) if the electrons form a spin triplet?

(c) What is the differential cross section at \( \theta = 90^\circ \) if the electrons are unpolarized?

36. Two electrons (electron mass = \( m \), spin = \( \frac{1}{2} \)) are confined to move on a circle of radius \( R \) (as for example in a model for the electronic states of [ring-shaped] benzene). The Hamiltonian is

\[
H_0 = -\frac{\hbar^2}{2mR^2} \left( \frac{\partial^2}{\partial \theta^2_i} + \frac{\partial^2}{\partial \theta^2_j} \right)
\]

where \( \theta_i \) is the angular position of particle \( i \) on the circle.

(a) Find the lowest energy, as well as the corresponding eigenfunctions(s). Give the degeneracy, if any, explicitly. Do the same for the first excited state.

(b) Include in the energy of part (a) an electron interaction term of the form

\[
V = V_0 \cos(\theta_1 - \theta_2), \quad V_0 \ll \frac{\hbar^2}{2mR^2}
\]

Find the corrected energy and degeneracy (if any) of the ground state(s) to lowest order of perturbations in \( V_0 \). Discuss the first order correction to the wave function(s) of the ground state(s)—in particular, how does the zeroth order wave function(s) of the first excited state(s) contribute?

37. It can be shown that the amplitude for the decay of an initial atomic state \( |i\rangle \) (wave function \( \psi_i \)) to a final state \( |f\rangle \) (wave function \( \psi_f \)) with the emission of a single photon is proportional to the integral (the reduction of a matrix element)

\[
\int d^3r \, e^{-i\mathbf{k}\cdot\mathbf{r}} \left( \psi_f^* \nabla \psi_i - \psi_i^* \nabla \psi_f \right)
\]
where in this expression the photon wave number is $\tilde{k}$. If the wavelength of the emitted photon is large compared to the atomic dimensions, then one can set the harmonic factor $e^{-ikr} \approx 1$. This is known as the dipole approximation, and the emitted photon is dipole radiation.

(a) Consider hydrogen-like atoms (arbitrary $Z$ for the nucleus but a single electron). For which such atoms is the dipole approximation justified? You may suppose that the relevant measure of atomic size is provided by the size of the smaller of the two states in question.

(b) Assume that the dipole approximation is justified. You can then integrate the expression above by parts to find the integral

$$\int d^3r \psi_j^* \tilde{\nabla} \psi_i.$$

Show that this integral is proportional to the matrix element of the operator $\tilde{r}$, i.e.

$$\int d^3r \psi_j^* \tilde{r} \psi_i.$$

(That is why the term dipole approximation is used.)

(c) By writing $\tilde{r}$ in spherical coordinates, show that the dipole radiation is emitted such that the orbital angular momentum quantum number changes by one.

$Hint$: Spherical harmonics have the general property that

$$\sin \theta e^{\pm ip} Y_{\ell m} = A_{\ell \pm m} Y_{\ell+1,m \pm 1} + B_{\ell \pm m} Y_{\ell-1,m \pm 1}$$

$$\cos \theta Y_{\ell m} = C_{\ell m} Y_{\ell+1,m} + D_{\ell m} Y_{\ell-1,m}$$

38. Two identical spin-1/2 fermions of mass $m$ are placed in the same one-dimensional harmonic oscillator potential $\frac{1}{2} \alpha x^2$. The only interaction between the two fermions is a small quadratic one of the form $\delta H = g \cdot (x_1 - x_2)^2$, where $g$ is a constant.

(a) Determine the lowest three distinct possible total energies for the 2-fermion system to first order in $g$. [That means you may ignore $O(g^2)$ effects.]

Consider all possible spin states.

(b) What are the corresponding normalized 2-particle wave functions to zeroth-order in $g$? You must show the spin as well as spatial parts of the wave functions. [You may express your wave function in terms of normalized 1-particle eigenfunctions $\psi_n(x)$ of the 1-particle harmonic oscillator problem and needn't give explicit formulas for $\psi_n(x)$.]
39. A particle of mass $m$ starts ($t = -\infty$) in the ground state of a one-dimensional harmonic oscillator $V(x) = \frac{1}{2}m\omega^2x^2$. At time $t = 0$, the harmonic oscillator potential suddenly disappears, and then it reappears later at a time $t = \tau$. That is, the potential is

$$V(x,t) = \begin{cases} \frac{1}{2}m\omega^2x^2, & t < 0; \\ 0, & 0 < t < \tau; \\ \frac{1}{2}m\omega^2x^2, & \tau < t. \end{cases}$$

A long time later ($t \to \infty$), the energy of the particle is measured. Solve exactly for the probability that this measurement yields $\frac{1}{2}\hbar\omega$.

The ground state wave function of a harmonic oscillator is $\psi_0(x) = (m\omega/\pi\hbar)^{1/4}\exp(-m\omega x^2/2\hbar)$. An integral you may or may not find useful is

$$\int_{-\infty}^{+\infty} dx
e^{-a^2/2}e^{-ikx} = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}.$$
\[
\langle R^2 \rangle_\psi = \alpha \langle r^2 \rangle_{aa} + \beta \langle r^2 \rangle_{bb} + \gamma \langle r \rangle_{aa} \cdot \langle r \rangle_{bb} + \varepsilon \langle |r|_{ab} \rangle^2 ,
\]
where \( \langle f \rangle_{ij} \) (with \( i, j = a \) or \( b \)) denotes the one-particle spatial matrix elements
\[
\langle f \rangle_{ij} = \int d^3 r \psi^*_i f \psi_j
\]

As part of your derivation, find the numerical values of the coefficients \( \alpha, \beta, \gamma, \) and \( \varepsilon \).

(d) Do the same as part (c) but for the \( s = 1 \) states.

(e) Show that electrons with \( s = 1 \) are generically further apart on average (and never closer on average) than electrons with \( s = 0 \), as measured by \( \langle R^2 \rangle_\psi \). This is an example of what fundamental physical effect?

41. A long-wavelength, polarized, electromagnetic wave is incident on a hydrogen atom. Long wavelength means \( \lambda >> r_{\text{atom}} \). Ignoring spin, the electron's interaction with the external radiation field can be treated as a perturbation,
\[
H_{\text{int}} = -\frac{e}{mc} \mathbf{A} \cdot \mathbf{p} + \frac{e}{2mc^2} |\mathbf{A}|^2 
\]
(in the Coulomb gauge, \( \nabla \cdot \mathbf{A} = 0 \)), to the atomic Hamiltonian. Neglect the \( A^2 \) term in parts (a)-(c). Treat the electromagnetic wave \textbf{classically} and the electron quantum mechanically.

(a) Explain why, under these circumstances, the probability for the electron to make a transition from an initial (unperturbed) atomic eigenstate \( |\psi_i \rangle \) to a final one \( |\psi_f \rangle \) is proportional to the square of some linear combination of the matrix elements \( \mathbf{W}_{fi} \equiv \langle \psi_f | \mathbf{p} | \psi_i \rangle \).

What determines which linear combination of
\[
\langle \psi_f | p_x | \psi_i \rangle, \quad \langle \psi_f | p_y | \psi_i \rangle,
\]
and \( \langle \psi_f | p_z | \psi_i \rangle \) is relevant?

(b) Derive an expression for \( \mathbf{W}_{fi} \) in terms of \( \mathbf{r}_{fi} \equiv \langle \psi_f | \mathbf{r} | \psi_i \rangle \) and the (unperturbed) energy eigenvalues of the states, \( E_i \) and \( E_f \). As always, show your work.

(c) Use parity, and the fact that \( \mathbf{r} \) is a vector operator, to derive the selection rules, in the above approximations, for radiative transitions between states of angular momentum \( |l_i m_i \rangle \) and \( |l_f m_f \rangle \). Explain your reasoning: a simple statement of the selection rules is \textit{not} an adequate response.
(d) When the intensity of the electromagnetic wave is small enough to treat $H_{\text{int}}$ as a perturbation to the atomic problem, then one often ignores the $A^2$ term in $H_{\text{int}}$ compared to the $A \cdot \mathbf{p}$ term, as you have above. Justify this. Explain any caveats to your justification.

42. The standard $l = 1$ spherical harmonics $Y_l^{m_z}(\hat{\mathbf{r}})$ can be written in the form

$$Y_l^0(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \hat{r}_z = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_l^{\pm 1}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} (\hat{r}_x \pm i\hat{r}_y) = \mp \sqrt{\frac{3}{8\pi}} \left( \frac{x \pm iy}{r} \right),$$

where the unit vector $\hat{r} = \mathbf{r}/|\mathbf{r}|$ has been used to specify direction, and

$$(\hat{r}_x, \hat{r}_y, \hat{r}_z) = (x/r, y/r, z/r)$$ refer to its components.

(a) What are the $l = 1$ eigenfunctions of $L_x$ in terms of $\hat{r}_x$, $\hat{r}_y$, and $\hat{r}_z$? Clearly specify the eigenvalue of $L_x$ associated with each eigenfunction you write down. [Hint: Simply use symmetry and/or simply rotate coordinates.]

(b) Consider the electron of an idealized hydrogenic atom. [“Idealized” means the simple non-relativistic Coulomb problem: ignore spin-orbit interactions, center of mass corrections, hyperfine splitting, the Lamb shift, etc.] An experimenter has a large collection of these atoms. For each atom, he measures $L_z$ and $L_x$, and he throws the atom away if the result does not correspond to $l = 1$ and $m_z = 1$. For each atom he keeps, he then does the following:

- Simultaneously measure $L_x^2$ and $L_x$ (an instant after the earlier measurement). What are the possible values of $L_x^2$ and $L_x$, and what fraction of his large number of measurements has each value?

(c) Another experimenter has a similar collection of atoms. For each atom, she also measures $L_x^2$ and $L_z$ and throws the atom away unless the result corresponds to $l = 1$ and $m_z = 1$. For each atom she keeps, she then does the following additional sequence of measurements:

[1.] Simultaneously measure $L_x^2$ and $L_x$ (an instant after the earlier measurement). An instant later, make measurement 2 below.

[2.] Simultaneously measure $L_z^2$ and $L_z$. What are the possible values of $L_z^2$ and $L_z$ from the final measurement (measurement 2) above, and what fraction of her large number of measurements has each value?
43. Consider a system of three spin-$\frac{1}{2}$ particles. Let $\hat{S} = \hat{S}_1 + \hat{S}_2 + \hat{S}_3$ be the total spin.

(a) What are the eigenvalues of $S^2$?

(b) How many independent states (of the spins) are there with each of those eigenvalues?

(c) Explicitly write a complete set of orthonormal $S^2$ eigenstates, writing each as a superposition of the $|m_1, m_2, m_3\rangle$ eigenstates $|\uparrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$, $|\uparrow\downarrow\downarrow\rangle$, etc. Normalize your states.

A formula which may or may not be useful to you:

$$J_{\pm}|j, m\rangle = \hbar \sqrt{j(j+1)-m(m\pm1)} |j, m\pm1\rangle.$$ 

44. Consider the bound-state problem for the attractive one-dimensional delta-function potential $-g\delta(x)$. You can convert the usual stationary Schrödinger equation for this into a momentum space equation by multiplying the Schrödinger equation by $f = \frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}$ and integrating over $x$.

(a) Using the fact that the momentum-space wave function $\varphi(p)$ is the integral over $x$ of $f\psi(x)$, find the equation obeyed by $\varphi(p)$.

(b) Solve this equation.

(c) Once you have solved for $\varphi(p)$, find $\psi(x)$ through the inverse transform.

(d) Show that evaluating $\psi(0)$ provides a relation for the energy. How many bound states are there? Solve the energy relation (if you can) to find the allowed bound state energy (or energies).

45. Use the Rayleigh-Ritz variational method to estimate the lowest energy level of the one-dimensional anharmonic oscillator defined by the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \beta^2 x^4$$

(*Hint: try a Gaussian trial wave function with one variational parameter.*)
46. An electron with charge $e$ and mass $m$ is confined to move on a circle of radius $r$. It is perturbed by a uniform electric field $F$ parallel to one of the diameters of the circle. Find the perturbation of the energy levels up to terms of the order of $F^2$. Notice in particular the anomalous behavior of the first excited state.

47. (a) Derive the commutation relations of $J_\pm = J_x \pm iJ_y$ with $J_z$, where $\vec{J}$ is an angular momentum operator. (You may assume the standard results for $J_i, J_j$ or, if necessary, derive everything you need from scratch.)

(b) Derive the normalization of $J_\pm |jm\rangle$. (That is, what is $\langle \psi | \psi \rangle$ if $|\psi\rangle = J_\pm |jm\rangle$ and $\langle jm | jm \rangle = 1$? Then phrase your final result as a formula for $J_\pm |jm\rangle$ in terms of other $|jm\rangle$’s (using any consistent phase convention). In your derivation, you may assume the standard result for the eigenvalues of $\vec{J}^2$.

(c) A two-particle system consists of a spin 1 particle and a spin $\frac{1}{2}$ particle. The total spin state can, of course, be represented either by a basis with good $S_{1z}$ and $S_{2z}$ quantum numbers or alternatively by a basis with good $\vec{S}^2 = S_{1z}^2 + S_{2z}^2$ and $S_z = S_{1z} + S_{2z}$ quantum numbers. Write the expansion of each possible $|sm\rangle$ state in terms of $|m_1m_2\rangle$ states. [Hint: Start with the state with highest $m_1$ and $m_2$.]

48. We define the Heisenberg representation as the one in which time dependence is put into operators and not into states. The time-dependent operator $A(t)$ is defined through the expectation value, as

$$\langle \psi(t)|A|\psi(t)\rangle = \langle \psi(0)|A(t)|\psi(0)\rangle,$$

where $A$ is the (time-independent) operator that appears in the Schrödinger representation.

(a) Give an explicit expression for $A(t)$ in terms of $A$ and any other operators or algebraic quantities you might like to use.

(b) Find the equation of motion obeyed by the operator $A(t)$.

(c) Consider a harmonic oscillator, for which the hamiltonian is

$$H = \hbar \omega (a^\dagger a + 1/2),$$
where $a^\dagger$ and $a$ are the creation and destruction operators. Show that in the Heisenberg picture the commutation relations between the time-dependent operators are the same as those at $t = 0$, namely

$$[a^\dagger(t), a(t)] = -1.$$

(d) Solve the equations of motion for $a^\dagger(t)$ and $a(t)$ in terms of their values at $t = 0$.

(e) Calculate the commutator $[a^\dagger(t), a(0)]$.

49. Consider a one-dimensional potential consisting of two identical square wells side by side. The depth of each well is $V_0$, and the width of each well is $a$, with $V_0$ and $a$ having values such that several bound states would be possible in a single well with those parameters. The separation between the wells is $b$, as in the figure.

(a) Sketch the ground state wave function $\psi_0$ and the first excited state wave function $\psi_1$ in the three cases that $b = 0$, $b = O(a)$, and $b >> a$.

(b) How do the corresponding energies $E_0$ and $E_1$ vary as $b$ varies between the possibilities of part (a)? Sketch $E_0$ and $E_1$ as functions of $b$.

(c) If you view this as a model of the potential energy of the electron in $H_2^+$, the nuclei can move to minimize the energy. Does the electron serve to bring the nuclei closer or to push them apart?

50. $F$ is the scattering amplitude of a very low energy plane wave, wave number $k$, with $kL << 1$ (where $L$ is the range of the potential), traveling along the positive $z$-axis, scattering from a single scattering center.

(a) What is the $\theta$-dependence of $F$, where $\theta$ is the scattering angle, measured from the positive $z$-direction?

(b) If there are two such scattering centers located respectively at $(x, y, z) = (-a, 0, 0)$ and $(+a, 0, 0)$ what is the scattering amplitude at the same $k$ as in part (a)?

(c) What is your answer to (b) if the two scattering centers are at $(0, 0, -a)$ and $(0, 0, +a)$?
Atoms are made up of nuclei that consist of protons and neutrons with masses \( M \approx 940 \text{ MeV}/c^2 \) and electrons with masses described by \( m_e \approx 0.5 \text{ MeV}/c^2 \). Suppose that we lived in a world in which the electrons had mass 5.0 \text{ MeV}/c^2 but with the nucleons retaining the masses they have now. You can continue to regard the nucleons as essentially infinitely heavy, and you should also pretend that there are no weak interactions in this hypothetical world (if there were, there would be no world!).

(a) What is the density of liquid water?

(b) At what temperature would paper burn? (normal paper burns at 451° Fahrenheit)

(c) Given that the rate for the \( 2p \rightarrow 1s \) transition in real hydrogen is \( 0.6 \times 10^9 \text{ s}^{-1} \), what would be the rate for this transition in our hypothetical world?

(d) Would the Planck radiation law be changed in the hypothetical world, and if so, how?

\[(\text{Hint: just figure out how atomic sizes and energies depend on the electron mass.})\]

A three-dimensional harmonic oscillator is perturbed by a constant force. Calculate the energies \( E_n \) to second order and compare with the exact results for energies and states. Explain.

Consider a rotator consisting of two identical particles (bosons) of mass \( m \) connected by a massless rigid rod of length \( 2R \). The center of the rod is attached to a point, so that the system is constrained to move in the \( xy \)-plane. When we ask about angular momentum, therefore, we are really asking about the \( z \)-component of angular momentum. What is the angular wave function, and what are the possible values of the angular momentum?

We can use an extension of the above result to make a "perfectly smooth" cylinder by considering a large chain of atoms aligned in a circle. (We consider only rotations about the axis perpendicular to the plane of the ring.) Therefore suppose you have an array of \( N \) identical equally spaced particles, each of mass \( m \), forming a ring of radius \( R \). The total mass of the ring is \( M = Nm \). Assume throughout that \( M \) and \( R \) are fixed. What are the possible values of the angular momentum of this system? \((\text{Hint: Think about invariance of the wave function under rotations of the system})\)

Find the energy difference between the ground state (that of zero angular momentum) and the first excited rotational state—in particular, what is the difference in the "smooth" limit, the case \( N \rightarrow \infty \)?

Repeat part (c) for a marked cylinder, in which, say, one of the atoms is missing or is different from the others. Explain the difference, if any, between your answers to parts (d) and (c).
54. Consider the bound states of two massive particles whose interaction \( V \) is central, attractive, and everywhere finite.

(a) Show by variation that the ground state of the system is an \( s \)-state.

(b) Use the WKB approximation to show that, if \( V(r) \sim 1/r^2 \) with \( \lambda > 2 \) for \( r \to \infty \), then the system has only a finite number of bound states.

\( \text{Hint: Use the variational reasoning of a) as a justification for considering only } s \text{-states and recall } \int_0^\infty R(r) \, dr \sim (n + \frac{1}{2}).\)\)

55. Consider an electron moving in the periodic potential

\[ V(x) = V_0 \sum_{n=-\infty}^{+\infty} \delta(x-n) \]

where \( V_0 > 0 \). There is a solution to Schrödinger’s equation having the form

\[ \Psi(x) = e^{ikx}U_k(x), \]

where \( U_k(x) \) is a periodic function, \( U_k(x + 1) = U_k(x) \).

(a) Show that \( \Psi(x) \) in the interval \( 0 < x < 1 \) has the form \( \Psi(x) = Ae^{iqx} + Be^{-iqx} \) and give \( q \) in terms of the electron energy \( E \).

(b) From the continuity of the wavefunction at the origin (and the periodicity of \( U \)) find an equation relating \( A, B, q \) and \( k \).

(c) Find another relationship among \( V_0, A, B, q \) and \( k \) by considering the derivatives of the wavefunction on approaching the origin from either side.

(d) Deduce that the \( E, k \) relation is determined by

\[ \cos k = \cos q + \frac{mV_0}{q\hbar^2} \sin q. \]

(e) Do real \( k \) solutions exist for energies \( E \) such that \( q \) is close to but slightly larger than \( \pi \)? Explain.

56. In the ammonia molecule \( \text{NH}_3 \), the three H’s form an equilateral triangle and the nitrogen atom lies on a line through the centroid of the triangle and perpendicular to its plane. Points of minimum energy are at distances \( \pm a \) from the plane. Refer to the state with the N at \( +a \) as \( |1\rangle \), at \( -a \) as \( |2\rangle \). Suppose the Hamiltonian has the form \( H = H_0 + V \) where

\[ H_0 |1\rangle = E_0 |1\rangle, \quad H_0 |2\rangle = E_0 |2\rangle, \quad \text{and} \quad \langle 1|V|1\rangle = 0, \quad \langle 2|V|2\rangle = 0, \quad \langle 1|V|2\rangle = -A. \]
(a) Find the energy of the ground state and first excited state of the system. What is the form of the corresponding wavefunctions?

(b) Give a very rough estimate for the electric dipole moment of NH$_3$ in configuration $|1\rangle$.

(c) Derive an expression for the energies of the two eigenstates of NH$_3$ in an electric field, and sketch your result as a function of field strength.

57. Derive an expression for the differential scattering cross section for the elastic scattering of electrons by hydrogen atoms. Use the Born approximation. For hydrogen,

$$\phi_{1s} = \left(\frac{4\pi}{a_0}\right)^{\frac{1}{2}} a_0^{-\frac{3}{2}} e^{-r/a_0}$$

here $a_0$ is the Bohr radius. (Ignore exchange.)

58. A charged particle in a potential well (for instance, an atom with one electron outside of closed shells) has the spectrum shown in the figure: If the particle is in the second excited state (energy $E_2$), and emits a photon, which transition is most likely? Calculate the transition rate for this photon emission. The radial parts of the wave functions are:

$$R_0 = N_0 e^{-ar}$$

$$R_2 = N_2 (1 - r) e^{-3r}$$

$$R_1 = N_1 r e^{-br}$$

$$R_3 = N_3 r^2 (1 - \frac{1}{2}r) e^{-dr}.$$

Carry out the integrations if time permits.

59. (a) Considering only Coulomb interactions, what is the binding energy of the hydrogen atom in the ground state?

(b) The effective Hamiltonian for the ground state of hydrogen may be written

$$H = H_0 + A\hat{S}_z \cdot \hat{S}_p + \mu_o B S_{ex} - \mu_p B S_{ps}$$
where $H_0$ is the Coulomb Hamiltonian, $A$ is a constant, $B$ is a magnetic field in the $Z$ direction, $\mathbf{S}_e$ and $\mathbf{S}_p$ are respectively the electron’s and proton’s spin vectors. What physical quantities are represented by $\mu_o$ and $\mu_p$? What is the order of magnitude of the ratio $\mu_o/\mu_p$?

(c) If $B = 0$, what is the splitting of the ground state in terms of $A$? What is the splitting called?

(d) What is the splitting of the ground state in the limit $B \to \infty$?

(e) Sketch qualitatively how the energy levels of the system change as $B$ is increased from 0 to $\infty$.

60. (a) A plane wave is incident from the left on the potential shown here. Calculate the reflection amplitude (not its square, the reflection probability) for all incident energies.

(b) What happens to a wavepacket that is incident on this potential? Give a qualitative discussion of what happens as a function of the average energy of the wavepacket, including the possibility of resonance with quasibound states in the well.